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Heuristics and Mathematical Representation in Einstein's Search for a Gravitational Field Equation

Jürgen Renn and Tilman Sauer

EINSTEIN FIRST DEALT WITH THE PROBLEM of a relativistic theory of gravitation in 1907 (Einstein 1907). He was then confronted with the task of revising the classical Newtonian theory of gravitation in the light of the relativity theory of 1905 since he had to write a review paper that would have to cover the implications of this theory for various areas of physics. Indeed, contrary to the field theory of electrodynamic interactions, Newton's theory of gravitation implies an instantaneous action at a distance, incompatible with the requirement of special relativity that the propagation speed of physical interactions is limited by the speed of light. Hence the revision of Newtonian gravitation theory entered Einstein's intellectual horizon as a necessity of the day, imposed by the need to integrate new results with the traditional body of knowledge.

The advent of the theory of special relativity had sharpened the conflict, but an incompatibility of classical mechanics and electrodynamics had widely been recognized by physicists before. Hence not only Einstein but also several of his contemporaries addressed the problem of formulating a field theory of gravitation that was to be in agreement with the principles suggested by the theory of the electrodynamic field, and most importantly with the new kinematics of relativity theory. In addressing this problem, however, Einstein quickly decided to take a step different from that of his contemporaries. Instead of formulating a special relativistic law of gravitational interaction, he searched for a generalization of

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This article is based on joint work with Michel Janssen, John Norton, and John Stachel. Preliminary versions of this paper were presented at the Fourth International Conference on the History of General Relativity held in Berlin, 31 July–3 August 1995, and at the Boston Colloquium for Philosophy of Science "Einstein in Berlin: The First Ten Years," Boston, 3–4 March 1997. A larger joint work is in preparation.

H. Goenner, J. Renn, J. Ritter, T. Sauer (eds.)
The Expanding Worlds of General Relativity
(Einstein Studies, volume 7) pp. 87–125

the relativity principle and for a new theory of gravitation that closely associated gravitation with inertia. The peculiarity of Einstein's approach is embodied in the two heuristic principles that guided his search from the beginning and that will be designated here throughout as the *equivalence principle* and the *principle of generalized relativity*, although these principles did not bear any specific names in Einstein's first paper on the subject. In his 1907 paper the first principle was formulated as the assumption of physical indistinguishability between a homogeneous gravitational field and a uniformly accelerated frame of reference. The second principle was adumbrated by the question:

> Is it conceivable that the principle of relativity also applies to systems that are accelerated relative to each other?[1] (Einstein 1907: 454).

These principles do not express features of classical or special relativistic physics as it was then understood but rather requirements to be satisfied only by the new theory of gravitation. And indeed, they are since the time of the completion of the theory of general relativity in 1915, with its generally covariant field equation, often claimed to capture the most essential aspects of this theory, although this claim has not remained undisputed (Norton 1993).

If accepted, this claim seems to imply, however, that precisely what later turned out to be the essential and most innovative conceptual developments brought about by general relativity were already anticipated by Einstein long before he found the field equations of his new theory. Is general relativity hence the result of a conceptual leap which belongs to a context of discovery that can only be further elucidated by studying the psychological roots of Einstein's creativity but will ultimately remain inexplicable? The history of the discovery of the field equations of general relativity seems to present a strong argument in favor of this conjecture. In fact, Einstein formulated his heuristic guidelines at the very beginning of his search for a new theory of gravitation and he stubbornly held on to them for roughly eight years, in spite of the considerable difficulties he found in implementing his original ideas. But which exactly were the difficulties that Einstein encountered in formulating the field equations of general relativity?

At first glance there seem to be two alternatives: If Einstein's difficulties were predominantly of a technical nature, due either to the complexity of the mathematical language to be learned and adapted to the problem at hand or to some false working hypotheses about the mathematical representation to be eliminated along the way, then the conceptual innovation brought about by general relativity would indeed be contained *in nuce* in Einstein's starting point. The discovery of the field equations of general relativity could then be recounted as a comedy of errors, with a brilliant beginning, some deviations, and a happy conclusion, yet without explanation of the crucial steps of conceptual development involved. The role of the formalism for this discovery would then be merely that of a medium in which preconceived physical conceptions are expressed more or less appropriately.

If Einstein's difficulties were, however, themselves of a conceptual nature then it

[1] "Ist es denkbar, daß das Prinzip der Relativität auch für Systeme gilt, welche relativ zueinander beschleunigt sind?"

rather seems that his eventual success must be interpreted as a case of serendipity, as a lucky finding that cannot be accounted for by the quality of his heuristic starting point. The role of the formalism for Einstein's discovery of the field equations would then be rather that of an ill-mastered language expressing a meaning different from the intended one. In this version, conceptual development would hence be accounted for by a lucky misunderstanding. In other words, these two extreme alternatives both fail to provide any rational account of the conceptual development from classical and special relativistic physics to the physics of general relativity. They also fail to take into account any non-trivial interaction between heuristics and representation. In the present paper we will analyze this interaction more closely and propose, on the basis of this analysis, a third, alternate account of conceptual development in the case of general relativity.

Fortunately, Einstein's research in the crucial years 1907 to 1915 is not only documented by his publication of preliminary versions of general relativity but also by research notebooks, as well as by his contemporary correspondence. In order to analyze the relationship between heuristics and formalism in the emergence of general relativity we will make use of preliminary results of an extensive study of these documents in the context of a cooperative research project involving Michel Janssen, John Norton, John Stachel, as well as the authors.[2]

1. Einstein's "Zurich Notebook" and the genesis of general relativity

The task of reconstructing Einstein's discovery of the theory of general relativity has challenged historians of science for a long time.[3] Major progress in this endeavour is due to John Stachel's and John Norton's path-breaking investigations.[4]

In the course of preparing the editorial project of the *Collected Papers of Albert Einstein* John Stachel first realized the significance of the so-called Zurich Notebook (Einstein *CP4*: doc. 10) for the reconstruction of the genesis of the theory of general relativity. In 1984, John Norton published a comprehensive reconstruction of Einstein's discovery process (Norton 1984). In this paper, he was able to correct some widespread prejudices about Einstein's path towards general relativity by an analysis of some key pages of the Zurich Notebook. But even after this pioneering work large parts of the Notebook remained obscure and poorly understood. In the course of a research project involving Peter Damerow, Werner Heinrich, Michel Janssen, John Norton, John Stachel, as well as the authors[5] the Zurich Notebook

[2] While the reconstruction of the Zurich Notebook on which we base our analysis is a joint achievement of this project, the authors alone bear the responsibility for the views expressed in this paper.

[3] A (certainly incomplete) list of the older secondary literature includes Hoffmann 1972; Lanczos 1972; Mehra 1974; Earman & Glymour 1978a, 1978b; Vizgin & Smorodinskii 1979; Pais 1982. More recent literature may be found in the volumes of the "Einstein Studies" series, Howard & Stachel 1989; Eisenstaedt & Kox 1992; Earman, Janssen & Norton 1993.

[4] See Stachel 1980, 1982, 1989 and Norton 1984.

[5] The research project was begun under the direction of Peter Damerow and Jürgen Renn with the Working Group Albert Einstein, funded by the Senate of Berlin and affiliated with the Center for Development and Socialization headed by Wolfgang Edelstein at the Max Planck Institute for Human

could now be fully reconstructed and comprehensively analyzed. It turned out that this research manuscript allows for a detailed understanding of Einstein's working procedures and the heuristic principles guiding them. The analysis presented in the following is based on the results of this collaboration.

The Zurich Notebook originally comprised 96 pages, the cover bearing the title "Relativität" in Einstein's hand. Eighty-four pages of this Notebook contain calculations or short notes on various problems of physics, mainly on gravitation theory. Due to the character of a research notebook, most of the calculations are extremely sketchy, display false starts, and come with almost no explanatory text. The dating of these entries to the critical period between summer 1912 and spring 1913, as well as the identification of their chronological sequence, must hence rely on the mathematical notation used, and, most importantly, on a detailed reconstruction of their meaning.

Let us briefly review the principal steps of Einstein's discovery of the theory of general relativity and locate the Zurich Notebook in this context. As mentioned above, in 1907 when Einstein was still an employee at the patent office in Bern he had already laid out crucial elements of the heuristics he would follow in the years to come, in particular the equivalence principle and the principle of generalized relativity. As early as 1907 he also considered three possible physical consequences of the equivalence principle, the gravitational red-shift, the bending of light in a gravitational field (Einstein 1907: sect. V), and the perihelion advance of Mercury.[6] He inferred from the equivalence principle that the speed of light must be variable, in contrast to one of the fundamental principles of the special theory of relativity of 1905. In spite of this rapid and impressive initial progress, however, he did not yet begin to work out a theory of gravitation based on a generalized principle of relativity. In fact, when the history of Einstein's work on his theory in the following years is judged from hindsight, it may appear as a sequence of missed opportunities, characterized by some reluctance to react to work on gravitation by his colleagues and in particular to that work which actually paved the way for further development. Not only did Einstein fail to follow up on his own first successful steps towards a relativistic theory of gravitation, he also failed for some time to adopt Minkowski's reformulation of special relativity in terms of a four-dimensional space-time manifold, a crucial instrument for the further development of a relativistic theory of gravitation. A pivotal element of Minkowski's four-dimensional representation of special relativity was the invariance under linear,

Development and Education in Berlin. It was continued under the direction of Jürgen Renn as part of the project of studies of the integration and disintegration of knowledge in modern science at the Max Planck Institute for the History of Science in Berlin.

[6] See Einstein to Conrad Habicht, 24 December 1907: "At the moment I am working on a relativistic analysis of the law of gravitation by means of which I hope to explain the still unexplained secular changes in the perihelion of Mercury." ("Jetzt bin ich mit einer ebenfalls relativitätstheoretischen Betrachtung über das Gravitationsgesetz beschäftigt, mit der ich die noch unerklärten Aenderungen der Perihellänge des Merkur zu erklären hoffe.") Einstein *CP5*: 82. Although Einstein did not say *how* he tried to calculate the perihelion advance it seems rather unlikely that he would not have based his consideration on the equivalence principle expounded shortly before in the 1907 review paper.

orthogonal transformations of the quantity

$$x^2 + y^2 + z^2 - (ct)^2,$$

formed from Cartesian coordinates x, y, z, t, with c denoting the speed of light (Minkowski 1908). This quantity, when taken infinitesimally as

$$ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2, \qquad (1)$$

in fact later turned out to be suitable not only for representing the 'flat' space-time of special relativity but also the 'curved' space-times of general relativity.

Only in 1911, now a professor in Prague, did Einstein come back to the problem of gravitation, when it occurred to him that a physical consequence of the equivalence principle, the bending of light in a gravitational field, might be observable for light rays passing near the sun (Einstein 1911). The publication of this remarkable physical prediction drew the attention of other scientists to Einstein's approach, with important consequences for the further development. In January 1912, it was Max Abraham who published a theory of gravitation (Abraham 1912a) with a gravitational field equation, formulated in terms of Minkowski's four-dimensional space-time formulation, which yielded Einstein's basic relation between the variable velocity of light and the gravitational potential as a special case.

Einstein responded with the publication of a different theory of gravitation (Einstein 1912a), based on the equivalence principle, but still not formulated in a four-dimensional formulation and restricted to the special case of static fields. He also attacked Abraham's theory because of what Einstein saw as the incompatibility between Abraham's simultaneous use of both Minkowki's formalism and Einstein's variable speed of light. To this, Abraham in turn responded, in an extremely brief note of correction (Abraham 1912b),[7] with the suggestion that his use of Minkowski's geometry is to be understood as being restricted to an infinitesimally small local region of the four-dimensional space-time, with the line element (1) (and hence the geometry of space-time) in general being dependent on the gravitational field through the coefficient $c^2 = c^2(x, y, z)$ in front of dt^2. It was thus Abraham who effectively introduced the line element of a non-flat space-time into gravitation theory.[8]

Although this insight would shortly become the basis of Einstein's further search for the gravitational field equation, it was at this point taken up neither by Abraham himself nor by Einstein—another missed opportunity. Only after further elaborating his theory for the special case of static fields, and after familiarizing himself with Minkowski's formalism more carefully, did Einstein, at some point in the summer of 1912 (Stachel 1980), eventually adopt a 10-component metric tensor

[7] Abraham's note suggests that Einstein expressed his criticism not only in print but also in private correspondence. Unfortunately, it seems that no such correspondence has been preserved.

[8] For a discussion of the historical evidence for this period, see Pais 1982: sect. 12b; see also Stachel 1980 and Maltese & Orlando 1995.

$g_{\mu\nu}$, describing a variable line-element in four dimensions,

$$ds^2 = \sum_{\mu,\nu=1}^{4} g_{\mu\nu}\, dx^\mu dx^\nu,$$

as the representation of the gravitational potential, and hence as the basis for his further search for a relativistic field theory of gravitation.

Earlier, probably in the spring of 1912, Einstein had conceived of yet another possibility for gaining observational support for a new theory of gravitation—gravitational lensing, at the time so distant from observational possibilities that Einstein did not even publish his prediction of this effect (Renn, Sauer & Stachel 1997).

In August 1912 Einstein left Prague, where he had stayed for one and a half years, for Zurich. There he remained, as a professor of the Eidgenössische Technische Hochschule (ETH), until he left for Berlin in the spring of 1914. During his time in Zurich he collaborated with his former ETH classmate, now professor of mathematics at the ETH, Marcel Grossmann, in his attempts to find a gravitational field equation for the metric tensor. Grossmann was particularly helpful in making the literature on invariant theory and on the absolute differential calculus accessible to Einstein. He also suggested to Einstein some mathematically plausible candidates for a gravitational field equation.

It has long been known that during this period of time, Einstein missed, so at least it appears from hindsight, another important opportunity. In fact, he had come close to the later field equation of general relativity by considering the Ricci tensor,

$$R_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} - \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\kappa} + \Gamma^\kappa_{\mu\nu}\Gamma^\lambda_{\kappa\lambda},$$

given to him by Grossmann as a possible candidate for the left-hand side of such a field equation, whose right-hand side is given by the energy-momentum tensor $T_{\mu\nu}$.[9] Numerous speculations have been ventured for why he needed another two and a half years before he took up this promising line of attack once more.[10] The detailed examination of the Zurich Notebook has meanwhile increased the list of missed opportunities. John Norton found in his first pioneering analysis of the Notebook that Einstein not only considered the Ricci tensor in 1912, about three years before he came back to it in November 1915, but that Grossmann showed him also a way of deriving another candidate for the left-hand side of the field equation closely related to the Ricci tensor. Indeed, one may subtract from the Ricci tensor a part which transforms tensorially under a restricted group of coordinate transformations and take the remaining part,

$$N_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} - \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\kappa},$$

as a new candidate for the gravitational field equation.

This 'tensor'[11] was also reconsidered by Einstein in November 1915 (Einstein 1915a), and for this reason will in the following simply be called the "November tensor."

The analysis of the Zurich Notebook carried out in the context of our joint research project has revealed another surprising missed opportunity. At the end of 1912 or in the beginning of 1913 Einstein even happened to consider, in linearized form, the final version of the field equation of general relativity. Adding a trace term ($R \equiv g^{\mu\nu}R_{\mu\nu}$) to the Ricci tensor, he actually considered what is now called the Einstein tensor,

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R,$$

as the left-hand side of a field equation, and discarded it as well. But in spite of pursuing these promising candidates, the Notebook instead ends with a short derivation of the left-hand side of the curious *Entwurf* field equations,

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\, g^{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x^\beta}\right) - g^{\alpha\beta} g^{\tau\rho}\frac{\partial g_{\mu\tau}}{\partial x^\alpha}\frac{\partial g_{\nu\rho}}{\partial x^\beta} - \frac{1}{2}\frac{\partial g_{\tau\rho}}{\partial x^\mu}\frac{\partial g^{\tau\rho}}{\partial x^\nu} + \frac{1}{4}\, g_{\mu\nu} g^{\alpha\beta}\frac{\partial g_{\tau\rho}}{\partial x^\alpha}\frac{\partial g^{\tau\rho}}{\partial x^\beta}, \tag{2}$$

so called because Einstein and Grossmann published field equations with this differential expression in a paper entitled "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation" in the spring of 1913 (Einstein & Grossmann 1913).[12]

The *Entwurf* field equations are represented by a complex mathematical differential operator which is covariant only under some restricted class of coordinate transformations, and for this reason make no sense from the point of view of modern general relativity. Clearly, the *Entwurf* equations are covariant at least under linear transformations but the precise transformational properties were unknown to Einstein and Grossmann when they published their paper and hence also the

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9 Here

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\left(\frac{\partial g_{\mu\kappa}}{\partial x^\nu} + \frac{\partial g_{\kappa\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right)$$

denotes the Christoffel symbols of the second kind. Summation over repeated indices is implied throughout this paper albeit in some equations below we will write out redundant summation signs in order to be closer to the formulas as written in the Zurich Notebook.

10 See the references cited in (Norton 1984: nn. 8, 9).

11 For the restricted group of unimodular coordinate transformations which leave $g \equiv \det(g_{\mu\nu})$ invariant, the quantity $\Gamma^\lambda_{\mu\lambda} = \partial(\ln g)/\partial x^\mu$ transforms as a vector, and hence its covariant derivative $\Gamma^\lambda_{\mu\lambda;\nu} = \partial\Gamma^\lambda_{\mu\lambda}/\partial x^\nu - \Gamma^\kappa_{\mu\nu}\Gamma^\lambda_{\kappa\lambda}$ transforms as a tensor. This is just the quantity by which the Ricci tensor differs from the November tensor $N_{\mu\nu}$ which therefore transforms also covariantly under unimodular coordinate transformations.

12 The article was published jointly by Einstein and Grossmann according to the title page. In our context it may be noteworthy, however, that the paper itself was divided into a 'physical' part for which Einstein alone took the responsibility, and a 'mathematical' part which was written by Grossmann. For an historical discussion of the *Entwurf* theory, see Norton 1984.

extent to which their field equation represented a realization of the principle of generalized relativity.

Until the fall of 1915 Einstein continued to elaborate on and refine the *Entwurf* theory, at first together with his friends Marcel Grossmann and Michele Besso, then, for the most part, on his own. He explored the transformational properties of its field equation and developed arguments that were intended to explain its failure to be generally covariant, i.e., not to realize the generalized principle to the utmost extent.[13] Einstein also developed new ways of deriving the *Entwurf* field equation from fundamental assumptions and probed the consequences of the theory, in particular—albeit again without success—the possibility of explaining on its basis the anomalous perihelion advance of Mercury.[14] He also studied the relation of the *Entwurf* theory to alternative relativistic theories of gravitation, in particular to that formulated by Gunnar Nordstrøm.[15] Nordstrøm's theory was, however, as little successful as Einstein's in explaining the perihelion shift of Mercury, the only direct empirical consequence not explained by Newtonian gravitation theory that was then accessible to observational control.

Although Einstein knew of the failure of the *Entwurf* theory to pass the test of explaining the Mercury anomaly since the middle of 1913, he nevertheless continued to hold on to this theory for more than two years. When in September 1915 he definitely found out that its field equation is not satisfied by the gravitational field of a uniformly rotating system, spoiling temporary hopes that this might be, after all, the case, he at first also missed the opportunity of giving up this theory.[16] In the fall of 1915 Einstein also had to realize that an earlier, mathematical derivation of its field equation was flawed. But even the discovery of this flaw did not imply the immediate demise of the *Entwurf* field equation as Einstein was quick to add additional assumptions to make its derivation work.[17]

It may thus almost come as a surprise that, in November 1915, Einstein did give up the *Entwurf* theory, after all. It was at this point that he once more turned to the discarded candidate field equations considered in the Zurich Notebook. But he still did not take up the correct field equation involving the Einstein tensor, the field equation whose linearized version he had considered three years earlier when working with the Zurich Notebook. Instead, he returned to the non-generally covariant candidate which he had also abandoned earlier and which we have called the "November tensor" (Einstein 1915a). For some days, he was convinced he had solved all the problems which he had found earlier with this tensor. He then

[13] The most prominent of these is the so-called "hole argument." See Stachel 1980 and Norton 1987 for a discussion of this argument.

[14] For an account of Einstein's computation of the perihelion shift on the basis of the *Entwurf* theory, see Earman & Janssen 1993 and Einstein *CP4*: 344–359.

[15] For a discussion of Nordstrøm's theory, see Norton 1992.

[16] For an account of Einstein's discovery of the incompatibility of the *Entwurf* equations with Minkowski space-time in rotating Cartesian coordinates (in itself a sequence of missed opportunites), see Janssen 1997.

[17] See Einstein to H. A. Lorentz, 12 December 1915 (EA 16–442).

abandoned the November tensor once again but still did not return to the correct Einstein tensor. He rather examined the Ricci tensor first (Einstein 1915b), only to give it up as well, also for a second time, after about a week. Only then did he at last come back to the Einstein tensor and he presented the correct field equations based on it to the Prussian Academy on 25 November 1915 (Einstein 1915d).

In order to analyze Einstein's willingness to give up generally covariant field equations and his motives behind these rather erratic looking leaps from one candidate field equation to another, and in order to understand the reasons for what appear to be so many missed opportunities on his part, one has to take a closer look at his heuristic principles and their effect on his research. More than any of Einstein's published papers, the Zurich Notebook provides an ideal source for such an analysis.

2. Elements of Einstein's heuristics

As pointed out above, the early summer of 1912 represents a crucial turning point in the history of Einstein's discovery of the gravitational field equation because it is then that he realized the significance of the metric tensor and the general line element for a generalized theory of gravitation. Following this insight Einstein started to study (again) the mathematics of Gaussian surface theory, and in the course of a more systematic survey of the relevant mathematical literature, undertaken in the collaboration with Grossmann, became acquainted with Beltrami invariants and the "absolute differential calculus" of Ricci and Levi-Civita.[18] The mathematical formalism provided by the theory of differential invariants and tensor calculus for Einstein presented an as yet unfamiliar network of deductive possibilities and constraints which had to be explored and endowed with physical meaning.

Before illustrating in Section 3 the interactive process of representation and interpretation, of deductivity and heuristics, by analyzing an episode of this process documented in the Zurich Notebook, we briefly review the building blocks out of which Einstein wanted to construct his new theory of gravitation and discuss the crucial principles guiding his heuristics.

2.1. Metric tensor, special relativity, and theory of static fields

Einstein assembled the building blocks for his new theory of gravitation on an early page (39L, see Figure 1) of the Zurich Notebook: the metric tensor, Minkowski's four-dimensional reformulation of special relativity, and the left-hand side of the gravitational equation of his second theory of static gravitational fields.

The general line element with a metric tensor $g_{\lambda\mu}$,

$$ds^2 = \sum g_{\lambda\mu}\, dx^\lambda dx^\mu,$$

for the first time altogether appears on this early page of the Zurich Notebook.

[18] For a historical survey of the relevant mathematics, including a discussion of Einstein's pertinent mathematical background, see Reich 1994.

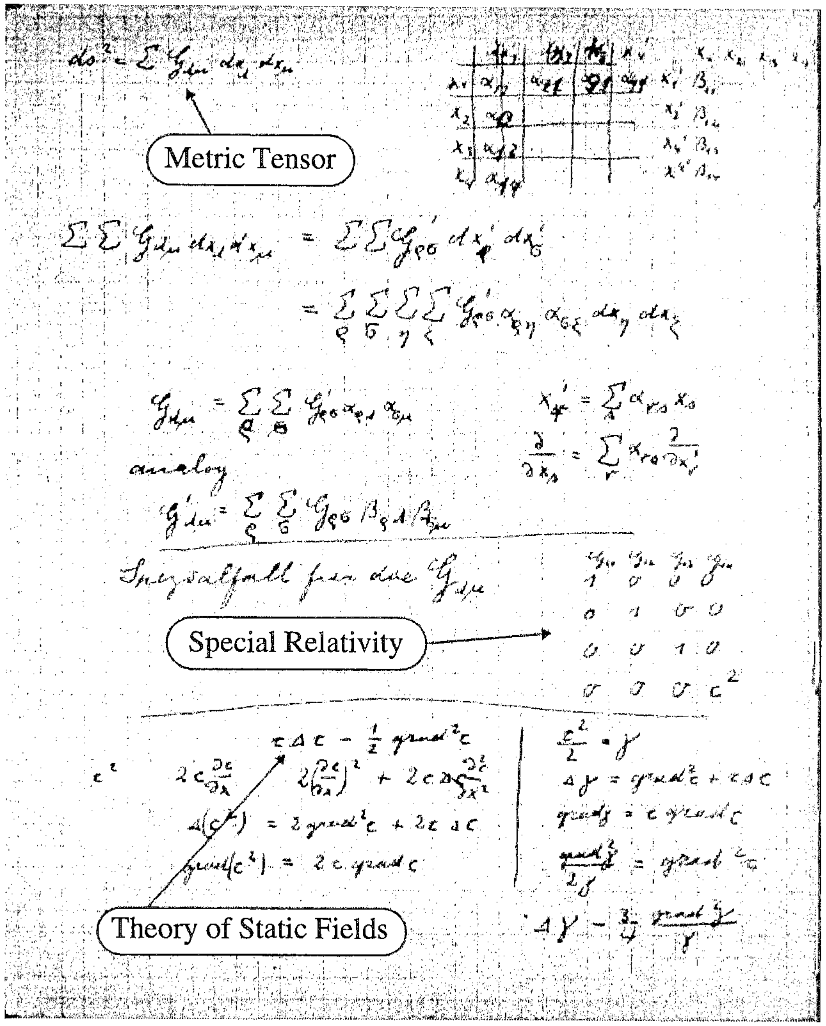

Figure 1. Page 39L of the Zurich Notebook. This early page of the Notebook contains Einstein's basic elements for a relativistic theory of gravitation.

The identification of p. 39L as the earliest page in the Notebook dealing with gravitation is corroborated both by Einstein's initial use of the majuscule G for denoting the metric tensor (in the remainder of the manuscript he uses the usual minuscule g) and by the elementary character of the calculations. Here in fact Einstein was still checking the tensorial character and the transformational properties of the metric tensor by explicit coordinate transformations. In later parts of the manuscript his techniques became much more sophisticated.

The special form of a diagonal metric,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix}, \tag{3}$$

with the g_{44}-component given by the square of the velocity of light refers to the four-dimensional reformulation of special relativity with an imaginary time coordinate. It represented the special case to which tentative generalizations had to reduce in the absence of gravitational fields. Einstein expected that his theory of static gravitational fields with the (spatially variable) velocity of light representing the gravitational potential could also be embedded into a generalized tensorial theory by means of this particular form of a metric, as becomes clear from the third and last building block displayed on this page.

The differential expression for the velocity of light c written on the bottom of the page can be identified as the left-hand side of the "field equation" for static gravitation,

$$c\Delta c - \frac{1}{2}(\operatorname{grad} c)^2 = kc^2\sigma, \tag{4}$$

which Einstein had published in March 1912 (Einstein 1912b). Equation (4) was obtained as a nonlinear generalization of the classical scalar Poisson equation with c representing the potential of the static gravitational field, k being a constant, and σ denoting the field-generating mass and (non-gravitational) energy density.

Einstein's intention for the transformation $c^2/2 = \gamma$, performed on the bottom of p. 39L, obviously was to put the differential equation (4) into a form that allowed its interpretation as one particular component of a 10-component tensorial field equation for the metric tensor.

This calculation therefore closely pertains to the central question governing all calculations of the Zurich Notebook dealing with the problem of gravitation: *What is the appropriate differential expression $\Gamma_{\mu\nu}$ which is formed from the metric tensor and its first and second derivatives and which enters a field equation of the form*

$$\Gamma_{\mu\nu} = \kappa T_{\mu\nu}, \tag{5}$$

with the stress-energy tensor $T_{\mu\nu}$ of matter as the source term on the right-hand side?

2.2. The heuristic requirements

Finding an answer to this question had two different aspects. On the one hand, Einstein had to find suitable candidates for the differential expression acting on and involving the metric tensor. On the other hand, he had to check, for each such candidate, whether it actually satisfied all his heuristic requirements.

Einstein's heuristic criteria were, of course, variable to some extent and were indeed modified in the course of and adapted to the explorative experiences accumulated in the research process. As we will illustrate below, the interplay of these general heuristic ideas with the deductive structure materialized in the mathematical representation governed the calculations contained in the Zurich Notebook and make their sequence understandable. The reconstruction of these calculations, together with an analysis of Einstein's publications and his contemporary correspondence, allows the identification of four distinct heuristic requirements which had to be checked for each of the candidate gravitational field equations.

2.2.1. *The Requirement of Equivalence*

Already in his review of 1907, Einstein formulated the assumption of complete physical equivalence between a uniformly accelerated reference frame and a constant homogeneous gravitational field. He asserted:

> The heuristic value of this assumption rests on the fact that it permits the replacement of a homogeneous gravitational field by a uniformly accelerated reference system, the latter case being to some extent accessible to theoretical treatment.[19] (Einstein 1907: 454).

Einstein's assumption was, however, much more than a means useful to make gravitational fields accessible to theoretical treatment. It was motivated by the requirement of keeping inertial and gravitational mass equal even in a relativistic gravitation theory, a requirement that was by no means self-evident in the contemporary discussion. In classical mechanics, the proportionality of inertial and gravitational mass, which guarantees Galileo's principle of the equality of accelerations in free fall, is merely an empirical statement. Einstein's approach to the new theory of gravitation, on the other hand, is characterized by transforming this empirical statement into a heuristic principle of fundamental importance for this new theory.[20]

Whereas the equivalence between a uniformly accelerated reference frame and a homogeneous static gravitational field forms the essence of the equivalence principle,[21] Einstein also attempted to extend this relation to other accelerated frames. Many years later, he once gave the following general characterization of the equivalence principle:

> The equivalence principle asserts that the qualities of physical space, as they present themselves from an accelerated coordinate system, represent a special case of the gravitational field. (Einstein to A. Rehtz, 12 July 1953, EA 27–134)

The paradigmatic case for such an extension of the equivalence principle to other accelerated frames was the "rotating bucket," i.e., a uniformly rotating system of reference. The rotating bucket had been used by Newton and much later by Mach in order to discuss the question of whether inertia is a property of absolute space or, as Mach suggested, caused by the interaction of masses. If the acceleration

[19] "Der heuristische Wert der Annahme liegt darin, daß sie ein homogenes Gravitationsfeld durch ein gleichförmig beschleunigtes Bezugssystem zu ersetzen gestattet, welch letzterer Fall bis zu einem gewissen Grade der theoretischen Behandlung zugänglich ist."

[20] For a discussion of the peculiarity of Einstein's approach in its historical context, see Renn 1994.

[21] For a historical discussion of the equivalence principle, see Norton 1985.

field of such a rotating frame of reference could be interpreted as a gravitational field, then rotation could be conceived as a state of rest, as Einstein once put it.[22] In other words, locally, a rotating frame of reference could be considered, according to Einstein, as being equivalent to an inertial frame equipped with a special gravitational field somehow caused by the presence of masses. In this way, the equivalence between acceleration fields and special cases of the gravitational field could be exploited in order to realize Mach's idea of interpreting inertial effects as being due to the interaction of masses. As a consequence, the inertial frames of reference of classical mechanics and special relativity no longer had to be considered as privileged frames of reference, fundamentally distinct from accelerated frames of reference. For Einstein this idea became the core of his search for a generalization of the relativity principle.

2.2.2. *The Requirement of Generalized Relativity*

Einstein attempted to generalize the principle of relativity by requiring that the covariance group of his new theory of gravitation be larger than the group of Lorentz transformations of special relativity. In his understanding, this requirement was optimally satisfied if the field equation of the new theory could be shown to possess the mathematical property of general covariance.

Einstein saw a generalized principle of relativity as guaranteeing the satisfaction of the equivalence principle as well. In fact, according to the equivalence principle, an arbitrarily accelerated frame of reference in Minkowski space-time can precisely then be considered as being physically equivalent to an inertial frame if a gravitational field can be introduced which accounts for the inertial effects in the accelerated frame. If now gravito-inertial fields are described by a generally covariant field equation, this must (at least locally) always be possible, since the metric tensor describing the inertial effects in the arbitrarily accelerated frame is then obviously a solution of this field equation and hence represents, in Einstein's understanding, a particular instance of a gravitational field. But in the period under consideration, the generalized relativity principle was not yet explicitly distinguished from what Einstein later introduced as "Mach's principle."[23] It was only the later development that necessitated this distinction; Einstein initially hoped that accounting for inertial effects in terms of gravitational fields would automatically provide an explanation of these effects by the interaction of masses.

2.2.3. *The Requirement of Correspondence*

Einstein further required that the new theory would describe, under certain limiting conditions, the gravitational effects familiar from Newtonian physics. For this

[22] "You see that I am still far from being able to conceive of rotation as rest!" ("Du siehst, dass ich noch weit davon entfernt bin, die Drehung als Ruhe aufassen zu können!") Einstein to M. Besso, 26 March 1912 (Einstein *CP5*: 436).

[23] The term "Mach's principle" was introduced in Einstein 1918a in the context of a controversy with E. Kretschmann on whether or not the general covariance of the field equations implied a generalization of the relativity principle of classical mechanics and special relativity or would only be a mathematical property. In response to this argument, Einstein explicitly distinguished between the principle of relativity and Mach's principle. For further historical discussion of Mach's principle, see Hoefer 1994; Renn 1994; and Barbour & Pfister 1995.

reason, he expected that the unknown gravitational field equation for the metric tensor would reduce to the Poisson equation for the scalar gravitational potential of the classical theory and that, under the same limiting conditions, the equation of motion of his new theory would yield Newton's second law with the force derived from this classical potential. Finally, he assumed that this Newtonian limit can be obtained from the full field equation via an intermediate step characterized by weak static fields leading to a linearized field equation for the metric tensor. Specifically, Einstein expected that the 'Newtonian limit' should be obtained via a metric of the form (3) so that a spatially variable g_{44}-component would guarantee the link to his earlier theory of static gravitational fields (Stachel 1980, 1982, 1989).

We subsume these various demands under what we call Einstein's *correspondence principle* of general relativity. The realization of this principle was a crucial condition for conveying physical meaning to the various mathematical constructs he elaborated since only in this way could they be brought into contact with the empirical knowledge embodied in Newtonian gravitation theory.

2.2.4. *The Requirement of Conservation*

When Einstein began his systematic search for a field equation in 1912, the development of special relativistic dynamics, including the four-dimensional formulation of electrodynamics and of continuum mechanics,[24] offered formulations of basic physical laws which suggested a plausible generalization to a generally relativistic setting. In particular, special relativistic dynamics displayed a generalizable model for the formulation of the conservation of energy and momentum centered upon a four-dimensional stress-energy tensor. By the fall of 1912, Einstein had indeed found a plausible, generally covariant equation involving this tensor, which he interpreted as representing both a generalization of the special-relativistic formulation of the conservation of energy and momentum as well as of the Newtonian law of motion for continuous matter in a gravitational field. It was therefore natural to use this equation as a touchstone for the gravitational field equation to be found; we call this requirement the *conservation principle*. In the Zurich Notebook this equation appears in the following form, valid for a symmetric stress-energy tensor $T^{\mu\nu}$

$$\frac{\partial}{\partial x^{\nu}}(\sqrt{-g}\, g_{\mu\lambda} T^{\mu\nu}) - \frac{1}{2}\sqrt{-g}\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} T^{\mu\nu} = 0. \tag{6}$$

In modern terms, this equation corresponds to the vanishing of the covariant divergence of the stress energy tensor of matter, $T^{\nu}_{\lambda;\nu} = 0$.

The principle of conservation of energy also motivated the search for an expression which could be interpreted as a (not necessarily generally covariant) stress-energy 'tensor' of the gravitational field, allowing one to rewrite the second term of the above equation as the coordinate divergence of this tensor. The conservation of energy and momentum would then be expressed by the vanishing of the coordinate divergence of the sum of the energy-momentum tensors of matter and field.

[24] For a contemporary review, see von Laue 1911.

These four heuristic requirements played a dominant role in Einstein's search for a relativistic theory of gravitation, acting in an oscillating manner either as starting points or as touchstones for tentative field equations. As it turned out, they actually overdetermined the problem and Einstein in his investigation of concrete field equations was forced to weigh the different requirements against each other. Attempting to make ends meet he was faced with the necessity of weakening or modifying one or more of them.

2.3. Einstein's double strategy

In the course of Einstein's work on the problem of gravitation two distinct and complementary strategies emerged for the construction of suitable candidates for the differential operator entering the left-hand side of a tentative gravitational field equation. These two strategies take the above-introduced heuristic requirements in a complementary manner either as points of departure or as touchstones. The strategies and the role of the heuristic requirements in following these strategies can be identified most distinctly in the reconstruction of the calculations of the Zurich Notebook.

A 'mathematical' strategy started from the requirement of the generalized relativity principle, The ground for pursuing this strategy had to be prepared by scanning the mathematical literature for suitable differential expressions with a well-known covariance group. The entries in the Notebook indeed reveal Einstein's increasing familiarity with the relevant mathematics of invariant theory and of Riemann calculus. Tentative field equations with candidate differential operators thus obtained from the mathematical literature then of course had to be tested for the principle of conservation of energy. It also had to be checked whether the Newtonian limit could be realized in the manner expected by Einstein.

The complementary 'physical' strategy started from the well-known limiting case of special relativity and the apparently also well established and firmly founded special theory of static gravitational fields. Along this strategy Einstein sought to construct physically plausible generalizations whose specialization to the Newtonian limit was obvious. Here again conservation of energy was an independent heuristic requirement which had to be checked for each candidate field equation. But most importantly, the covariance group of the differential expressions constructed along this strategy was unknown from the beginning. It hence remained to be investigated to what extent the principle of generalized relativity had actually been realized.

The identification of these two complementary strategies turned out to be the key for understanding many of Einstein's considerations and calculations documented in the Zurich Notebook, as well as for understanding many of his considerations from the period between 1913 and 1915. In the reconstruction of the genesis of the theory of general relativity the identification of these strategies allows in particular a detailed understanding of the complicated process of expressing physical concepts by means of a mathematical framework with its own logical and deductive structure and, vice versa, of endowing the objects of the mathematical representation with a meaning in the conceptual framework of physics.

3. A case study: discovery and rejection of the Einstein tensor in the Zurich Notebook

In this Section we illustrate the interaction between heuristics and representation with an episode of calculations documented in the Zurich Notebook. The episode culminates in Einstein's discovery and rejection of the Einstein tensor in late 1912 or early 1913. The reconstruction is based on the comprehensive analysis of the Notebook undertaken by our group.[25]

In the episode presented in the following, Einstein first investigated a plausible generalization of the Laplacian operator along the 'physical' strategy but was then distracted from this path and driven toward the 'mathematical' strategy when Grossmann showed him the Riemann tensor. Investigating differential expressions derived from the Riemann tensor along the 'mathematical' strategy, Einstein came first to consider the Ricci tensor, the very same object he would consider as the left-hand side of a gravitational field equation three years later in his second memoir of November 1915 (Norton 1984). In the Notebook, Einstein then turned, guided by his heuristic principles and the conflicts generated by their mathematical implementation, to the gravitation tensor of the final theory, the Einstein tensor—if only in linearized approximation. The same heuristic principles that led him to the Einstein tensor then forced him, however, to give it up. Eventually, he returned again to the physical strategy.

3.1. The heuristic ideas meet the mathematical representation

3.1.1. *The core operator—the ideal starting point for the physical strategy*

Both directions of Einstein's heuristic double strategy had their natural starting points: the correspondence principle in the case of the physical strategy, the relativity principle in the case of the mathematical strategy. Each of these alternative starting points required a concretization in the mathematical representation. Along the 'physical' strategy, an obvious way of generalizing the Laplacian operator of the classical Poisson equation for the Newtonian gravitational field was to apply the Laplace-Beltrami operator directly to the components of the metric:

$$K_{\mu\nu} \equiv \sum_{\alpha\beta} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{-g}\, g^{\alpha\beta} \frac{\partial g^{\mu\nu}}{\partial x^\beta} \right). \tag{7}$$

We call this object the *core operator* since it is the starting point and the core of many of Einstein's considerations in the years 1912–1915.[26] Obviously, the core operator reduces to the d'Alembertian operator for weak fields, and it reduces to the Laplacian operator for the special case of the static metric (3). Hence it satisfies

[25] The discovery that Einstein actually considered the (linearized) Einstein tensor in the Notebook, while unknown in the literature, has meanwhile become the basis for the annotation of the relevant pages of the Notebook in Einstein *CP4*; it has been summarized in Castagnetti, Damerow et al. 1994 and Renn & Sauer 1996.

[26] Strictly speaking, the expression as written in the text is not an operator since it contains the metric components on which operators act.

the heuristic Requirement of Correspondence, i.e., it allows taking the Newtonian limit in the expected manner.

For the special case in which the determinant of the metric is constant and the "Hertz-condition":[27]

$$\sum \frac{\partial g^{\mu\nu}}{\partial x^\mu} = 0, \tag{8}$$

holds,[28] the core operator reduces to the simple form

$$g^{\alpha\beta} \frac{\partial^2 g^{\mu\nu}}{\partial x^\alpha \partial x^\beta}. \tag{9}$$

While it was clear that the core operator satisfies the correspondence principle, Einstein had to check its compatibility with his other heuristic principles. Several pages of the Zurich Notebook can be reconstructed as documenting just such checks.

On page 13R of the Notebook (Figure 2), e.g., Einstein started an investigation of the compatibility of this operator with his heuristic condition of conservation of energy. Specifically, he checked the consequences of energy-momentum conservation (6) for a tentative field equation which had expression (9) on its left-hand side by substituting it for the energy-momentum tensor. Such a substitution amounts to a consistency check between field equation and energy-momentum conservation. This consistency check led him to consider the expression[29]

$$\sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x^\mu} \left(g_{m\nu} g^{\alpha\beta} \frac{\partial^2 g^{\mu\nu}}{\partial x^\alpha \partial x^\beta} \right) - \frac{1}{2} \sum_{\alpha\beta\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^m} g^{\alpha\beta} \frac{\partial^2 g^{\mu\nu}}{\partial x^\alpha \partial x^\beta}. \tag{10}$$

Considered as an equation representing energy-momentum conservation (i.e., set equal to 0), this expression should not introduce any new conditions on the gravitational field in addition to the field equation, at least if the core operator is a plausible candidate for the left-hand side of the field equation. Indeed, in the correct field equation of general relativity no such new conditions are introduced by the requirement of energy-momentum conservation due to the contracted Bianchi identities. Furthermore, it should be possible to rewrite the above expression in terms of a stress-energy tensor of the gravitational field.

In the Notebook, Einstein observed, however, that the above expression would normally contain derivatives of third order, unless condition (8) holds. In other words, he found that energy-momentum conservation amounts to the introduction

[27] This condition is mentioned in a letter by Einstein to Paul Hertz on 22 August 1915 (EA 12–203). The letter and its relevance for the history of general relativity are discussed in extenso in Howard & Norton 1993.

[28] Both conditions together imply the condition $\partial/\partial x^\mu (\sqrt{-g}\, g^{\mu\nu}) = 0$ which is equivalent to the harmonic condition (12), to be discussed below.

[29] In the Notebook, contravariant components of the metric are denoted by a Greek letter, $\gamma_{\mu\nu}$, rather than by the modern 'upstairs' indices, $g^{\mu\nu}$. Also, indices to coordinates and coordinate differentials were always written 'downstairs'.

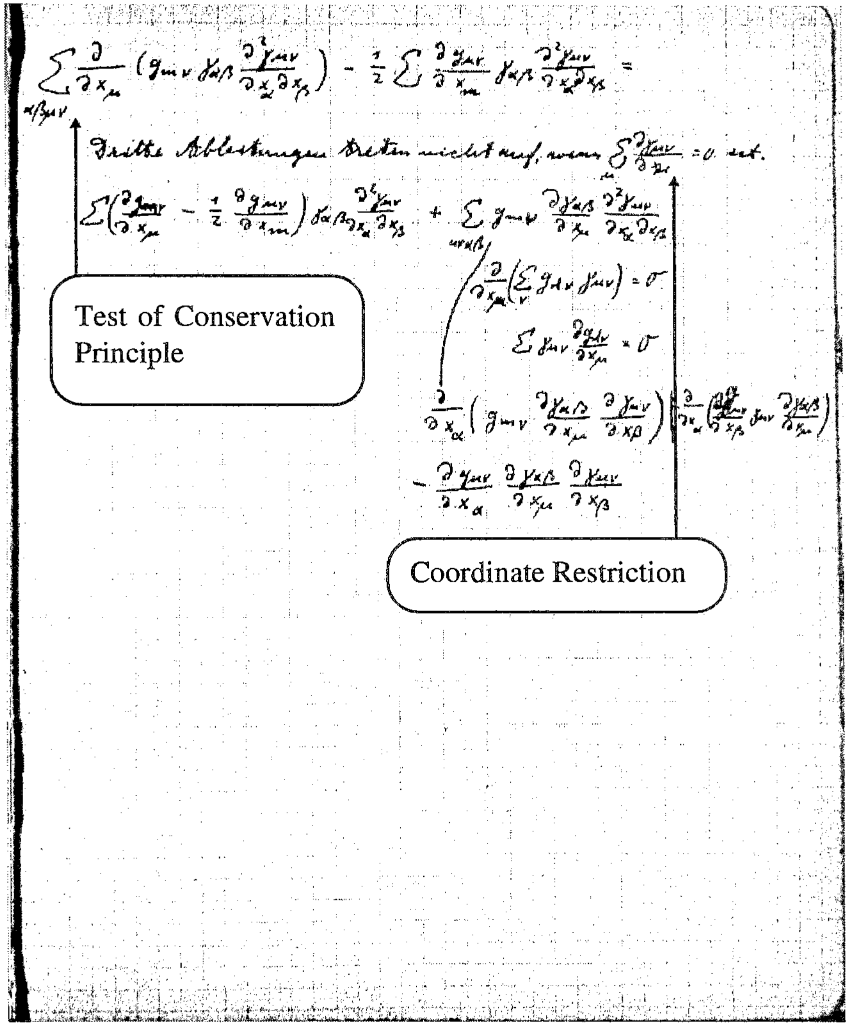

Figure 2. Page 13R of the Zurich Notebook. Here Einstein checked the compatibility of the *core operator* with his heuristic requirement of energy-momentum conservation.

of additional requirements. After an attempt at further simplifying expression (10) under the assumption of Equation (8), Einstein broke off. At this point, he apparently found no obvious way of interpreting Equation (10) and of showing the compatibility of the core operator with energy-momentum conservation. Probably for this reason he did not pursue this line of thought any further.

3.1.2. *The Riemann tensor—the ideal starting point for the mathematical strategy*

Following the mathematical strategy, it is the Riemann tensor which could be taken as a natural starting point because it opens up the perspective of finding generally covariant field equations by employing mathematically well understood objects and techniques. Einstein became aware of the Riemann tensor through the help of his mathematician friend Marcel Grossmann. In October 1912, Einstein wrote to Arnold Sommerfeld:

> I am now working exclusively on the gravitation problem and I believe that I can overcome all difficulties with the help of a mathematician friend of mine here.[30] (Einstein *CP5*: 505).

This letter is the only external evidence for dating a page of the Notebook which has Grossmann's name on it next to the definition of what is called a "tensor of fourth manifold,"[31]

$$(i\kappa, lm) = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^\kappa \partial x^l} + \frac{\partial^2 g_{\kappa l}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^\kappa \partial x^m} - \frac{\partial^2 g_{\kappa m}}{\partial x^i \partial x^l}\right) + \sum_{\rho\sigma} g^{\rho\sigma}\left(\begin{bmatrix} im \\ \sigma \end{bmatrix}\begin{bmatrix} \kappa l \\ \rho \end{bmatrix} - \begin{bmatrix} il \\ \sigma \end{bmatrix}\begin{bmatrix} \kappa m \\ \rho \end{bmatrix}\right),$$

readily recognized to be the fully covariant Riemann tensor (Figure 3).[32]

With such a generally covariant object at hand, the 'mathematical' strategy must have appeared quite promising since the heuristic Requirement of Generalized Relativity could be fulfilled to the utmost extent. But as such the Riemann tensor was, of course, not a suitable object. In order to obtain a field equation of the form (5), Einstein had to extract a two-index object from the Riemann tensor, in particular since the energy-momentum tensor on the right-hand side has only two indices. Indeed, on p. 14L, Einstein contracts the Riemann tensor once,

$$\sum g^{\kappa l}(i\kappa, lm),$$

in order to obtain such a two-index object, the covariant Ricci tensor, which might be taken as a candidate for the left-hand side of a tentative field equation.

As with the core operator, Einstein now had to examine this new candidate against the list of his heuristic requirements. For objects constructed along the 'mathematical' strategy the requirement of the correspondence principle was the first and most important one on the list. A short consideration, however, made Einstein realize that for weak fields the Ricci tensor does not reduce to the required form of the d'Alembertian acting on the weak field components. In addition to a core operator term of the form (9), the Ricci tensor contains three other terms with

30 "Ich beschäftige mich jetzt ausschliesslich mit dem Gravitationsproblem und glaube nun mit Hilfe eines befreundeten Mathematikers aller Schwierigkeiten Herr zu werden."

31 "Tensor vierter Mannigfaltigkeit," see Figure 3.

32 $\begin{bmatrix} im \\ \sigma \end{bmatrix} \equiv \frac{1}{2}\left(\frac{\partial g_{i\sigma}}{\partial x_m} + \frac{\partial g_{\sigma m}}{\partial x_i} - \frac{\partial g_{im}}{\partial x_\sigma}\right)$ denotes the Christoffel symbol of the first kind.

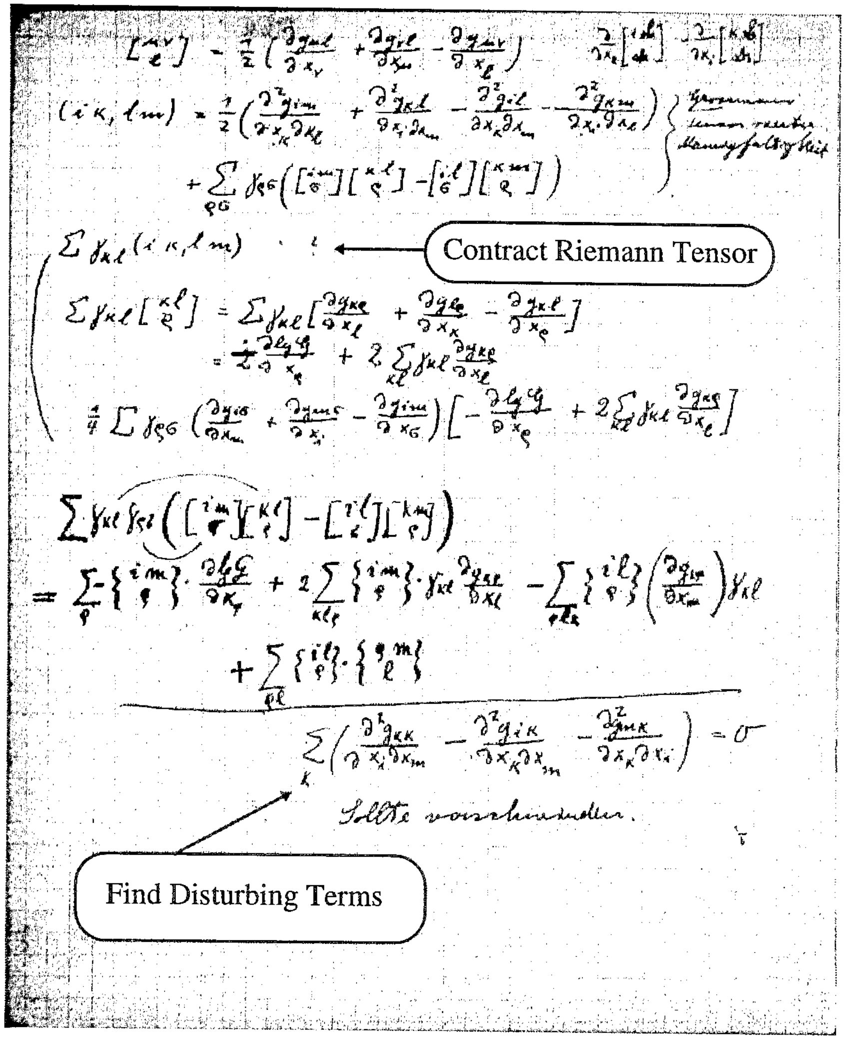

Figure 3. Page 14L of the Zurich Notebook. On this page Einstein started his investigation of the Riemann tensor following the 'mathematical' strategy.

second derivatives of the metric. These terms obstruct a smooth transition to the Newtonian limit for weak, static fields in the manner expected by Einstein. In the Notebook, Einstein set these terms equal to zero,

$$\sum_{\kappa}\left(\frac{\partial^2 g_{\kappa\kappa}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{i\kappa}}{\partial x^\kappa \partial x^m} - \frac{\partial^2 g_{m\kappa}}{\partial x^\kappa \partial x^i}\right) = 0, \tag{11}$$

with the comment: "should vanish."[33]

The upshot of this first investigation was that the Ricci tensor violated Einstein's heuristic Requirement of Correspondence.

3.2. The mathematical representation generates conflicts and novelties

3.2.1. *Coordinate restrictions as a novelty suggested by the conflict between the Requirements of Correspondence and Relativity*

We have seen that the consideration of the Ricci tensor revealed a conflict between the Requirements of Generalized Relativity and of Correspondence. This conflict did, however, not force Einstein to give up the Riemann tensor as his starting point, as the formalism still provided more paths, the exploration of which made good sense from the perspective of Einstein's heuristics.

But the mathematical representation did not only suggest alternative objects that could be tested against Einstein's heuristics, it also offered possibilities for modifying the heuristics, e.g., by restricting or reinterpreting one of the heuristic requirements. One obvious candidate for such a modification was the requirement of general relativity because it represented a 'maximalist' goal of Einstein's search.

The expression of the conflict between correspondence and relativity principles by the 'disturbing terms' (11) suggested, in particular, that these terms could be brought to vanish by imposing an additional restrictive condition. On p. 19L, Einstein in fact restricted the group of admissible coordinate systems to those which satisfy the following condition (see Figure 4)

$$\sum_{k,l} g^{kl}\begin{bmatrix} kl \\ i \end{bmatrix} = 0. \tag{12}$$

Coordinates satisfying this condition at the time were called "isothermal"[34] and are called 'harmonic' coordinates today. For these coordinates the Ricci tensor reduces to the form

$$\sum_{k,l} g^{kl}\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \Delta_{im}, \tag{13}$$

where we have defined Δ_{im} by

$$\Delta_{im} \equiv -\frac{1}{2}\frac{\partial g^{kl}}{\partial x^m}\frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g^{kl}}{\partial x^m}\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g^{kl}}{\partial x^i}\frac{\partial g_{mk}}{\partial x^l} - g^{\rho\sigma}g^{kl}\frac{\partial g_{i\rho}}{\partial x^l}\frac{\partial g_{m\sigma}}{\partial x^k} + g^{\rho\sigma}g^{kl}\frac{\partial g_{il}}{\partial x^\rho}\frac{\partial g_{m\sigma}}{\partial x^k},$$

a term that contains only products of first derivative terms, which do not affect the Newtonian limit in Einstein's understanding of weak and static fields.

33 "Sollte verschwinden," see Figure 3.

34 See Bianchi 1910: § 36–37.

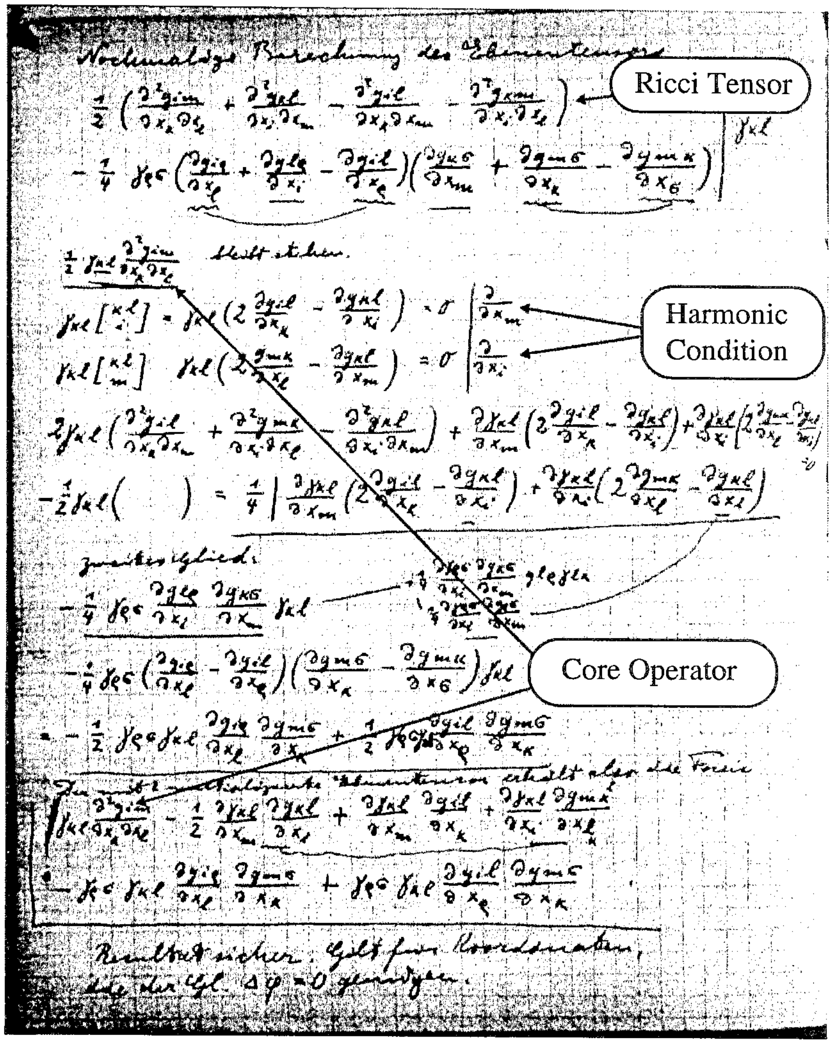

Figure 4. Page 19L of the Zurich Notebook. Einstein succeeded in reducing the Ricci tensor using harmonic coordinates.

It is unclear how exactly Einstein understood the additional condition, whether indeed as a restriction of the relativity principle as we suggest here or as a coordinate condition in the modern sense.[35] Nowadays coordinate conditions such as

[35] Looking only at p. 19L, it is quite possible that Einstein understood the harmonic coordinate condition appearing on this page in the modern sense. This is the way the appearance of Equation (12) was interpreted in Norton 1984. But a comprehensive reconstruction of the Notebook has revealed hints that the interpretation given in the text may be in better agreement with other parts of the Notebook.

Equation (12) are invoked if a generally covariant equation is applied to a physical situation which suggests the specialization of the coordinate system. The obvious example is Newtonian gravitation theory. In whichever way Einstein saw the restrictive condition (12), it is clear that the mathematical concretization of Einstein's heuristic requirements created a conflict which in turn led to a new insight, the possibility of avoiding this conflict by adding coordinate restrictions to a generally covariant field equation.

The reduced object (13) must have looked promising to Einstein. The only term containing second derivative terms was a term of the form of the core operator

$$\gamma_{kl}\frac{\partial^2 g_{im}}{\partial x_k \partial x_l}$$

and the other terms could be neglected for weak fields. The reduced object (13) was consequently in agreement with the heuristic Requirement of Correspondence and one might proceed to check the corresponding field equation against the other heuristic requirements. Since the Ricci tensor could only be brought into the form (13) for coordinates satisfying condition (12), however, it also remained to be checked whether this additional condition was compatible with his other heuristic requirements.

3.2.2. *A conflict between the Requirements of Correspondence and Conservation*

Einstein now considered the case of weak fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the Minkowski metric with imaginary time coordinate is given by $\eta_{\mu\nu} = \mathrm{diag}(1, 1, 1, 1)$. In this case the object (13) reduces to the core operator (9), which further simplifies to the d'Alembertian operator acting on the components $h_{\mu\nu}$. In linearized approximation one therefore obtains the following field equation with an energy-momentum tensor $\rho_0 u^\mu u^\nu$ for incoherent matter ("dust") as source term

$$\sum_\kappa \frac{\partial^2 h_{\mu\nu}}{\partial x^{\kappa 2}} = \kappa \rho_0 u_\mu u_\nu, \tag{14}$$

and this linearized field equation was written down explicitly on p. 19R of the Notebook (Figure 5).

Einstein first convinced himself that this equation satisfied the principle of conservation since, for the approximation at hand, it allowed the derivation of an energy-momentum 'tensor' of the gravitational field.[36]

However, there was another problem which immediately arose from the consideration of energy-momentum conservation. In linearized approximation, energy-momentum conservation is expressed by the vanishing of the simple coordinate

In particular, Einstein repeatedly performed coordinate transformations of such coordinate conditions (e.g., p. 22L), which does not make sense from a modern point of view. This question will be discussed in detail elsewhere.

[36] "The energy and momentum theorem holds to the relevant approximation." ("Energie- und Impulssatz gilt mit der in Betr[acht] kommenden Annäherung," see Figure 5.

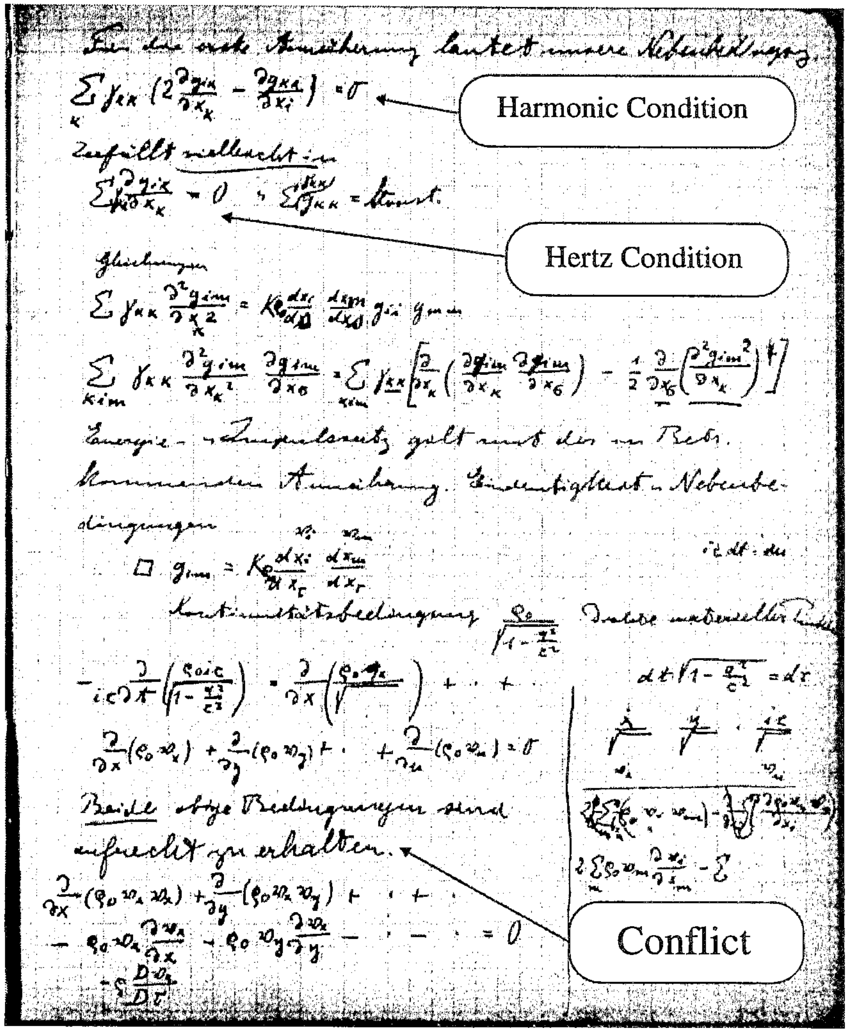

Figure 5. Page 19R of the Zurich Notebook. Einstein discovered a conflict between the harmonic coordinate restriction, necessary to satisfy the Requirement of Correspondence, and the "Hertz condition," necessary to satisfy the requirement of energy and momentum conservation.

divergence of the energy-momentum tensor of matter. Together with the linearized field equation (14) this condition immediately implies the linearized Hertz condition,

$$\sum_{\nu} \frac{\partial h_{\mu\nu}}{\partial x^{\nu}} = 0, \qquad (15)$$

as an additional restriction on the coordinates. Taking into account his two heuristic Requirements of Correspondence and Conservation, Einstein at this point thus had to deal with two restrictive conditions on the coordinates. Comparing condition (15) with the "isothermal" or "harmonic" condition (12), which in linearized approximation reads

$$\sum_{\nu} \left(\frac{\partial h_{\kappa\lambda}}{\partial x^{\kappa}} - \frac{1}{2} \frac{\partial h_{\kappa\kappa}}{\partial x^{\lambda}} \right) = 0, \qquad (16)$$

one recognizes that the simultaneous stipulation of the two conditions (15) and (16) further implies the condition

$$\sum_{\nu} \frac{\partial h_{\kappa\kappa}}{\partial x^{\lambda}} = 0. \qquad (17)$$

This latter condition, however, was unacceptable for two reasons. First, it implied together with the linearized field equation (14) the constancy of the trace of the energy-momentum tensor of matter. This consequence is obviously violated even by the simple case of dust. Second, Equation (17) implies that for weak fields the tensorial field equation does not reduce to a simple equation for the g_{44} component representing the scalar gravitational potential in the Newtonian limit. Instead, one has to take into account also non-trivial spatial g_{ii}-components at the same level of weak field approximation. This last consequence means that the Newtonian limit could not be attained in the expected way in the special case of weak, static fields, via the metric (3) for Einstein's theory of static gravitation.

In summary, Einstein had, on the one hand, succeeded at this point in extracting from the Ricci tensor a linearized field equation that was compatible with his heuristic Requirement of Correspondence. But his difficulty was that he had only been able to do so by imposing the harmonic condition (12) resp. (16). Energy-momentum conservation on the other hand implied the restrictive condition (15) and the simultaneous stipulation of these two conditions implied the untenable restriction (17).

3.2.3. *The Einstein tensor as a novelty suggested by the conflict between two heuristic requirements*

This dilemma of two incompatible restrictions triggered a further exploration of the formalism. On the next page of the Notebook Einstein considered a modification of the linearized field equation by which he hoped to avoid this dilemma (see Figure 6). This modification effectively made him consider the linearized Einstein tensor as a possible candidate for the left-hand side of the field equation.

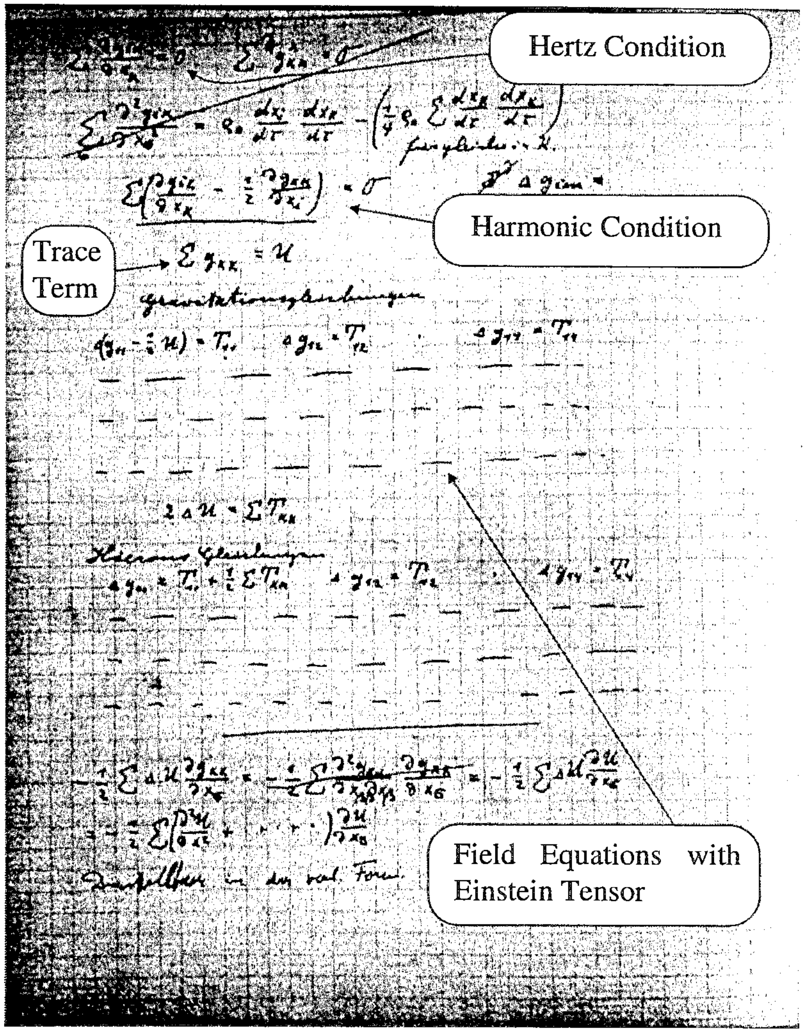

Figure 6. Page 20L of the Zurich Notebook. Einstein tried to solve the conflict between the harmonic coordinate restriction and the Hertz restriction by an ad hoc modification of the field equations. Einstein here effectively wrote down the field equations of the final theory in linear approximation.

Looking at the linearized field equation (14), he noticed that the first of the two above-mentioned consequences, i.e., that the trace of $T_{\mu\nu}$ had to be a constant, could be fixed by adding a trace term to the source term on the right-hand side of

the field equation,[37]

$$\sum_\sigma \frac{\partial^2 h_{i\kappa}}{\partial x^{\sigma 2}} = K\rho_0 u_i u_\kappa - K\frac{1}{4}\delta_{i\kappa}\rho_0 \sum_k u_k u_k. \tag{18}$$

He must then have remembered that his current considerations were made under the general assumption, formulated on the top of p. 19R, that the harmonic restriction (16) "*perhaps* splits up"[38] into the two stronger conditions (15) and (17). Now that this train of thought had led him into severe difficulties Einstein reconsidered the harmonic coordinate restriction. This reconsideration probably suggested the idea of adding a trace term $\sum h_{\kappa\kappa} = U$ to the left-hand side of the linearized field equation (15) resulting in the following field equation[39]

$$\sum_\sigma \frac{\partial^2}{\partial x^{\sigma 2}} \left\{ h_{i\kappa} - \frac{1}{2}\delta_{i\kappa} U \right\} = T_{i\kappa}. \tag{19}$$

Such an additional trace term has the consequence that energy-momentum conservation, expressed in terms of the divergence of $T_{\mu\nu}$, no longer implies the Hertz restriction (15) but rather the harmonic restriction (16). Hence the coordinate restriction arising from energy-momentum conservation does not impose an additional restriction but coincides with the restrictive harmonic condition.

With this modification, Einstein had now arrived at a new candidate field equation by considering what we now identify as the linearized Einstein tensor. The new field equation, generated by a further elaboration of the mathematical representation triggered by a conflict between two heuristic principles, now had to be tested again with respect to Einstein's heuristic requirements.

Before Einstein turned to a consideration of his other heuristic principles, he added a multiple of the traced equation and thus transformed the equation to an alternative form which has the trace term on the same side as the source term. This might have been done in order to check whether his new modification was equivalent to his previous adding of a trace term. Einstein at this point made an algebraic error which, however, was inconsequential. He formed the trace of the gravitational field equation (19) erroneously obtaining $2\partial^2 U/\partial x^{\sigma 2} = \sum T_{\kappa\kappa}$ instead of $-\partial^2 U/\partial x^{\sigma 2} = \sum T_{\kappa\kappa}$. Adding $1/2$ of the traced equation to the field equation (19) he obtained (again erroneously)

$$\sum_\sigma \frac{\partial^2 h_{i\kappa}}{\partial x^{\sigma 2}} = T_{i\kappa} + \frac{1}{2}\delta_{i\kappa} T, \tag{20}$$

37 In the Notebook, Einstein explicitly noted that the trace term was to be subtracted only "in the case of the same i and κ" ("für gleiche i u[nd] κ," see Figure 6) instead of using the compact notation of the 'Kronecker delta' $\delta_{i\kappa}$ as is done in our transcription. In the manuscript these lines were immediately deleted, possibly because Einstein at some point realized that such a modification had to address the second of the above-mentioned problems.

38 "Zerfällt *vielleicht* in," see Figure 6.

39 In the Notebook, Einstein explicitly put down the 11-, 12-, and 14-component of this equation, expressing the d'Alembertian $\sum_\sigma \frac{\partial^2}{\partial x^{\sigma 2}}$ by a triangle Δ, and indicated the other components by horizontal dashes.

whereas the version which would correctly be equivalent to (19) would have $-1/2$ instead of $+1/2$. The algebraic error was again inconsequential because in any case equation (20) was different from Equation (18). This was an expected result since the field equation (18) was obtained by requiring that the trace of the source term vanish on the basis of the assumption that the linearized harmonic condition (16) splits up into the two stronger conditions (15) and (17). The field equation (20) was obtained by reverting to the harmonic condition (16) and requiring that the vanishing of the coordinate divergence of $T_{i\kappa}$ was consistent with it.

3.2.4. *The conflict between the Einstein tensor and the Requirement of Correspondence*

As with the simple linearized field equation (14), the first thing to check for the modified field equation (19) was to see whether one could still satisfy the requirement of energy conservation by deriving a gravitational energy-momentum tensor. By a brief consideration Einstein found that also with the additional trace term one could find such a gravitational energy tensor and he noted that it would be "representable in the required form."[40]

Adding the trace term to the linearized field equation either in the form of Equation (18) or in the form of (19) did not solve, however, the second of the two problems mentioned above, i.e., the incompatibility with the correspondence principle. In either case, the trace term implied that the metric (3) for static fields could not be a solution of the field equation and was hence incompatible with it. The trace term precluded that the tensorial multi-component field equation reduces to a simple one-component Poisson equation for the g_{44}-component of the metric representing the Newtonian gravitational potential. Hence, the Newtonian limit could not be realized in the way expected by Einstein and this was probably the reason why he deleted Equation (18) in the Notebook. In view of this consequence, he either had to reconsider his correspondence principle or he had to give up the whole approach of reducing the Ricci tensor using the harmonic condition (12). On the next pages of the Notebook (20R–21R), he indeed reconsidered this requirement and in particular his assumption that the metric of static gravitational fields has to be of the form (3). But he apparently found his arguments in favor of the correspondence principle strong enough to reject the Einstein tensor.[41] The same heuristic principles that had led Einstein in 1912–1913 to consider the final field equation of general relativity, thus also forced him to reject it.

In spite of the problems with the Ricci and the Einstein tensors, as the Notebook documents, Einstein continued for some time to search for a suitable gravitation tensor on the basis of the Riemann tensor (22L–25R). A couple of pages later, however, Einstein eventually gave up the idea, after having explored all the possibilities that the formalism appeared to offer. The 'mathematical strategy' of

[40] "Darstellbar in der verl[angten] Form," see Figure 6.

[41] One such argument was explicitly formulated as late as 1915 in a letter to Freundlich (EA 11-208). Here Einstein reports that he examined the question whether matter at rest produces metric components other than g_{44}. Apparently assuming a weak-field condition of the form (14), he concluded that this cannot be the case. A more detailed discussion of these arguments will be presented elsewhere.

starting from a mathematically well defined object with well-known covariance group had definitely failed to yield a field equation that was compatible also with the more physically motivated heuristic Requirements of Correspondence and Conservation.

3.3. The *Entwurf* operator—a metastable state in the interaction of heuristics and mathematical representation

After the failure of the 'mathematical' strategy, Einstein, at the end of the Notebook (26L ff), returned to the 'physical' strategy. He now started from the core operator (7) which, as we have discussed, reduces to the Laplacian operator without any additional constraints. For this operator, the satisfaction of the correspondence principle represented no difficulty. In order to also comply with the conservation principle, Einstein then introduced higher-order correction terms to this core operator, constructed in just such a way that the divergence equation (6) could be satisfied. The conservation principle was now employed in a constructive manner rather than only being used as a criterion for accepting or rejecting candidate field equations. The procedure yielded expression (2) for the left-hand side of the field equation (5), and the problem for the core operator which we discussed earlier could thus be resolved from the beginning. In March 1913, Einstein and Grossmann published this expression as the left-hand side of a gravitational field equation in their joint *Entwurf* article, Einstein & Grossmann 1913. The field equation of this theory is compatible with all of Einstein's heuristic requirements—save the relativity principle. In fact, the transformational behavior of the *Entwurf* field equation was not exactly known and left as an open problem when Einstein and Grossmann published their paper.[42] But since Einstein anyhow was not sure to what extent this principle could be realized, his search for a new theory of gravitation had thus reached a state in which it made sense to elaborate the consequences of this theory instead of continuing to produce alternative versions. Historically, this state of the theory proved to be metastable, bound to decay after a couple of years.

4. **The interplay between heuristics and deductivity**

Among the missed opportunities in Einstein's long and winding path towards general relativity, his rejection of gravitational field equations based on the Einstein tensor certainly is the gravest. With the field equations of his final theory already before his eyes—if only in linear approximation—he could happily, or so at least it may appear, have achieved his most brilliant scientific achievement already in 1912 or 1913. The fact that Einstein missed this opportunity seems to present a strong case against the interpretation of his final achievement as a case of serendipity,

[42] "But we do not know whether there exists a general transformation group under which the equations are covariant. The question of the existence of such a group ... is the most important one which follows up on the considerations expounded here." ("Wir wissen aber nicht, ob es eine allgemeine Transformationsgruppe gibt, der gegenüber die Gleichungen kovariant sind. Die Frage nach der Existenz einer derartigen Gruppe ... ist die wichtigste, welche sich an die hier gegebenen Ausführungen anknüpft.") (Einstein & Grossmann 1913: 18)

which would have had Einstein find the field equation by mere luck. If in late 1915 Einstein had accidentally come back to general covariance and, equally by accident, to the Einstein tensor, had found it acceptable and had published it, he might well have done so two years earlier.

But the alternative interpretation, according to which Einstein's return to general covariance in 1915 might be due to a last minute *conceptual* breakthrough by which he redirected his *heuristics* along a more successful line of attack has other problems. The 1915 paper itself (Einstein 1915d), as well as the other papers of that period, do not bear evidence of such a radical change. Also the commentaries available from Einstein's correspondence, or from the papers he published shortly afterwards, do not indicate a fundamental revision of his heuristic principles. The equivalence principle is still understood as being included in the generalized principle of relativity which Einstein believed to be satisfied precisely because of the general covariance of the new field equation. The conservation principle is, even in Einstein's conclusive 1915 paper, still considered as an additional requirement to be imposed on the field equation rather than as being implied by it via the contracted Bianchi identity. The fundamental features of Einstein's understanding of the correspondence principle had not changed either. He still did not have a mathematical framework in which the dynamics and the space-time structure of classical gravitation theory could be obtained from general relativity by a consistent limiting procedure.[43]

Nevertheless, some years later, in late November 1915, Einstein formulated the 'final' field equation of general relativity (Einstein 1915d). How then did Einstein succeed in formulating the field equation of general relativity if not by a conceptual breakthrough? First of all, from a modern perspective, his heuristics did cover the mathematical requirements that uniquely define the Einstein tensor. If one looks for a generally covariant, second-rank tensor that is linear in the second derivatives of the metric components, contains no higher derivatives, vanishes in flat space-time and satisfies the Bianchi identity, then no other alternative is available. Whatever else was implied by Einstein's heuristics, these mathematical requirements are in fact suggested by his generalized principle of relativity, by his correspondence principle, and by his conservation principle. Einstein's problem was that these requirements did not exhaust his heuristics and that their consequences were not fully compatible with it. In the face of such conflicts, he was hence forced to weigh the various components of his heuristics against each other, and he had to be prepared to reduce his ambitious goals, for instance by restricting the demand for general covariance or by weakening his criteria for the satisfaction of energy-momentum conservation. But his judgement about the proper equilibrium between his conflicting heuristic components depended on the state of elaboration of their deductive consequences. This is why the balance between the heuristic components turned out to favor different candidates in the course of Einstein's first examination of the Ricci, the November and the Einstein tensor in the years 1912–1913 than in the course of his second examination of these tensors

[43] For a modern discussion of this limiting procedure, see Ehlers 1981.

in 1915. The interplay between heuristics and deductivity is the crucial intellectual process that prepared the ground for the discovery of the field equation of general relativity.

Let us recapitulate. At the time of his considerations documented by the Zurich Notebook, it was his heuristic Requirement of Correspondence, together with the requirement of conservation, that made Einstein reject the Einstein tensor. Einstein's understanding of the Newtonian limit was that for weak and static fields the 10-component tensorial field equations would reduce to the Poisson equation for the time-time component of the metric, representing the single, scalar potential of Newtonian gravitation. Similarly, this metric component would have to govern the equations of motion for a material point moving in a Newtonian gravitational field. But the trace term introduced on p. 20L of the Notebook (see Equation (19) above) inevitably implied that non-trivial spatial components had to be taken into account on the same level of approximation as that which yielded the time-time component as the Newtonian gravitational potential, and this violation of his heuristic requirements made it inacceptable for Einstein.

It was thus Einstein's heuristic requirements that prevented an early breakthrough in 1912 or 1913. On the other hand, as we have seen, it was the very same heuristic requirements that had led to the introduction of the trace term in the first place since this term had only been considered in an attempt to solve a conflict between the Requirements of Correspondence and Conservation. In hindsight, it must be observed, however, that the motivation for adding a trace term to the Ricci tensor in 1912 was markedly different from the justification of the Einstein tensor in the final theory. In the Zurich Notebook, Einstein had to resolve the conflict between the harmonic coordinate condition on the one hand, necessary to satisfy the Requirement of Correspondence, and the Hertz condition on the other hand, necessary to satisfy the requirement of energy conservation. He had tried to resolve this conflict by adding a trace term to the field equation which would turn the coordinate restriction stemming from the requirement of conservation from the Hertz condition to the harmonic condition as well. Thus it was the trace term appearing in the harmonic condition that was the basis for the trace term in the field equation. From a modern perspective, however, the trace term in the harmonic coordinate condition has nothing to do with the fact that the Einstein tensor satisfies the contracted Bianchi identity. It was thus a contingent feature of the mathematical representation of his heuristic ideas that led Einstein to consider the correct field equation in linearized approximation in 1912. When Einstein stumbled, so to say, upon this field equation in an early stage of his exploration of the deductive structure implied by the mathematical representation, and found problems in its physical interpretation, he chose to give up the mathematical strategy rather than his more physically motivated heuristic requirements.

In the years following the publication of the *Entwurf* field equations, Einstein had explored many consequences of this theory. In the course of this exploration, he had become familiar with more aspects of the mathematical representation, and in particular began to employ a mathematical representation, the variational formulation, which allowed him to draw far-reaching conclusions about the transformational properties of the theory. The use of variational techniques for some time helped to stabilize the conceptual framework of the *Entwurf* theory, in particular through the identification of the covariance group of the field equations and its interpretation in terms of the hole argument. Eventually, however, properties of the theory were elaborated to a point where they led to manifestly contradictory consequences in various instances. As we have mentioned, already in 1913 Einstein and Besso had found that the *Entwurf* equations did not account for the anomalous advance of the perihelion of Mercury. In an epistolary controversy with Levi-Civita in the spring of 1915, Einstein had to realize that an alleged proof of the covariance properties of the *Entwurf* equations was faulty.[44] He then found that the equations would not hold for Minkowski space-time in rotating Cartesian coordinates. Finally, he had to realize that an allegedly unique derivation of the field equations was faulty.[45] Turning to a reconsideration of alternatives to the *Entwurf* equations, Einstein was now able to examine the candidates he had previously considered in the Zurich Notebook with a new level of knowledge about deductive consequences and equipped with powerful tools such as the calculus of variations. In the course of this reconsideration, he first returned to the November tensor, then to the Ricci tensor, and only then published the final field equations with the Einstein tensor.

But what precisely turned the balance in 1915 in favor of the Einstein tensor, in spite of the remaining conflicts with Einstein's heuristic principles and in spite of the availability of alternatives? The solution came through a technical loophole, which until November 1915 had remained unnoticed, but which then opened up the way if not to a final resolution but at least to a circumvention of the most serious objection against the Einstein tensor. As we have seen, this objection resulted from the correspondence principle and consisted in the observation that the weak field equation resulting from the Einstein tensor involves a metric with more than one variable component. Such a weak field equation can hence not simply be reduced to the classical Poisson equation for one scalar potential.

Surprisingly, a closer inspection of the Newtonian limit of the equation of motion and, in particular, its interpretation as an independent postulate of the theory—contrary to the modern understanding—opened up a way to avoid this dilemma. The geodesic equation reduces in fact, under the mathematical conditions which correspond to classical physics (weak static fields, low velocities, appropriate coordinates), to the Newtonian equation of motion. Under these circumstances, only one component of the metric tensor actually matters, while the other components of the metric can be neglected. As Einstein explicitly noted some years later in his Princeton lectures:

> We see ... that even in first approximation the structure of the graviational field is fundamentally different from the one according to Newton's theory; this is a consequence of the tensorial, non-scalar character of the gravitational potential. The fact that this had not been noticed before is due to the fact that exclusively the component g_{44} enters the equations of motion in first approximation.[46] (Einstein 1956: 89).

[44] For a discussion of the controversy between Einstein and Levi-Civita, see Cattani & De Maria 1989.

[45] See Einstein to Sommerfeld, 28 November 1915, published in Hermann 1968: 32–36.

In other words, here was the loophole through which Einstein could escape from the argument that weak static fields have to be described by a metric tensor with only a single variable component, corresponding to the Newtonian gravitational potential. If other variable components could exist without affecting the equation of motion in the Newtonian limit, this anomaly became tolerable and now shifted the balance in favor of the Einstein tensor. The fact that the metric associated with this peculiar way of attaining the Newtonian limit also explained the perihelium shift of Mercury protected it against the criticism of being just a dubious technical trick and stabilized Einstein's network of conclusions well beyond what, at that point, could have been achieved by its still fragile internal coherency.

5. General relativity—an accidental discovery?

The triumph of November 1915 was hence not the victory of new concepts over old ones but just the temporary stabilization of a complex network made up of still largely traditional concepts, of Einstein's original heuristic arguments with only slight adjustments, and of unforeseen results on the level of the mathematical representation of the new theory. If this interpretation is indeed correct, then the conflicts between the original heuristics and the deductive consequences of the new theory could not have been settled yet. Einstein's continued search for a realization of his original ideas after 1915, beyond the field equation of general relativity, provides in fact strong support in favor of this interpretation (Renn 1994). He recognized, for instance, that his strategy of implementing Machian ideas in the new theory via the requirement of general covariance did not work, since general relativity allows solutions in which inertial effects are present even without being caused by masses. As a reaction to this unexpected difficulty, Einstein did not simply give up his original goal as part of a context of discovery that had been superseded by his results. He rather strengthened his heuristic requirements by now demanding that the metric tensor be "completely" determined by the masses of the bodies that act as a source of the gravitational field. While he had previously simply assumed that such a determination would be an automatic consequence of his heuristic program, in 1918 he felt compelled to introduce this requirement as an additional condition, complementing his original heuristics (Einstein 1918a). He gave it the name "Mach's principle" and made clear that he had hitherto included this requirement in his understanding of the generalized principle of relativity.

That Einstein's reinforcement of his original heuristics was not a matter of philosophical dispute over what had already been achieved becomes clear from

[46] "Man sieht ..., daß auch in erster Näherung die Struktur des Gravitationsfeldes von derjenigen gemäß NEWTONS Theorie prinzipiell abweicht; es liegt dies daran, daß das Gravitationspotential tensoriellen und nicht skalaren Charakter hat. Daß sich dies nicht längst bemerkbar gemacht hat, kommt davon, daß in die Bewegungsgleichung des Massenpunktes in erster Näherung ausschließlich die Komponente g_{44} eingeht."

the drastic consequences he was still prepared to draw from it. In 1917 he was ready to modify the field equation of general relativity by the introduction of the cosmological term, with the intention of confining the solutions of his theory precisely to those that satisfy his demand for a Machian explanation of inertia (Einstein 1917). But even when it turned out that the modified theory also had non-Machian solutions, Einstein continued to believe that

> the General Theory of Relativity only forms a satisfactory system if, according to it, the physical qualities of space are *completely* determined by matter alone.[47] (Einstein 1918b: 271; emphasis in the original)

He now hoped that, even if other solutions would be mathematically possible, at least nature would have a preference for a static cosmological model compatible with his Machian understanding of inertia. Only in early 1931 was Einstein forced to recognize that this hope was not borne out by the astronomical data in favor of an expanding universe.[48]

Did Einstein, after this apparently definite failure of his heuristic program, now finally restrict himself to the technical and conceptual exploration of general relativity as he had established it in 1915? Surprisingly not, because he had other reasons to remain unsatisfied with this theory, reasons that were apparently in flat contradiction with his Machian heuristics. He pursued, at least since 1919, an alternative strategy to modify or further develop general relativity in such a way that space is not conceived as a field effect ultimately caused by matter, but so that, vice versa, matter is rather being constructed in terms of a universal field representing the physical qualities of space.[49] Einstein's hopes to reach a unified field theory that would integrate gravitation theory with electromagnetism and also explain the quantum structure of matter remained as unfulfilled as his Machian program. But if we take into account these more ambitious goals as a context for Einstein's work on gravitation, it seems that the formulation of the field equation of general relativity in 1915 was, after all, nothing more than an accidental discovery in the framework of a research project that was guided by concepts, heuristic principles, and expectations which cannot be reconciled with those associated with general relativity as we understand it today.

6. Einstein's vision redeemed

The opposition between Einstein's goals of explaining space in terms of matter or matter in terms of space resembles somewhat today's alternative between the program to quantize gravitation in the framework of quantum field theory and the program to revise the concepts of space and time as they are used in quantum field theory according to general relativity. The striking fact, however, that one

[47] "Bildet die allgemeine Relativitätstheorie nämlich nur dann ein befriedigendes System, wenn nach ihr die physikalischen Qualitäten des Raumes allein durch die Materie *vollständig* bestimmt werden."

[48] See Einstein 1931; for further discussion, see Renn 1994: 49–50.

[49] For a discussion of Einstein's investigation of modified field equations in the context of his program of finding a unified field theory, see Jim Ritter's forthcoming article.

and the same man could entertain such contrary options raises the question of whether there may not be some deeper level to Einstein's heuristics that might also explain its contribution to his success with general relativity, without reducing this achievement to a pure case of serendipity. In fact, on closer inspection, Einstein's Machian program, as well as the apparently opposite goal of a unified field theory, turn out both to be the result of his striving for a conceptual unity of physics that would overcome the dualism between particle and field physics. That this fundamental problem occupied his mind is not new. More interesting is Einstein's particular approach in addressing this issue, which, in our view, accounts to a large extent for the success of his heuristics.

Instead of relying on the concepts of either mechanics or electrodynamics, Einstein took the knowledge accumulated in both branches of physics and the structures organizing them as equally fundamental. Contrary to Lorentz's electrodynamics, for instance, Einstein's special theory of relativity treated the relativity principle of mechanics as just as foundational as the laws of the propagation of light established by electrodynamics, at the price of a revision of the concepts of classical physics.[50] Similarly, instead of attempting to resolve Mach's paradox of the privileged role of inertial frames in the context of a revised version of classical mechanics, as did several followers of Mach, Einstein addressed this problem in the context of a field theory of gravitation in which inertial forces could be understood as an aspect of a unified gravito-inertial field. And finally, instead of constraining the understanding of this gravito-inertial field by the chronogeometry of special relativity, which to most of his contemporaries had quickly acquired a universal, almost a priori status for physics, Einstein realized that chronogeometry and gravito-inertial structure are equally universal for physics.

Mathematical tools such as the affine connection for describing the gravito-inertial structure were still lacking when Einstein began his work. But even in the absence of such tools, Einstein's uncommon openness to philosophical questions had helped him to recognize the universality of the gravito-inertial structure where others just saw one more force to be subsumed in a special relativistic treatment. In spite of the mathematical and conceptual difficulties that he encountered, his heuristic principles guided a reconciliation between gravito-inertial structure and chronogeometry which excluded any prior geometry, thus effectively determining the characteristic features of general relativity, even as we understand it today. In other words, Einstein's heuristics was so successful not because it anticipated the conceptual novelties of general relativity but because it molded the knowledge accumulated in the different branches of classical physics in such a way that its integration into one coherent framework became possible. This makes Einstein's philosophical openness and his integrative outlook on the foundations of physics into an historical lesson from which even today's unifying ventures may profit.

When studying this lesson it is, however, important to keep in mind the very specific historical circumstances under which a single scientist could make such an integrative achievement. Just as the development of Einstein's *political* thinking

[50] For a further discussion of Einstein's perspective on Lorentz' electrodynamics, see Renn 1993.

was dependent on its *social* context (Scheideler & Goenner 1997), Einstein's integrative capabilities in the *conceptual* realm were dependent on historically specific features of his *intellectual* biography.[51]

And already the contemporary attempt to exploit Einstein's integrative capabilities in the *conceptual* realm for an *organization* of science—aimed at fostering an integration of physics and chemistry—unmistakably failed (Castagnetti & Goenner 1997).

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[51] A thorough discussion of this dependence would go beyond the scope of this article. For a brief discussion of some aspects of this issue, see Renn & Schulman 1992: introduction.

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