

# Partial Identification in an Ecological Inference Model\*

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## Abstract

We provide partial identification results and inference procedures for structural prediction/counterfactual analysis in *ecological inference* models, in which aggregate and individual-level data must be combined in order to draw inferences about individual-level behavior; this arises naturally in many applications in industrial organization, labor economics, and political science. We interpret an ecological inference model as a treatment effect model in which the outcomes of interest and the conditional covariates are only available in separate datasets. Under a “selection on observables” assumption, the counterfactual distribution and policy parameters of interest are partially identified. We present an application to recent US presidential elections.

**Keywords:** Ecological inference, partial identification, copula, treatment effects

**JEL Codes:** C14, C31

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# 1 Introduction

Combining aggregate and individual-level data arises naturally in many empirical applications. These include the “classic” *ecological inference* (EI) problem from political science on predicting (individual) voting behavior in which one dataset consists of aggregate vote shares and the other consists of individual’s characteristics; demand analysis in economics in which one dataset consists of market shares and the other consists of consumer demographics; poverty analysis in which one dataset consists of program participation and the other consists of demographic attributes.

Early work on partial identification in the EI problem focus on predicting individual-level behavior when outcomes are binary and individual characteristics are also binary, see Duncan and Davis (1953) and Goodman (1953). King (1997), most notably, and Cross and Manski (1999, 2002) formalize and extend this work to the case of *continuous* outcome variables, with *discrete* individual characteristics.

In this paper, we generalize this model to the case where both outcomes and covariates can be continuous or discrete. In order to do this, we develop a novel approach, which borrows and combines tools and insights from the treatment effect and copula literatures. Specifically, we reinterpret the forecasting problem in Cross and Manski (1999) as a program evaluation problem; by doing so, the structural prediction exercise in the EI model becomes analogous to a problem of identifying mean counterfactual outcomes in a treatment effect model, in which the outcomes of interest are observed only at the aggregate level, but the conditioning covariates are observed at the individual-level. Because the data are contained in two separate datasets, the usual formulas for the mean counterfactual outcomes cannot be used; however, by applying results from the copula literature, we obtain *bounds* on these counterfactual outcomes. To our knowledge, such an approach is new in both the treatment effects and EI literatures.

The need to combine two separate datasets distinguishes this paper from existing work in the treatment effect literature in which outcomes and covariates are observed in the same dataset, see e.g., Rosenbaum and Rubin (1983a, b), Hahn (1998), Heckman, Ichimura, Smith, and Todd (1998), Dehejia and Wahba (1999), and Hirano, Imbens, and Ridder (2000), Chernozhukov, Fernandez-Val, Melly (2009), Rothe (2010, 2011), and Firpo, Fortin, and Lemieux (2010), to name only a few. Under the “selection-on-observables” assumption, the mean counterfactual outcomes (and hence the average treatment effects for the treated) are point-identified via a propensity-score reweighting of the data.

This point identification disappears when the model variables are observed only in two separate datasets. However, by applying the Cambanis-Simons-Stout inequality<sup>1</sup>, which follows from the Frechet-Hoeffding copula inequalities, we can obtain sharp bounds on the mean counterfactual outcomes. Other parameters, such as counterfactual distributions and average treatment effect parameters, can be similarly bounded. The Cambanis-Simons-Stout inequality (and, more generally, related results from the probability or copula literature which provide bounds on the expectation of functionals of random variables with fixed marginal distributions) have recently been used to evaluate distributional treatment effects under selection-on-observables assumption in Fan and Park (2009, 2010), Fan and Wu (2010), Firpo and Ridder (2009), Heckman, Smith, and Clements (1997), and Fan and Zhu (2009). These papers assume that outcomes and covariates are observed in the same dataset so that the *marginal* and counterfactual marginal distributions are point identified; hence, these papers study the partial identification of treatment effect parameters that depend on the *joint* distribution of the potential outcomes.

The literature on data combination is much smaller. Manski (2000; esp. Section 5) considers bounds for a treatment effect model when the aggregate treatment outcomes and agent demographics are separately observed. Ridder and Moffitt (2007; section 3.1) present a thorough discussion of the use of Frechet (copula) bounds in data combination contexts.<sup>2</sup> Relatedly, Hoderlein and Stoye (2009) use the Frechet bounds to bound violations of the revealed-preference axioms in a repeated cross-section context. To our knowledge, our approach to structural prediction in EI models, based on combining insights from the treatment effects and copula literatures, is new.

The lower and upper bounds of mean counterfactual outcomes and average treatment effect parameters established in this paper are functionals of quantile functions of generated variables. To establish asymptotic properties of the plug-in estimators of the lower and upper bounds, we make heavy use of modern empirical process methods similar to Linton, Song, and Whang (2010). Inference procedures then follow existing work for partially identified parameters including Chernozhukov, Hong, and Tamer (2007), Stoye (2009), Andrews and Soares (2010), and Fan and Park (2010). A numerical example and an application to recent US presidential elections illustrate the usefulness of our results.

The rest of this paper is organized as follows. Section 2 introduces the EI model, the traditional approach to the structural prediction or counterfactual analysis in the EI model

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<sup>1</sup>see Theorem 2 in Cambanis, Simons, and Stout (1976) or Lemma 3.1 in Section 3 of this paper

<sup>2</sup>For a reference on the Frechet bounds, see Joe (1997).

and its limitations, and our new approach. In Section 3, we present the main identification results. Section 4 presents estimators of the bounds on mean counterfactual outcomes and results from a numerical example. An empirical application is presented in Section 5. Section 6 establishes asymptotic properties of the plug-in estimators of the bounds on mean counterfactual outcomes. The last section concludes. Technical proofs are presented in the Appendix.

Throughout the rest of this paper, we use  $F_{A|B}(\cdot|b)$  and  $f_{A|B}(\cdot|b)$  to denote the distribution function and density function of the random variable  $A$  conditional on  $B = b$ . For a distribution function  $F$ , we use  $F^{-1}(\cdot)$  to denote its quantile function.

## 2 Ecological Inference Model

### 2.1 Set-up and the Traditional EI Approach

We start by describing the formal setup we use for the EI model. Since our paper departs from the standard EI tradition (e.g. King (1997) and Cross and Manski (1999, 2002)) and applies tools and insights from the treatment effect literature to the EI problem, we will present three running examples.

**Example A: vote outcomes across election years.** This is the “classic” ecological inference problem from political science. Let  $D$  be the indicator for election year,  $Y_D$  be an indicator for the party voted for in election year  $D$ , and  $Z$  denote a vector of demographic variables for voters in the different election years. In this setting, the counterfactual outcome distributions are of particular interest; for US presidential election data, the counterfactual distribution  $F_{Y_{2000}|D}(\cdot|1980)$  describes how the 1980 US population would have voted if they had to choose among the candidates (GW Bush vs. Gore) in the 2000 presidential election. Indeed, Cross and Manski (1999) analyze the prediction of the counterfactual vote outcomes  $E(Y_{1996}|D = 2000)$ , using our notation. ■

**Example B: changes in wage distribution across time.** This example is drawn from DiNardo, Fortin, and Lemieux (1996). As in the voting example above,  $D$  is a binary indicator for two different years:  $D = 0$  for the baseline year 1988, and  $D = 1$  for the counterfactual year 1979.  $Y_D$  denotes wages in year  $D$ , and DiNardo, Fortin and Lemieux focus on estimating  $f_{Y_0|D}(\cdot|1)$ , which they interpret as the counterfactual density of wages “if individual attributes had remained at their 1979 levels and workers had been paid according

to the wage schedule observed in 1988". ■

**Example C: demand across different geographic markets.** Let  $D$  be an indicator for geographic market, and  $Y_D \in \{0, 1, \dots, J\}$  be a multinomial indicator of choice among one of the  $J$  competing products (with  $Y_D = 0$  indicating purchase of no product).  $Z$  denotes a vector of demographic variables for consumers in the different geographic markets. In this setting, the prices of the products  $P_D$  vary over geographic markets, so that the counterfactual distribution  $F_{Y_d|D}(\cdot|d')$  can be interpreted as the counterfactual demand of the consumers in market  $d$  (who faced prices  $P_d$ ) if their prices were changed to  $P_{d'}$ , the prices prevailing in market  $d'$ . ■

To incorporate all three examples in our set-up, we let  $D \in \{0, 1\}$  denote two time periods or geographic markets (“treatments”), and let  $Y_D$  denote the corresponding outcome variable of interest for  $D = 0, 1$ . Using the treatment effect interpretation,  $Y_0$  and  $Y_1$  are considered “potential outcomes”; that is, each individual agent, regardless of the treatment group that he belongs to, is associated with values for *both* of the potential outcomes  $Y_0$  and  $Y_1$ . However, because each agent belongs to only one of the treatment groups, his observed outcome is  $Y = Y_1 D + Y_0(1 - D)$ .

In typical EI models, the outcome variables are discrete variables, taking  $M \geq 2$  distinct values. In the *aggregate dataset*, we assume that the joint distribution function of  $(Y, D)$  is observed; since both of these are discrete variables, this joint distribution function takes the form of a  $M \times 2$  matrix with  $(m, d)$ -element equal to the aggregate probability  $\Pr(Y = m, D = d)$ . The second dataset is an *individual-level* sample, containing a sample of individual-level demographic variables  $Z \in \mathcal{Z} \subset \mathbb{R}^{d_z}$  and treatment indicators  $D$ .<sup>3</sup>

Let  $F_{Y_d|D}(\cdot|d')$  denote the conditional distribution function of  $Y_d$  given  $D = d'$  and  $F_{Y_d}(\cdot)$  the marginal distribution function of  $Y_d$ . Obviously, in the aggregate data, the two conditional distributions  $F_{Y_d|D}(\cdot|d)$  for  $d = 0, 1$  are observed. However, because of inherent differences between the individuals in the two aggregate units, we want to disentangle differences between the two distributions due to inherent behavioral differences (i.e., inherent differences between  $Y_d$  and  $Y_{d'}$ ) versus demographic differences (i.e., as captured in the distribution of the covariates  $Z$ ) between the two units.<sup>4</sup> In the EI literature (as well as

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<sup>3</sup>For simplicity of notation, we assume that the only common variable to both datasets is  $D$ . It is straightforward to extend the identification results in this paper to incorporate covariates that are observable in both data sets.

<sup>4</sup>Firpo, Fortin and Lemieux (2010) refer to these as, respectively, the “structure” and compositional effects.

the treatment effects literature), the typical parameters of interest such as mean counterfactual outcomes are (functionals of) the (i) marginal potential outcome distributions  $F_{Y_d}(\cdot)$ ,  $d = 0, 1$ ; as well as the (ii) counterfactual outcome distributions  $F_{Y_d|D}(\cdot|d')$ , for  $d, d' \in \{0, 1\}$ ,  $d \neq d'$ .

The **traditional EI approach** is a two-step procedure in which we first infer individual-level behavior and then draw inferences for counterfactual outcomes from the individual-level behavior; see Cross and Manski (1999, 2002) and Cho and Manski (2008). To illustrate, suppose the individual-level covariates  $Z$  are discrete-valued and the interest is in the mean counterfactual outcome  $E(Y_0|D = 1)$ . The EI approach starts with the identification or partial identification of the *long regression*  $E(Y_0|D = 0, Z = z)$  from two identified marginal distributions:  $F_{Y_0|D}(\cdot|0)$  identified from the aggregate data and  $F_{Z|D}(\cdot|0)$  identified from the individual data. With *the invariance assumption*:  $E(Y_0|D = 0, Z = z) = E(Y_0|D = 1, Z = z)$ , the identified set for  $E(Y_0|D = 1)$  can be deduced from that of the long regression  $E(Y_0|D = 0, Z = z)$  and  $F_{Z|D}(\cdot|1)$  identified from the individual data, since

$$\begin{aligned} E(Y_0|D = 1) &= E[E(Y_0|D = 1, Z) | D = 1] \\ &= E[E(Y_0|D = 0, Z) | D = 1]. \end{aligned} \tag{2.1}$$

When the covariates  $Z$  are continuous, however, the identification of the long regression  $E(Y_0|D = 0, Z = z)$  no longer obtains using the arguments in Cross and Manski, so that a different EI approach is needed in these cases.

## 2.2 A New Approach

To allow for both discrete and continuous covariates in  $Z$ , we develop a completely new approach to the counterfactual analysis in the EI model in this paper. Our approach draws on, and combines, tools and insights from the treatment effect literature and the copula literature. In the treatment effect literature, the *selection on observables* assumption is often invoked to identify and evaluate average treatment effects.<sup>5</sup> It consists of the following two conditions:

**(C1)** Let  $(Y_1, Y_0, D, Z)$  have a joint distribution. For all  $z \in \mathcal{Z}$ ,  $(Y_1, Y_0)$  is jointly independent of  $D$  conditional on  $Z = z$ .

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<sup>5</sup>See e.g., Rosenbaum and Rubin (1983a, b), Hahn (1998), Heckman, Ichimura, Smith, and Todd (1998), Dehejia and Wahba (1999), and Hirano, Imbens, and Ridder (2000), to name only a few.

**(C2)** For all  $z \in \mathcal{Z}$ ,  $0 < p(z) < 1$ ,  $0 < p_1 < 1$ , where  $p(z) = \Pr(D = 1|Z = z)$  and  $p_d = \Pr(D = d)$  for  $d = 1, 0$ .

(C1) is a conditional independence assumption and (C2) is a support assumption. Note that (C1) strengthens the invariance assumption  $E(Y_0|D = 0, Z = z) = E(Y_0|D = 1, Z = z)$  underlying the traditional EI approach (see above). Strengthening the invariance assumption to a full conditional independence assumption allows us to address interesting policy parameters besides the structural prediction problem in the EI framework.

**Example B (cont'd):** The analysis in DiNardo, Fortin, and Lemieux (1996; hereafter DFL) can be reinterpreted in light of assumption (C1); see Firpo, Fortin, and Lemieux (2010). Under this assumption, the counterfactual wage density is:

$$\begin{aligned} f_{Y_0|D}(y|1) &= \int f_{Y_0,Z|D}(y, z|1) dz \\ &= \int f_{Y_0|Z,D}(y|z, 1) f_{Z|D}(z|1) dz \\ &= \int f_{Y_0|Z,D}(y|z, 0) f_{Z|D}(z|1) dz \end{aligned} \tag{2.2}$$

where the third line applies (C1). Eq. (2.2) above corresponds directly to Eq. (3) in DFL (subject to changes in notation), showing how DFL's analysis fits into the framework described here. ■

**Example C (cont'd):** To understand the conditional independence assumption (C1) further, we consider a model  $Y_d = g(P = P_d, Z, \eta_d)$  where  $\eta_d$  captures all other unobservables in demand. Note that conditional on  $Z$ , the only randomness in demand  $Y_d$  is due to  $\eta_d$ . The conditional independence assumption essentially boils down to  $(\eta_0, \eta_1 \perp P)|Z$ . This can be interpreted as an “exogeneity” assumption on prices, conditional on  $Z$ . That is, conditional on  $Z$ , the observed demographic variables, the unobservables  $\eta_d$  do not affect prices. This is an undoubtedly strong assumption – it rules out, for instance, unobserved demand shocks which affect firms' pricing decisions – but, as we will see, identification is already non-trivial even in this simple case. ■

When  $(Y, D, Z)$  are all observed in a single dataset (so that there is no EI problem), it is well known that, under (C1) and (C2), the conditional distribution functions of  $Y_d$  given  $Z$  denoted as  $F_{Y_1|Z}(y|z)$  and  $F_{Y_0|Z}(y|z)$  are identified:

$$\begin{aligned} F_{Y_1|Z}(y|z) &= P(Y_1 \leq y|Z = z) = P(Y_1 \leq y|Z = z, D = 1) \\ &= P(Y \leq y|Z = z, D = 1), \end{aligned} \tag{2.3}$$

$$F_{Y_0|Z}(y|z) = P(Y \leq y|Z = z, D = 0), \quad (2.4)$$

and the marginal distributions  $F_{Y_1}(y)$ ,  $F_{Y_0}(y)$  are identified as well:

$$F_{Y_1}(y) = E \left[ \frac{D}{p(Z)} I\{Y \leq y\} \right] \text{ and } F_{Y_0}(y) = E \left[ \frac{1-D}{1-p(Z)} I\{Y \leq y\} \right]. \quad (2.5)$$

Moreover under (C1) and (C2), the counterfactual distribution function  $F_{Y_0|D}(y|1)$  is also identified:

$$F_{Y_0|D}(y|1) = \frac{1}{p_1} E \left[ \frac{(1-D)p(Z)}{1-p(Z)} I\{Y \leq y\} \right].$$

Thus parameters that are functionals of  $F_{Y_1|Z}(y|z)$ ,  $F_{Y_0|Z}(y|z)$ ,  $F_{Y_0|D}(y|1)$ , and the marginal distributions are identified.

In the EI model, however,  $(Y, D)$  and  $(Z, D)$  are observed in separate datasets, and we face a fundamental identification problem: the conditional expectations in Eqs. (2.3) and (2.4) can no longer be identified from the available data; neither can the unconditional distributions in Eq. (2.5). In fact, we will see that once the variables  $(Y, D, Z)$  are not simultaneously observed in the same dataset, then we no longer have point identification of these distributions, even under assumptions (C1) and (C2). To tackle this problem, we resort to the Cambanis-Simons-Stout inequality (see Lemma 3.1 below) to establish sharp bounds on some of these quantities, which is the topic of the next section.

### 3 Counterfactual Analysis in the EI Model

In this section, we present our main identification results. These include identification/partial identification results for the marginal and counterfactual marginal distributions of the potential outcomes  $Y_0, Y_1$  and for functionals of these distributions, including the traditional program evaluation parameters such as the average treatment effect and treatment effect for the treated. Our main identification results exploit the result in the next lemma, which is a special case of Theorem 2 in Cambanis, Simons, and Stout (1976).

**Lemma 3.1 (The Cambanis-Simons-Stout inequality)** *Let  $S$  and  $T$  denote two random variables with known marginal distribution functions  $F_S$  and  $F_T$ . Assume  $S$  and  $T$  have finite variances. Then*

$$\int_0^1 F_S^{-1}(1-u) F_T^{-1}(u) du \leq E(ST) \leq \int_0^1 F_S^{-1}(u) F_T^{-1}(u) du.$$

*Without additional information, the bounds are sharp.*

It is worth pointing out that the Cambanis-Simons-Stout inequality provides sharp bounds on  $E(ST)$  when the marginal distributions of  $S, T$  are known, while an application of the Cauchy-Schwartz inequality to  $E(ST)$  in this case leads to bounds that are in general not sharp.

Throughout the rest of this paper, we assume Assumption (I) below:

**Assumption (I).** Let  $W = 1/p(Z)$  and  $V = 1/[1 - p(Z)]$ . Assume  $Var(W) < \infty$ ,  $Var(V) < \infty$ , and  $Var(V/W) < \infty$ . In addition, let  $g$  denote a measurable function such that  $Var(g(Y_d)) < \infty$  for  $d = 1, 0$ .

Our first series of results establishes sharp bounds on the mean of  $g(Y_d)$ :

**Theorem 3.2** (i) Let  $\mu_d(g) \equiv E(g(Y_d))$ . Then  $\mu_d^L(g) \leq \mu_d(g) \leq \mu_d^U(g)$ , where  $d = 1, 0$  and

$$\begin{aligned}\mu_1^L(g) &= E \left[ D \int_0^1 F_{g(Y)|D}^{-1}(1-u|D) F_{W|D}^{-1}(u|D) du \right], \\ \mu_1^U(g) &= E \left[ D \int_0^1 F_{g(Y)|D}^{-1}(u|D) F_{W|D}^{-1}(u|D) du \right], \\ \mu_0^L(g) &= E \left[ (1-D) \int_0^1 F_{g(Y)|D}^{-1}(1-u|D) F_{V|D}^{-1}(u|D) du \right], \\ \mu_0^U(g) &= E \left[ (1-D) \int_0^1 F_{g(Y)|D}^{-1}(u|D) F_{V|D}^{-1}(u|D) du \right].\end{aligned}$$

Without additional information, the bounds are sharp.

(ii) Let  $\mu_{d|1}(g) \equiv E(g(Y_d)|D=1)$ . Then  $\mu_{1|1}(g)$  is identified:  $\mu_{1|1}(g) = E(Dg(Y)/p_1)$  and  $\mu_{0|1}^L(g) \leq \mu_{0|1}(g) \leq \mu_{0|1}^U(g)$ , where

$$\begin{aligned}\mu_{0|1}^L(g) &= \frac{1}{p_1} E \left[ (1-D) \int_0^1 F_{g(Y)|D}^{-1}(1-u|D) F_{\frac{V}{W}|D}^{-1}(u|D) du \right], \\ \mu_{0|1}^U(g) &= \frac{1}{p_1} E \left[ (1-D) \int_0^1 F_{g(Y)|D}^{-1}(u|D) F_{\frac{V}{W}|D}^{-1}(u|D) du \right].\end{aligned}$$

Without additional information, the bounds are sharp.

**Proof:** Consider  $\mu_1(g)$ . An analogue of Eq. (2.5) gives us an expression for  $\mu_1(g)$ , in terms of the variables  $(Y, D, Z)$ , but we cannot compute this because we do not observe the joint distribution  $(Y, D, Z)$ , but only the two separate distributions of  $(Y|D)$  and  $(D, Z)$ . The

individual-level data allows us to identify the propensity score  $p(z) = P(D = 1|z)$ . Then, rearranging the expression, we get

$$\mu_1(g) = E \left[ \frac{D}{p(Z)} g(Y) \right] = E(Dg(Y)W) = E(DE[g(Y)W|D]).$$

The rightmost quantity here contains the term  $E[g(Y)W|D]$ , which is the expectation of a product of two random variables  $g(Y)$  and  $W$ , which are observed in different datasets, so that the expectation cannot be computed feasibly. However, we can apply Lemma 3.1 to obtain bounds on the expectation of their product. This leads to the bounds for  $\mu_1(g)$  in part (i) of Theorem 3.2. Similarly, by using the expression:  $\mu_0(g) = E \left[ \frac{1-D}{1-p(Z)} g(Y) \right]$ , we obtain the bounds for  $\mu_0(g)$  in part (i). For part (ii), noting

$$\frac{V}{W} = \frac{p(Z)}{1-p(Z)},$$

we get  $p_1\mu_{0|1}(g) = E[(1-D)\left(\frac{V}{W}\right)g(Y)]$  and the bounds in part (ii).

The bounds for  $\mu_1(g)$  are sharp, in that there exist distributions of  $(D, Y, W)$  which attain these bounds. In fact, the upper bound on  $\mu_1(g)$  is achieved when, conditional on  $D$ ,  $(g(Y), W)$  are perfectly positively dependent on each other; the lower bound is achieved when, conditional on  $D$ ,  $(g(Y), W)$  are perfectly negatively dependent on each other. Analogously, the upper bound on  $\mu_0(g)$  is achieved when conditional on  $D$ ,  $(g(Y), V)$  are perfectly negatively dependent on each other and the lower bound is achieved when conditional on  $D$ ,  $(g(Y), V)$  are perfectly positively dependent on each other. ■

We note that  $\mu_1^L(g)$  and  $\mu_1^U(g)$  are identified from the sample information, as  $F_{g(Y)|D}(\cdot|d)$  is identified from the first dataset,  $F_{W|D}(\cdot|d)$  ( $F_{V|D}(\cdot|d)$ ) is identified from the second dataset, and the expectation in the expressions for  $\mu_1^L(g)$  and  $\mu_1^U(g)$  can be identified from either dataset (or both).

### 3.1 Structural Prediction and Treatment Effects

Let  $\Delta = Y_1 - Y_0$  denote the individual treatment effect. Let  $\mu_\Delta, \mu_{\Delta|1}$  denote, respectively, the average treatment effect (ATE) and the average treatment effect on the treated (TT), i.e.,  $\mu_\Delta = E(\Delta)$  and  $\mu_{\Delta|1} = E(\Delta|D = 1)$ . Bounds on  $\mu_\Delta$  and  $\mu_{\Delta|1}$  follow immediately from Theorem 3.2:

$$\mu_1^L - \mu_0^U \leq \mu_\Delta \leq \mu_1^U - \mu_0^L, \quad (3.1)$$

and

$$E \left[ \frac{D}{p_1} Y \right] - \mu_{0|1}^U \leq \mu_{\Delta|1} \leq E \left[ \frac{D}{p_1} Y \right] - \mu_{0|1}^L.$$

Let  $g(Y_d) = I\{Y_d \leq y\}$  in Theorem 3.2. Noting

$$F_{I_Y|D}^{-1}(u|D) = \begin{cases} 0 & \text{for } u \in [0, 1 - F_{Y|D}(y|D)) \\ 1 & \text{for } u \in [1 - F_{Y|D}(y|D), 1] \end{cases},$$

where  $I_Y = I\{Y \leq y\}$ , we obtain the identified sets for the marginal distribution functions of the potential outcomes,  $F_{Y_1}(y)$ ,  $F_{Y_0}(y)$  in part (i) of Theorem 3.3 below. The identified sets for the counterfactual marginal distribution functions  $F_{Y_1|D}(y|1)$ , and  $F_{Y_0|D}(y|1)$  are obtained similarly.

**Theorem 3.3** (i) For  $d = 0, 1$ , we have:  $F_d^L(y) \leq F_{Y_d}(y) \leq F_d^U(y)$ , where

$$\begin{aligned} F_1^L(y) &= E \left[ D \int_0^{F_{Y|D}(y|D)} F_{W|D}^{-1}(u|D) du \right], \\ F_1^U(y) &= E \left[ D \int_{1-F_{Y|D}(y|D)}^1 F_{W|D}^{-1}(u|D) du \right], \\ F_0^L(y) &= E \left[ (1-D) \int_0^{F_{Y|D}(y|D)} F_{V|D}^{-1}(u|D) du \right], \\ F_0^U(y) &= E \left[ (1-D) \int_{1-F_{Y|D}(y|D)}^1 F_{V|D}^{-1}(u|D) du \right]. \end{aligned}$$

Without additional information, the bounds are sharp (both pointwise and uniformly).

(ii)  $F_{Y_1|D}(y|1)$  is identified:  $F_{Y_1|D}(y|1) = E[DI\{Y \leq y\}/p]$  and  $F_{Y_0|D}(y|1)$  is partially identified:  $F_{0|D}^L(y|1) \leq F_{Y_0|D}(y|1) \leq F_{0|D}^U(y|1)$ , where

$$\begin{aligned} F_{0|D}^L(y|1) &= \frac{1}{p_1} E \left[ (1-D) \int_0^{F_{Y|D}(y|D)} F_{\bar{W}|D}^{-1}(u|D) du \right], \\ F_{0|D}^U(y|1) &= \frac{1}{p_1} E \left[ (1-D) \int_{1-F_{Y|D}(y|D)}^1 F_{\bar{W}|D}^{-1}(u|D) du \right]. \end{aligned}$$

Without additional information, the bounds are sharp (both pointwise and uniformly).

We note that the distribution bounds in Theorem 3.3 are not only pointwise sharp but also uniformly sharp, i.e., they are sharp for the corresponding distribution functions. To see

this, consider the bounds on  $F_1(\cdot)$ . Both  $F_1^L(\cdot)$  and  $F_1^U(\cdot)$  are distribution functions. The lower bound  $F_1^L(\cdot)$  is the distribution function of  $Y_1$  when conditional on  $D$ ,  $(Y, W)$  are perfectly positively dependent on each other; the upper bound  $F_1^U(\cdot)$  is the distribution function of  $Y_1$  when, conditional on  $D$ ,  $(Y, W)$  are perfectly negatively dependent on each other.

The uniform sharpness of the bounds in Theorem 3.3 allows us to establish sharp bounds on monotone functionals of the marginal or counterfactual marginal distribution functions. Such functionals include the quantile treatment effects (QTE) defined as

$$\begin{aligned} QTE_u &= F_{Y_1}^{-1}(u) - F_{Y_0}^{-1}(u) \text{ and} \\ QTE_{u|1} &= F_{Y_1|D}^{-1}(u|1) - F_{Y_0|D}^{-1}(u|1), \end{aligned}$$

where  $u \in (0, 1)$ , and the cumulative distribution functions

$$\begin{aligned} F_\Delta(\delta) &= \Pr(\Delta \leq \delta), \\ F_\Delta(\delta|D=1) &= \Pr(\Delta \leq \delta|D=1), \end{aligned}$$

and the corresponding quantile functions. For example, it may be of interest to evaluate the probability of a positive treatment effect: either  $\Pr(\Delta > 0)$  or  $\Pr(\Delta > 0|D=1)$ , and the median of  $\Delta$  for the treated. Sharp bounds on these parameters can be established using Theorem 3.3 and the lemma below adapted from Frank, Nelsen, and Schweizer (1987); see also Fan and Park (2009, 2010).

**Lemma 3.4** *Let  $F_\Delta(\delta|\cdot) = \Pr(\Delta \leq \delta|\cdot)$ . Then  $F_\Delta^L(\delta|\cdot) \leq F_\Delta(\delta|\cdot) \leq F_\Delta^U(\delta|\cdot)$ , where*

$$\begin{aligned} F_\Delta^L(\delta|\cdot) &= \max\left(\sup_y [F_1(y|\cdot) - F_0(y - \delta|\cdot)], 0\right), \\ F_\Delta^U(\delta|\cdot) &= 1 + \min\left(\inf_y [F_1(y|\cdot) - F_0(y - \delta|\cdot)], 0\right), \end{aligned}$$

where  $F_d(y|\cdot) = \Pr(Y_d \leq y|\cdot)$  for  $d = 1, 0$ .

Consider, for instance, the CDF  $F_\Delta(\delta|D=1)$ . From Theorem 3.3 and Lemma 3.4, we have:

$$F_\Delta^L(\delta|D=1) \leq F_\Delta(\delta|D=1) \leq F_\Delta^U(\delta|D=1),$$

where

$$\begin{aligned} F_\Delta^L(\delta|D=1) &= \max\left(\sup_y [F_{Y_1|D}(y|1) - F_{Y_0|D}^U(y - \delta|1)], 0\right), \\ F_\Delta^U(\delta|D=1) &= 1 + \min\left(\inf_y [F_{Y_1|D}(y|1) - F_{Y_0|D}^L(y - \delta|1)], 0\right). \end{aligned}$$

Sharp bounds on the quantile function of  $F_\Delta(\delta|D=1)$  follow from sharp bounds on  $F_\Delta(\delta|D=0)$ .

### 3.2 Group-Level Behavior

In some applications, group level behavior may be of interest, see Cho and Manski (2008). Let  $\mathcal{S}$  denote a subset of  $\mathcal{Z}$  such that  $\Pr(Z \in \mathcal{S}) > 0$ . Using our approach, we can establish sharp bounds on the group-level behavior for individuals with characteristics  $Z \in \mathcal{S}$  even when some of the components of  $Z$  are continuous. To illustrate, consider  $E(Y_0|Z \in \mathcal{S})$

$$\begin{aligned} E(Y_0|Z \in \mathcal{S}) &= \frac{1}{\Pr(Z \in \mathcal{S})} E((1-D)YVI\{Z \in \mathcal{S}\}) \\ &= \frac{1}{\Pr(Z \in \mathcal{S})} E((1-D)E[YVI\{Z \in \mathcal{S}\}|D]) \end{aligned}$$

leading to

$$\begin{aligned} &\frac{1}{\Pr(Z \in \mathcal{S})} E\left((1-D) \int_0^1 F_{Y|D}^{-1}(1-u|D) F_{V_S|D}^{-1}(u|D) du\right) \\ \leq E(Y_0|Z \in \mathcal{S}) &\leq \frac{1}{\Pr(Z \in \mathcal{S})} E\left((1-D) \int_0^1 F_{Y|D}^{-1}(u|D) F_{V_S|D}^{-1}(u|D) du\right), \end{aligned}$$

where  $V_S = VI\{Z \in \mathcal{S}\}$ .

## 4 Estimation of the Bounds

In describing the estimators, we simplify further to the case where the outcomes are binary:  $Y_0, Y_1 \in \{0, 1\}$ .<sup>6</sup> Let  $P_{01} = \Pr(Y = 0|D = 1)$  and  $P_{00} = \Pr(Y = 0|D = 0)$ . Thereby, the first (aggregate) dataset contains two values  $P_{01}$  and  $P_{00}$ , while the second (individual-level) dataset contains  $(Z_i, D_i)_{i=1}^n$ , where  $i$  denotes individual. The asymptotic properties of all the estimators are derived under the assumption that  $P_{01}$  and  $P_{00}$  are completely known (observed without sampling error). It is straightforward, but tedious, extension to consider the case when  $P_{01}$  and  $P_{00}$  are observed with noise.

In this section we present our estimators of the bounds on mean counterfactual outcomes (corresponding to the bounds defined in Theorem 3.2 above), and a numerical example. We delay the detailed discussion of the asymptotic theory for these estimators to Section 6 below. Moreover, estimation of other functionals of the counterfactual outcome distribution

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<sup>6</sup>The extension to more values of the outcomes, or to more than two treatments, is straightforward.

(as in Theorem 3.3) proceeds in analogous fashion as for the mean; for this reason, we do not discuss this in the paper, but details are available from the authors upon request.

## 4.1 Estimators

We begin by considering the estimation of the (bounds on) mean counterfactual outcomes. For binary outcomes,

$$F_{Y|D}^{-1}(1-u|D) = \begin{cases} 0 & \text{if } 1 - \Pr(Y=0|D) \leq u \leq 1 \\ 1 & \text{if } 0 \leq u < 1 - \Pr(Y=0|D) \end{cases}$$

so an application of Theorem 3.2 yields:

$$\begin{aligned} \mu_1^L &= E \left[ D \int_0^{1-\Pr(Y=0|D)} F_{W|D}^{-1}(u|D) du \right] = p_1 \int_0^{1-P_{01}} F_{W|D}^{-1}(u|1) du, \\ \mu_1^U &= E \left[ D \int_{\Pr(Y=0|D)}^1 F_{W|D}^{-1}(u|D) du \right] = p_1 \int_{P_{01}}^1 F_{W|D}^{-1}(u|1) du, \end{aligned}$$

and

$$\begin{aligned} \mu_0^L &= E \left[ (1-D) \int_0^{1-\Pr(Y=0|D)} F_{V|D}^{-1}(u|D) du \right] = (1-p_1) \int_0^{1-P_{00}} F_{V|D}^{-1}(u|0) du, \\ \mu_0^U &= E \left[ (1-D) \int_{\Pr(Y=0|D)}^1 F_{V|D}^{-1}(u|D) du \right] = (1-p_1) \int_{P_{00}}^1 F_{V|D}^{-1}(u|0) du. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} \mu_{0|1}^L &= \frac{1-p_1}{p_1} \int_0^{1-P_{00}} F_{V/W|D}^{-1}(u|0) du, \\ \mu_{0|1}^U &= \frac{1-p_1}{p_1} \int_{P_{00}}^1 F_{V/W|D}^{-1}(u|0) du. \end{aligned}$$

To estimate these bounds, we need to estimate  $p_1$  and the quantile functions:  $F_{W|D}^{-1}(u|d)$ ,  $F_{\frac{V}{W}|D}^{-1}(u|d)$ , and  $F_{V|D}^{-1}(u|d)$  for  $d = 1, 0$ . An estimator of  $p_1$  is  $\hat{p}_1 = n^{-1} \sum_{i=1}^n D_i$ . To estimate the quantile functions, we let

$$F_{P|D}(p|d) = \Pr(p(Z) \leq p|D = d)$$

and, assuming  $p(Z)$  is a continuous random variable, we obtain:

$$F_{W|D}(w|d) = \Pr \left( p(Z) \geq \frac{1}{w} | D = d \right) = 1 - F_{P|D} \left( \frac{1}{w} | d \right),$$

$$F_{V|D}(v|d) = \Pr\left(p(Z) \leq 1 - \frac{1}{v} | D = d\right) = F_{P|D}\left(1 - \frac{1}{v} | d\right),$$

and

$$F_{V/W|D}(a|d) = \Pr\left(p(Z) \leq \frac{a}{1+a} | D = d\right) = F_{P|D}\left(\frac{a}{1+a} | d\right).$$

Let  $Q_{A|D}(u|d) \equiv F_{A|D}^{-1}(u|d)$  and  $\widehat{Q}_{A|D}(u|d)$  denote a consistent estimator of  $Q_{A|D}(u|d)$  for  $A = W, V/W, V$ . Then

$$Q_{W|D}(u|d) = \frac{1}{F_{P|D}^{-1}(1-u|d)}, \quad Q_{V|D}(u|d) = \frac{1}{1 - F_{P|D}^{-1}(u|d)}, \quad Q_{V/W|D}(u|d) = \frac{F_{P|D}^{-1}(u|d)}{1 - F_{P|D}^{-1}(u|d)}.$$

Let  $\widehat{p}(z)$  denote any consistent estimator of the propensity score  $p(z)$  using dataset  $\{Z_i, D_i\}_{i=1}^n$ . Further let

$$\widehat{P}_i = \widehat{p}(Z_i), \quad i = 1, \dots, n.$$

Our quantile estimators are constructed from the generated dataset:  $\{\widehat{P}_i, D_i\}_{i=1}^n$ . Let

$$\widehat{F}_{P|D}^{-1}(u|d) = \inf\left\{a \in \mathcal{A} : \widehat{F}_{P|D}(a|d) \geq u\right\}, \quad (4.1)$$

where  $\widehat{F}_{P|D}(a|d)$  is defined as:

$$\widehat{F}_{P|D}(a|d) = \frac{n^{-1} \sum_{i=1}^n I\{\widehat{P}_i \leq a\} I\{D_i = d\}}{\widehat{p}_d}. \quad (4.2)$$

Using the quantile estimator  $\widehat{F}_{P|D}^{-1}(u|d)$ , we construct the following quantile estimators:

$$\widehat{Q}_{W|D}(u|d) = \frac{1}{\widehat{F}_{P|D}^{-1}(1-u|d)}, \quad \widehat{Q}_{V|D}(u|d) = \frac{1}{1 - \widehat{F}_{P|D}^{-1}(u|d)}, \quad \widehat{Q}_{V/W|D}(u|d) = \frac{\widehat{F}_{P|D}^{-1}(u|d)}{1 - \widehat{F}_{P|D}^{-1}(u|d)},$$

and propose the estimators of the bounds below:

$$\begin{aligned} \widehat{\mu}_1^L &= \widehat{p}_1 \left[ \int_0^{1-P_{01}} \widehat{Q}_{W|D}(u|1) du \right], \\ \widehat{\mu}_1^U &= \widehat{p}_1 \left[ \int_{P_{01}}^1 \widehat{Q}_{W|D}(u|1) du \right], \\ \widehat{\mu}_0^L &= (1 - \widehat{p}_1) \left[ \int_0^{1-P_{00}} \widehat{Q}_{V|D}(u|0) du \right], \\ \widehat{\mu}_0^U &= (1 - \widehat{p}_1) \left[ \int_{P_{00}}^1 \widehat{Q}_{V|D}(u|0) du \right], \end{aligned}$$

and

$$\begin{aligned}\widehat{\mu}_{0|1}^L &= \frac{1 - \widehat{p}_1}{\widehat{p}_1} \int_0^{1 - P_{00}} \widehat{Q}_{V/W|D}(u|0) du, \\ \widehat{\mu}_{0|1}^U &= \frac{1 - \widehat{p}_1}{\widehat{p}_1} \int_{P_{00}}^1 \widehat{Q}_{V/W|D}(u|0) du.\end{aligned}$$

## 4.2 Numerical Comparisons from a Test Model

For assessing the performance of our bounds, we consider a simple simulation example. We consider the following test model:

$$\begin{aligned}Y_1^* &= \gamma_1 Z + V_1, \quad Y_0^* = \gamma_0 Z + V_0, \\ Y_1 &= \frac{\exp(Y_1^*)}{1 + \exp(Y_1^*)}, \quad Y_0 = \frac{\exp(Y_0^*)}{1 + \exp(Y_0^*)},\end{aligned}\tag{4.3}$$

$$D = I\{Z\delta - \epsilon \geq 0\},\tag{4.4}$$

in which  $(Z, V_1, V_0, \epsilon) \sim N(0, I_4)$ .

For this model, we can compute our bounds for various values of the parameters  $(\delta, \gamma_1, \gamma_0)$ . For the average treatment effects, we compare our bounds to Manski's (1990) "worst case" benchmark, which are for the case when only the aggregate data are available. Specifically, suppose  $Y_1 \in [0, 1]$ . Then Manski's worst case bounds on  $\mu_1$  are:

$$E[Y|D = 1]p_1 \leq \mu_1 \leq E[Y|D = 1]p_1 + [1 - p_1].$$

The bounds are presented graphically in Figures 1 and 2. In these graphs, the solid blue line in the middle corresponds to the actual values of the ATE or ATT, which can be directly computed from the simulated data. One noteworthy feature in the ATE graphs in Figure 1 is how much our bounds shrink relative to the Manski bounds. When  $\delta = 0$ , our bounds point identify ATE and ATT.

[Figure 1 about here.]

[Figure 2 about here.]

Subsequently, we also consider the small-sample properties of our bounds. In Table 1, we show results from a Monte Carlo study. As we would expect, increasing the size of the sample leads to more precise estimates of the bounds (but, of course, cannot tighten the bounds).

[Table 1 about here.]

## 5 An Empirical Example

In this section, we consider an empirical application of our bounds to analyze voteshares in the US presidential elections of 1980 and 2000. We consider a binary outcome variable  $Y_i$  for individual eligible voter  $i$ :

$$Y_i = \begin{cases} 1 & \text{if } i \text{ votes Republican} \\ 0 & \text{if } i \text{ does not vote Republican} \end{cases}$$

(Note that  $Y_i = 0$  encompasses voting Democratic, voting for a third party candidate, or abstaining.)

Likewise, the variable  $D_i$  is defined as:

$$D_i = \begin{cases} 1 & \text{if } i \text{ was eligible to participate in 2000 election} \\ 0 & \text{if } i \text{ was eligible to participate in 1980 election} \end{cases}$$

The modeling of the years as distinct “treatments” is inspired by the analysis in DiNardo, Fortin, and Lemieux (1996), as discussed earlier. Accordingly, we define the two potential outcome variables: for  $D = 0, 1$ ,

$$Y_{D,i} = \begin{cases} 1 & \text{if } i \text{ votes Republican in election year } D \\ 0 & \text{if } i \text{ does not vote Republican in election year } D \end{cases}$$

and the relationship between the observed and potential outcomes is:

$$Y_i = \mathbb{1}_{D_i=1}Y_{1,i} + (1 - \mathbb{1}_{D_i=0})Y_{0,i}.$$

The propensity score was estimated using a probit regression on a random sample of individual-level census data from the 1980 and 2000 censuses. The specification was:<sup>7</sup>

$$P(D_i = 1|Z_i) = \Phi(\beta_0 + \beta_1 \text{LOGINC}_i + \beta_2 \text{COLLEGE}_i + \beta_3 \text{AGE}_i \\ + \beta_4 \text{NONWHITE}_i + \beta_5 \text{UNEMP}_i + \beta_6 \text{EMPL}_i)$$

where  $\Phi(\cdot)$  denotes the standard normal CDF. The definitions and summary statistics for the demographic variables are given in Table 2. This propensity score regression was estimated separately for each state, as well as for the nation as a whole.

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<sup>7</sup>The results were quite stable to different definitions and specifications of the conditioning variables in the propensity scores.

[Table 2 about here.]

To aid in the interpretation of the results, we make a simplifying expository assumption that the main intrinsic difference across election years is the identity of the presidential candidate – Ronald Reagan and Walter Mondale in 1980, and George W. Bush (GWB) and Al Gore in 2000. In this setting, the counterfactual outcomes  $Y_{1,i}$  (resp.  $Y_{0,i}$ ) indicate whether voter  $i$  would have voted for GWB in the 2000 election (resp. Reagan in the 1980 election). Moreover, the counterfactual probability  $P(Y_0 = 1|D = 1)$  provides an answer to the question: if Reagan and Mondale had run in 2000, would an average eligible voter in 2000 have voted for the Republican ticket? Analogously, the average treatment effect  $E(Y_1 - Y_0|D = 1)$  answers the question: fixing the voting population in the year 2000, how would the vote share have changed if Reagan and Mondale had run in 2000, instead of GWB and Gore?

## 5.1 Estimates of the Bounds

Bounds on these treatment effects are presented in Table 3. We calculated the bounds for each state, as well as for the nation as a whole. Looking at the TT column in the table, we see that, for all the states, the bounds on  $ATT = E(Y_1 - Y_0|D = 1)$  contain zero. Moreover, for all the states, and for the nation as a whole, the actual change in Republican voteshare between election years (in the “Diff.” column) lies between the upper and lower bounds of the ATT. This implies that, using the previous terminology: if we were to hold fixed the candidates in the 2000 election, we cannot reject the hypothesis that there would have been no discernible change in the vote outcomes.

To get more insight, we decompose this treatment effect into

$$ATT = P(Y_1 = 1|D = 1) - P(Y_0 = 1|D = 1).$$

The first term on the right-hand side is directly observed in the data, and reported in the column labelled “2000” in Table 3. We have derived bounds for the second term on the right-hand side, which are reported in Table 4 (the lower and upper bounds are, respectively, in the “ $\mu_{01}^L$ ” and “ $\mu_{01}^U$ ” columns). For all states, these bounds contain the actual Republican voteshare in 1980 ( $E(Y_0)$ , as reported in Table 3). These results show that we cannot reject the hypothesis that the candidates did not make a difference, in the sense that the Republican voteshares would have been the same in 2000 even if Reagan and Mondale were running instead of Bush and Gore.

While these bounds are wide, they shed light on one hypothesis, which is that appeal for Reagan, and his anti-government message, was especially strong in 1980 when the economy was in a recession. Our results show that the data do not allow us to reject the hypothesis that Reagan’s message would have resonated just as well with the voters in 2000, a year of economic prosperity (as evidenced in the summary statistics in Table 2).

At the same time, these wide bounds also imply that demographic changes in the US population between 1980 and 2000 are not enough to explain changes in voting behavior during that period. The analysis requires more powerful explanatory variables, such as advertising or campaign contributions, which have risen dramatically between 1980 and 2000. Also, instrumental variables, which can be excluded from the outcome equations, will also tighten the bounds. We will explore these directions in future extensions.

[Table 3 about here.]

[Table 4 about here.]

## 5.2 Confidence Sets

All the parameters of interest including the mean outcomes  $\mu_1, \mu_0$ , the counterfactual outcome  $\mu_{0|1}$ , and the treatment effect parameters  $\mu_\Delta, \mu_{\Delta|1}$  are interval identified and estimators of their bounds are jointly asymptotically normally distributed. Their inference falls within the framework of Andrews and Soares (2010), Fan and Park (2010), and Stoye (2009), and Chernozhukov, Hong, and Tamer (2007).

To compute the confidence sets for ATE and ATT, we followed the procedure in Stoye (2009). As an input into Stoye’s procedure, we computed the standard errors and correlation between the lower and upper bound estimates for each of the five parameters  $E(Y_0)$ ,  $E(Y_1)$ ,  $E(Y_0|D = 1)$ ,  $ATE = E(Y_1) - E(Y_0)$ , and  $ATT = E(Y_1|D = 1) - E(Y_0|D = 1)$ . Because the asymptotic variances of our estimators of the bounds on these parameters are very complicated due to the use of generated variables, we used bootstrap simulation to approximate the variances and correlations.<sup>8</sup>

In Figures 3 and 4, we present graphs of the estimated bounds, along with the confidence sets. We see that, across all the states, the confidence sets and the estimated bounds are

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<sup>8</sup>using 100 bootstrap resamples

very close, suggesting that sampling error is not a big concerns in these bounds estimates. Overall, then, this suggests that sampling error due to estimation is minor relative to the magnitude of the differences between the upper and lower bounds for the parameters.

[Figure 3 about here.]

[Figure 4 about here.]

## 6 Asymptotics for Bounds Estimators

In this section we present the asymptotic theory for the bounds estimators of the mean counterfactual outcomes defined in Section 4 above. Let  $\mathcal{W}$ ,  $\mathcal{V}$ , and  $\mathcal{A}$  denote the support of the random variables  $W$ ,  $V$ , and  $V/W$  respectively. To simplify the derivation of the asymptotic properties of our estimators, we introduce the following distribution functions:

$$\begin{aligned} G_{W|D}(a|d) &= F_{W|D}\left(\frac{a}{p_1}|d\right), \frac{a}{p_1} \in \mathcal{W} \\ G_{V|D}(a|d) &= F_{V|D}\left(\frac{a}{1-p_1}|d\right), \frac{a}{1-p_1} \in \mathcal{V} \\ G_{V/W|D}(a|d) &= F_{V/W|D}\left(\frac{ap_1}{1-p_1}|d\right), \frac{ap_1}{1-p_1} \in \mathcal{A} \end{aligned}$$

and their estimators:

$$\begin{aligned} \hat{G}_{W|D}(a|d) &= \hat{F}_{W|D}\left(\frac{a}{\hat{p}_1}|d\right) = 1 - \hat{F}_{P|D}\left(\frac{\hat{p}_1}{a}|d\right), \\ \hat{G}_{V|D}(a|d) &= \hat{F}_{V|D}\left(\frac{a}{1-\hat{p}_1}|d\right) = \hat{F}_{P|D}\left(1 - \frac{1-\hat{p}_1}{a}|d\right), \\ \hat{G}_{V/W|D}(a|d) &= \hat{F}_{V/W|D}\left(\frac{a\hat{p}_1}{1-\hat{p}_1}|d\right) = \hat{F}_{P|D}\left(\frac{a\hat{p}_1}{1-(1-a)\hat{p}_1}|d\right). \end{aligned}$$

Noting that

$$G_{W|D}^{-1}(u|d) = \inf \left\{ a : F_{W|D}\left(\frac{a}{p_1}|d\right) \geq u \right\} = p_1 Q_{W|D}(u|d)$$

and

$$G_{V|D}^{-1}(u|d) = \inf \left\{ a : F_{V|D}\left(\frac{a}{1-p_1}|d\right) \geq u \right\} = (1-p_1) Q_{V|D}(u|d),$$

we have

$$\begin{aligned}\widehat{\mu}_1^L &= \int_0^{1-P_{01}} \widehat{G}_{W|D}^{-1}(u|1) du, \quad \widehat{\mu}_1^U = \int_{P_{01}}^1 \widehat{G}_{W|D}^{-1}(u|1) du, \\ \widehat{\mu}_0^L &= \int_0^{1-P_{00}} \widehat{G}_{V|D}^{-1}(u|0) du, \quad \widehat{\mu}_0^U = \int_{P_{00}}^1 \widehat{G}_{V|D}^{-1}(u|0) du.\end{aligned}$$

Similarly,

$$\widehat{\mu}_{0|1}^L = \int_0^{1-P_{00}} \widehat{G}_{V/W|D}^{-1}(u|0) du, \quad \widehat{\mu}_{0|1}^U = \int_{P_{00}}^1 \widehat{G}_{V/W|D}^{-1}(u|0) du.$$

The alternative expressions for the bounds estimators reveal that they are smooth functionals of  $\widehat{G}_{W|D}(\cdot|d)$ ,  $\widehat{G}_{V|D}(\cdot|d)$ , and  $\widehat{G}_{V/W|D}(\cdot|d)$  which in turn are ‘empirical distribution functions’ depending on the estimator of the propensity score  $\widehat{P}_i$  and  $\widehat{p}_d$ . To characterize the effect of  $\widehat{P}_i$ , we assume a semiparametric model for the propensity score  $p(z)$ , i.e.,  $p(z) = p(z; \beta_o, \tau_o)$  for some  $\beta_o \in \mathcal{B}$  and  $\tau_o \in \mathcal{T}$ , where  $\mathcal{B}$  is a compact subset of an Euclidean space and  $\mathcal{T}$  is an infinite dimensional space (see Linton, Song, and Whang (2010)). This set-up subsumes the most commonly used parametric (such as probit or logit) and semiparametric models (including partially linear and single index models). Let  $(\widehat{\beta}, \widehat{\tau})$  denote consistent estimators of  $(\beta_o, \tau_o)$  using dataset  $\{Z_i, D_i\}_{i=1}^n$ . Then  $\widehat{p}(z) \equiv p(z; \widehat{\beta}, \widehat{\tau})$ .

To avoid any confusion, we use  $p_{1o}$  to denote the true value of  $p_1$ . Let  $\theta = (p_1, \beta)$  and  $\theta_o = (p_{1o}, \beta_o)$ . Further define  $B_{\mathcal{B} \times \mathcal{T}}(\delta) = \{(\beta, \tau) \in \mathcal{B} \times \mathcal{T} : \|\beta - \beta_o\| + \|\tau - \tau_o\|_\infty < \delta\}$  and  $B_{\Theta \times \mathcal{T}}(\delta) = \{(\theta, \tau) \in \Theta \times \mathcal{T} : \|\theta - \theta_o\| + \|\tau - \tau_o\|_\infty < \delta\}$  for  $\delta > 0$ , where  $\|\cdot\|$  denotes the Euclidean norm and  $\|\cdot\|_\infty$  denotes the sup-norm. Let  $N(\varepsilon, \mathcal{T}, \|\cdot\|_\infty)$  denote the  $\varepsilon$ -covering number of  $\mathcal{T}$  with respect to  $\|\cdot\|_\infty$ . Let  $\mathcal{P}$  be the collection of all the potential distributions of  $(Z_i, D_i)$  that satisfy Assumptions (s), (p) and (b) below.<sup>9</sup>

**Assumption (s).** (i)  $p(Z_i)$  is a continuous random variable. (ii) There exists small positive constants  $\epsilon_1, \epsilon_2$  such that  $p(z) \in [\epsilon_1, 1 - \epsilon_2]$  for all  $z \in \mathcal{Z}$ . (iii) Let  $f_{P|D}(\cdot|d)$  denote the conditional pdf of  $p(Z_i)$  given  $D_i = d$ . We assume  $f_{P|D}(\cdot|d)$  is bounded away from zero on its support.

**Assumption (p).** (i)  $\{(Z_i, D_i)\}_{i=1}^n$  is a random sample.

(ii)  $\log N(\varepsilon, \mathcal{T}, \|\cdot\|_\infty) \leq C\varepsilon^{-d_\tau}$  for some  $d_\tau \in (0, 1]$ .

(iii)  $f_{P|D}(\cdot|d)$  is bounded on its support and continuously differentiable with bounded derivatives on its support; and for some  $\delta > 0$ , there exists a functional  $\Gamma_P(\cdot|d)[\beta - \beta_o, \tau - \tau_o]$

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<sup>9</sup>In Assumptions (s), (p), and (b),  $C, C_1, C_2$  are generic constants which may take different values in different assumptions.

of  $(\beta - \beta_o, \tau - \tau_o)$ ,  $(\beta, \tau) \in B_{\mathcal{B} \times \mathcal{T}}(\delta)$ , such that for all  $p \in [\epsilon_1, 1 - \epsilon_2]$ ,

$$\begin{aligned} & |F_{P|D}(p; \beta, \tau|d) - F_{P|D}(p|d) - \Gamma_P(p|d) [\beta - \beta_o, \tau - \tau_o]| \\ & \leq C_1 \|\beta - \beta_o\|^2 + C_2 \|\tau - \tau_o\|_\infty^2; \end{aligned} \quad (6.1)$$

and for  $(\theta, \tau) \in B_{\Theta \times \mathcal{T}}(\delta)$ , for all  $w \in \mathcal{W}$ ,

$$\begin{aligned} & \left| \Gamma_P\left(\frac{p_1}{w}|1\right) [\beta - \beta_o, \tau - \tau_o] - \Gamma_P\left(\frac{p_{1o}}{w}|1\right) [\beta - \beta_o, \tau - \tau_o] \right| \\ & \leq C_1 \|\theta - \theta_o\|^2 + C_2 |p_1 - p_{1o}| \|\tau - \tau_o\|_\infty; \end{aligned} \quad (6.2)$$

for all  $v \in \mathcal{V}$ ,

$$\begin{aligned} & \left| \Gamma_P\left(1 - \frac{1 - p_1}{v}|0\right) [\beta - \beta_o, \tau - \tau_o] - \Gamma_P\left(1 - \frac{1 - p_{1o}}{v}|0\right) [\beta - \beta_o, \tau - \tau_o] \right| \\ & \leq C_1 \|\theta - \theta_o\|^2 + C_2 |p_1 - p_{1o}| \|\tau - \tau_o\|_\infty, \end{aligned} \quad (6.3)$$

with constants  $C_1$  and  $C_2$  that do not depend on  $P$ .

(iv) There exists  $\delta, C > 0$  and a subvector  $Z_1$  of  $Z$  such that: (a) the conditional density of  $Z$  given  $Z_1$  is bounded uniformly over  $(\beta, \tau) \in B_{\mathcal{B} \times \mathcal{T}}(\delta)$  and over  $P \in \mathcal{P}$ , (b) for each  $(\beta, \tau) \in B_{\mathcal{B} \times \mathcal{T}}(\delta)$  and  $(\beta', \tau') \in B_{\mathcal{B} \times \mathcal{T}}(\delta)$ ,  $p(Z; \beta, \tau) - p(Z; \beta', \tau')$  is measurable with respect to the  $\sigma$ -field of  $Z_1$ , and (c) for each  $(\beta_1, \tau_1) \in B_{\mathcal{B} \times \mathcal{T}}(\delta)$  and for each  $\delta > 0$ ,

$$\sup_{P \in \mathcal{P}} \sup_{z_1} E_P \left[ \sup_{(\beta_2, \tau_2) \in B_{\mathcal{B} \times \mathcal{T}}(\delta)} |p(Z; \beta_1, \tau_1) - p(Z; \beta_2, \tau_2)|^2 |Z_1 = z_1 \right] \leq C \delta^{2s},$$

for some  $s \in (d_\tau, 1]$  with  $d_\tau$  in Assumption (p) (ii), where the supremum over  $z_1$  runs in the support of  $Z_1$ .

**Assumption (b).** (i) For each  $\epsilon > 0$ ,  $\sup_{P \in \mathcal{P}} P\left(\|\hat{\beta} - \beta_o\| + \|\hat{\tau} - \tau_o\|_\infty > \epsilon\right) = o(1)$  and  $\sup_{P \in \mathcal{P}} P(\hat{\tau} \in \mathcal{T}) \rightarrow 1$  as  $n \rightarrow \infty$  such that  $\|\hat{\beta} - \beta_o\| = o_P(n^{-1/4})$  and  $\|\hat{\tau} - \tau_o\|_\infty = o_P(n^{-1/4})$  uniformly in  $P \in \mathcal{P}$ .

(ii) For each  $\epsilon > 0$ , uniformly  $p \in [\epsilon_1, 1 - \epsilon_2]$ ,

$$\sup_{P \in \mathcal{P}} P\left(\left|\sqrt{n} \Gamma_P(p|d) [\hat{\beta} - \beta_o, \hat{\tau} - \tau_o] - \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_i(p; \beta_o, \tau_o, d)\right| > \epsilon\right) \rightarrow 0;$$

where  $\psi_i(p; \beta_o, \tau_o, d)$  depends on  $(Z_i, D_i)$  such that there exists  $\eta > 0$  such that for all  $p \in [\epsilon_1, 1 - \epsilon_2]$ ,  $E_P[\psi_i(p; \beta_o, \tau_o, d)] = 0$ , and

$$\sup_{P \in \mathcal{P}} E_P \left[ \sup_{w \in \mathcal{W}} |\psi_i(p; \beta_o, \tau_o, d)|^{2+\eta} \right] < \infty.$$

(iii) There exist constants  $C > 0$  and  $s_1 \in (d_\tau/2, 1]$  with  $d_\tau$  in Assumption (p) (ii) such that for each  $p_1 \in [\epsilon_1, 1 - \epsilon_2]$ , and for each  $\epsilon > 0$ ,

$$E \left[ \sup_{p \in [\epsilon_1, 1 - \epsilon_2]: |p - p_1| \leq \epsilon} |\psi_i(p; \beta_o, \tau_o, d) - \psi_i(p_1; \beta_o, \tau_o, d)|^2 \right] \leq C \epsilon^{2s_1}.$$

**Theorem 6.1** *Suppose assumptions (s), (p) and (b) hold. Let  $\{\nu_W(\cdot), \nu_V(\cdot)\}$  be a bivariate Gaussian process with zero mean and covariance kernel given by  $C((w_1, v_1), (w_2, v_2))$  defined as:*

$$C((w_1, v_1), (w_2, v_2)) = \begin{pmatrix} E(\nu_W(w_1|1) \nu_W(w_2|1)) & E(\nu_W(w_1|1) \nu_V(v_2|1)) \\ E(\nu_W(w_2|1) \nu_V(v_1|1)) & E(\nu_V(v_1|1) \nu_V(v_2|1)) \end{pmatrix},$$

where

$$\begin{aligned} E(\nu_W(w_1|1) \nu_W(w_2|1)) &= \text{Cov}(V_{i,W}(w_1; \theta_o, \tau_o), V_{i,W}(w_2; \theta_o, \tau_o)), \\ E(\nu_W(w_1|1) \nu_V(v_2|1)) &= \text{Cov}(V_{i,W}(w_1; \theta_o, \tau_o), V_{i,V}(v_2; \theta_o, \tau_o)), \\ E(\nu_V(v_1|1) \nu_W(w_2|1)) &= \text{Cov}(V_{i,V}(v_1; \theta_o, \tau_o), V_{i,W}(w_2; \theta_o, \tau_o)), \\ E(\nu_V(v_1|1) \nu_V(v_2|1)) &= \text{Cov}(V_{i,V}(v_1; \theta_o, \tau_o), V_{i,V}(v_2; \theta_o, \tau_o)). \end{aligned}$$

in which

$$\begin{aligned} &V_{i,W}(w; \theta_o, \tau_o) \\ &= -\frac{1}{p_{1o}} [I\{p(Z_i; \beta_o, \tau_o)/p_{1o} \leq 1/w\} - [1 - G_{W|D}(w|1)]] I\{D_i = 1\} \\ &\quad + \frac{1}{w} f_{P|D}(p_{1o}/w|1) [I\{D_i = 1\} - p_{1o}] + \psi_i(p_{1o}/w; \beta_o, \tau_o, 1) \end{aligned} \quad (6.4)$$

and

$$\begin{aligned} &V_{i,V}(v; \theta_o, \tau_o) \\ &= \frac{1}{(1 - p_{1o})} \left[ I\left\{p(Z_i; \beta_o, \tau_o) \leq 1 - \frac{1 - p_{1o}}{v}\right\} - G_{V|D}(v|0) \right] I\{D_i = 0\} \\ &\quad + \frac{1}{v(1 - p_{1o})} f_{P|D}\left(1 - \frac{1 - p_{1o}}{v} | 0\right) (I\{D_i = 0\} - p_{1o}) \\ &\quad + \frac{1}{(1 - p_{1o})} \psi_i\left(1 - \frac{1 - p_{1o}}{v}; \beta_o, \tau_o, 0\right). \end{aligned} \quad (6.5)$$

Then, uniformly in  $P \in \mathcal{P}$ ,

$$\sqrt{n} \begin{pmatrix} \widehat{\mu}_1^L - \mu_1^L \\ \widehat{\mu}_1^U - \mu_1^U \\ \widehat{\mu}_0^L - \mu_0^L \\ \widehat{\mu}_0^U - \mu_0^U \end{pmatrix} \Longrightarrow N(0, \Sigma_\mu),$$

where  $\Sigma_\mu = E(Z_{10}Z'_{10})$  in which  $Z_{10} = (Z_{1L}, Z_{1U}, Z_{0L}, Z_{0U})'$  with

$$\begin{aligned} Z_{1L} &= \int_0^{1-P_{01}} \frac{\nu_W}{g_{W|D}} \circ G_{W|D}^{-1}(u|1) du, \\ Z_{1U} &= \int_{P_{01}}^1 \frac{\nu_W}{g_{W|D}} \circ G_{W|D}^{-1}(u|1) du, \\ Z_{0L} &= \int_0^{1-P_{00}} \frac{\nu_V}{g_{V|D}} \circ G_{V|D}^{-1}(u|0) du, \\ Z_{0U} &= \int_{P_{00}}^1 \frac{\nu_V}{g_{V|D}} \circ G_{V|D}^{-1}(u|0) du. \end{aligned}$$

Below we provide a sketch of the main steps in the proof of Theorem 6.1 and a discussion of the role of each of the Assumptions (s), (p) and (b) in the proof.

**Step 1.** We show that the random function,

$$(w, v) \mapsto \sqrt{n} \begin{pmatrix} \widehat{G}_{W|D}(w|1) - G_{W|D}(w|1) \\ \widehat{G}_{V|D}(v|0) - G_{V|D}(v|0) \end{pmatrix}$$

converges weakly to the Gaussian process  $\{\nu_W(\cdot), \nu_V(\cdot)\}$  uniformly in  $P \in \mathcal{P}$  and the convergence is in  $D(\mathcal{W}) \times D(\mathcal{V})$ . This is done using empirical process methods similar to the proof of Theorem 1 in Linton, Song, and Whang (2010); see also Andrews (1994), Chen, Linton, and van Keilegom (2003), and Linton, Maasoumi, and Whang (2005). Assumptions (p) and (b) are mainly used in this step. They are sufficient conditions for the Assumptions 1-3 needed for Lemma A.1 (in the Appendix).

Assumption p (ii) restricts the complexity of the class of functions that  $\tau_o$  belongs to. This and Assumption p (iv) ensure that the classes of functions:

$$\{I\{p(\cdot; \beta, \tau) \leq p_1/w\} : w \in \mathcal{W}, (\theta, \tau) \in B_{\Theta \times \mathcal{T}}(\delta)\}$$

and

$$\{I\{p(\cdot; \beta, \tau) \leq 1 - (1 - p_1)/v\} : v \in \mathcal{V}, (\theta, \tau) \in B_{\Theta \times \mathcal{T}}(\delta)\}$$

are uniformly Donsker so that the effect of estimating  $p_{1o}$  and  $(\beta_o, \tau_o)$  can be analyzed via the behavior of  $F_{P|D}(p_1/w; \beta, \tau|d)$  and  $F_{P|D}(1 - (1 - p_1)/v; \beta, \tau|d)$ . Eq. (6.1) in Assumption (p) (iii) implies that  $F_{P|D}(p; \beta, \tau|d)$  can be approximated by linear functionals uniformly in  $p$  and eqs. (6.2) and (6.3) in Assumption (p) (iii) impose further smoothness conditions on the relevant linear functionals. Together with Assumption (b) on uniform linear representations of these functionals evaluated at  $(\widehat{\beta}, \widehat{\tau})$ , they allow us to establish the

stated weak convergence result in this step. Example 6.2 below verifies these assumptions for a parametric model for the propensity score. Semiparametric models such as single-index models may be shown to satisfy these assumptions as well.

In general, suppose  $Z$  contains at least one component with an absolutely continuous distribution and  $p(\cdot; \beta, \tau)$  is monotonically increasing in that component. Without loss of generality, let the first component of  $Z$  be that variable and the inverse of  $p(\cdot; \beta, \tau)$  with respect to the first variable be  $q(\cdot, Z_{-1}; \beta, \tau)$ . Then

$$\begin{aligned} F_{P|D}(p; \beta, \tau|d) &= \Pr(p(Z; \beta, \tau) \leq p|D = d) \\ &= \Pr(Z_1 \leq q(p, Z_{-1}; \beta, \tau) | D = d) \\ &= E[F_{Z_1|D}(q(p, Z_{-1}; \beta, \tau)) | D = d]. \end{aligned}$$

Suppose  $q(\cdot, Z_{-1}; \beta, \tau)$  satisfies: for  $(\beta, \tau) \in B_{\mathcal{B} \times \mathcal{T}}(\delta)$ ,

$$\begin{aligned} &|q(p, Z_{-1}; \beta, \tau) - q(p, Z_{-1}; \beta_o, \tau_o) - \Gamma_q(p, Z_{-1}|d)[\beta - \beta_o, \tau - \tau_o]| \\ &\leq C_1 \|\beta - \beta_o\|^2 + C_2 \|\tau - \tau_o\|_\infty^2. \end{aligned} \quad (6.6)$$

Let

$$\Gamma_P(p|d)[\beta - \beta_o, \tau - \tau_o] = E[f_{Z_1|D}(q(p, Z_{-1}; \beta, \tau)) \Gamma_q(p, Z_{-1}|d)[\beta - \beta_o, \tau - \tau_o]]. \quad (6.7)$$

Then under some conditions, we can verify Assumption (p) (iii) and Assumption (b) (ii) for estimators of  $\beta, \tau$  satisfying Assumption (b) (i). Crucial to Assumption (b) (ii) in the presence of  $\tau$  is the existence of component  $Z_{-1}$  resulting in the smoothing operation in the expression for  $\Gamma_P(p|d)[\beta - \beta_o, \tau - \tau_o]$  in (6.7).

**Step 2.** We show that for all constants  $c_1, c_2, c_3, c_4$ , the map  $\phi_{F_1, F_0} : D(\mathcal{W}) \times D(\mathcal{V}) \rightarrow \mathbb{R}^2$  defined as

$$\begin{aligned} \phi_{F_1, F_0} &= c_1 \int_0^{1-P_{01}} F_1^{-1}(u) du + c_2 \int_{P_{01}}^1 F_1^{-1}(u) du \\ &\quad + c_3 \int_0^{1-P_{00}} F_0^{-1}(u) du + c_4 \int_{P_{00}}^1 F_0^{-1}(u) du \end{aligned}$$

is Hadamard-differentiable at  $(G_{W|D}(\cdot|1), G_{V|D}(\cdot|0))$  tangentially to  $C(\mathcal{W}) \times C(\mathcal{V})$  with the derivative

$$\begin{aligned} \phi'_{F_1, F_0}(\alpha_W, \alpha_V) &\mapsto c_1 \int_0^{1-P_{01}} \frac{\alpha_W}{g_{W|D}} \circ G_{W|D}^{-1}(u|1) du + c_2 \int_{P_{01}}^1 \frac{\alpha_W}{g_{W|D}} \circ G_{W|D}^{-1}(u|1) du \\ &\quad + c_3 \int_0^{1-P_{00}} \frac{\alpha_V}{g_{V|D}} \circ G_{V|D}^{-1}(u|0) du + c_4 \int_{P_{00}}^1 \frac{\alpha_V}{g_{V|D}} \circ G_{V|D}^{-1}(u|0) du, \end{aligned}$$

where  $g_{W|D}$  and  $g_{V|D}$  are pdfs corresponding to  $G_{W|D}$  and  $G_{V|D}$ . Assumption (s) is used in this step. It ensures that the quantile functions  $(F_1^{-1}, F_0^{-1})$  are Hadamard-differentiable at  $(G_{W|D}(\cdot|1), G_{V|D}(\cdot|0))$  tangentially to  $C(\mathcal{W}) \times C(\mathcal{V})$ ; see van der Vaart and Wellner (1996)<sup>10</sup> By the Functional Delta method (see van der Vaart and Wellner (1996)), we obtain that, uniformly in  $P \in \mathcal{P}$ ,

$$\sqrt{n} \left( c_1 \widehat{\mu}_1^L + c_2 \widehat{\mu}_1^U + c_3 \widehat{\mu}_0^L + c_4 \widehat{\mu}_0^U - [c_1 \mu_1^L + c_2 \mu_1^U + c_3 \mu_0^L + c_4 \mu_0^U] \right) \implies \phi'_{F_1, F_0}(\nu_W, \nu_V).$$

Theorem 6.1 can be used to construct confidence sets (CSs) for the mean outcomes  $\mu_1, \mu_0$ . It also allows us to construct CSs for the ATE  $\mu_\Delta$ , as an application of the Delta method leads to the asymptotic distributions of the estimators of the lower and upper bounds on  $\mu_\Delta$ :  $(\widehat{\mu}_1^L - \widehat{\mu}_0^U, \widehat{\mu}_1^U - \widehat{\mu}_0^L)$ .

The second term in the expression for  $V_{i,W}(w; \theta_o, \tau_o)$  ( $V_{i,V}(v; \theta_o, \tau_o)$ ) in (6.4) ((6.5)) accounts for the effect of estimating  $p_{1o}$  by  $\widehat{p}_1$  and the third term accounts for the effect of estimating  $(\beta_o, \tau_o)$  by  $(\widehat{\beta}, \widehat{\tau})$ ; these take the more familiar form when the model for the propensity score is parametric as illustrated below.

**Example: A parametric model for the propensity score.** Let  $p(z) = p(z; \beta_o)$  for some  $\beta_o \in \mathcal{B}$ . Such a parametric model for the propensity score was considered in the empirical application above. If  $F_{P|D}(p; \beta|d)$  is twice differentiable in  $(p, \beta)$  with bounded second order derivatives in  $p$  and a neighborhood of  $\beta_o$ , then Assumption (p)(iii) is satisfied with

$$\Gamma_P(p|d)[\beta - \beta_o] = \frac{\partial F_{P|D}(p; \beta_o|d)}{\partial \beta'} (\beta - \beta_o)$$

and

$$\begin{aligned} & \left| \Gamma_P \left( \frac{p_1}{w} | 1 \right) [\beta - \beta_o] - \Gamma_P \left( \frac{p_{1o}}{w} | 1 \right) [\beta - \beta_o] \right| \\ &= \left| \left[ \frac{\partial F_{P|D} \left( \frac{p_1}{w}; \beta_o | 1 \right)}{\partial \beta'} - \frac{\partial F_{P|D} \left( \frac{p_{1o}}{w}; \beta_o | 1 \right)}{\partial \beta'} \right] (\beta - \beta_o) \right| \\ &\leq C_1 \|\theta - \theta_o\|^2, \end{aligned}$$

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<sup>10</sup>We note that Assumption (s) may be relaxed at the expense of a more tedious proof analogous to the proof of Claim 1 in Bhattacharya (2007), see also Goldie (1977).

$$\begin{aligned}
& \left| \Gamma_P \left( 1 - \frac{1-p_1}{v} | 0 \right) [\beta - \beta_o] - \Gamma_P \left( 1 - \frac{1-p_{1o}}{v} | 0 \right) [\beta - \beta_o] \right| \\
= & \left| \left[ \frac{\partial F_{P|D} \left( 1 - \frac{1-p_1}{v}; \beta_o | 0 \right)}{\partial \beta'} - \frac{\partial F_{P|D} \left( 1 - \frac{1-p_{1o}}{v}; \beta_o | 0 \right)}{\partial \beta'} \right] (\beta - \beta_o) \right| \\
\leq & C_1 \|\theta - \theta_o\|^2.
\end{aligned}$$

Assumption (p)(iv) is satisfied if  $p(z; \beta)$  is twice differentiable with bounded second order derivatives in a neighborhood of  $\beta_o$ . Let  $\widehat{\beta}$  satisfy:

$$\sqrt{n} \left( \widehat{\beta} - \beta_o \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varphi_i(\beta_o) + o_P(1)$$

where  $\varphi_i(\beta_o)$  satisfies:  $E(\varphi_i(\beta_o)) = 0$  and  $E(\varphi_i^{2+\eta}(\beta_o)) < \infty$  for some  $\eta > 0$ . Then Assumption (b) is satisfied with

$$\psi_i(p; \beta_o, d) = \frac{\partial F_{P|D}(p; \beta_o | d)}{\partial \beta'} \varphi_i(\beta_o).$$

To establish the asymptotic distribution of  $(\widehat{\mu}_{0|1}^L, \widehat{\mu}_{0|1}^U)$ , we replace Assumption (p)(iii) with Assumption (p)(iii)' below.

**Assumption (p)(iii)'**:  $f_{P|D}(\cdot | d)$  is bounded on its support and continuously differentiable with bounded derivatives on its support; and for some  $\delta > 0$ , there exists a functional  $\Gamma_P(\cdot | d) [\beta - \beta_o, \tau - \tau_o]$  of  $(\beta - \beta_o, \tau - \tau_o)$ ,  $(\beta, \tau) \in B_{\mathcal{B} \times \mathcal{T}}(\delta)$ , such that for all  $p \in [\epsilon_1, 1 - \epsilon_2]$ , (6.1) holds, and for  $(\theta, \tau) \in B_{\Theta \times \mathcal{T}}(\delta)$ , all  $a \in \mathcal{A}$ ,

$$\begin{aligned}
& \left| \Gamma_P \left( \frac{ap_1}{1 - (1-a)p_1} | 0 \right) [\beta - \beta_o, \tau - \tau_o] - \Gamma_P \left( \frac{ap_{1o}}{1 - (1-a)p_{1o}} | 0 \right) [\beta - \beta_o, \tau - \tau_o] \right| \\
\leq & C_1 \|\theta - \theta_o\|^2 + C_2 |p_1 - p_{1o}| \|\tau - \tau_o\|_\infty,
\end{aligned} \tag{6.8}$$

with constants  $C_1$  and  $C_2$  that do not depend on  $P$ .

The following result (which is a special case of Theorem 6.1 above) obtains:

**Theorem 6.2** *Let  $\nu_{V|W}(\cdot)$  be a Gaussian process with zero mean and covariance kernel:*

$$E(V_{i,V|W}(a_1; \theta_o, \tau_o) V_{i,V|W}(a_2; \theta_o, \tau_o)),$$

where

$$\begin{aligned}
& V_{i,V/W}(a; \theta_o, \tau_o) \\
= & \frac{1}{(1-p_{1o})} \left[ I \left\{ p(Z_i; \beta_o) \leq \frac{ap_{1o}}{1-(1-a)p_{1o}} \right\} - G_{V/W|D}(v|0) \right] I\{D_i = 0\} \\
& + \frac{1}{(1-p_{1o})} \psi_i \left( \frac{p_{1o}a}{1-(1-a)p_{1o}}; \beta_o, \tau_o, d \right) \\
& + \frac{a}{(1-p_{1o})[1-(1-a)p_{1o}]^2} f_{P|D} \left( \frac{p_{1o}a}{1-(1-a)p_{1o}}; \beta_o, \tau_o | 0 \right) (I\{D_i = 1\} - p_{1o}).
\end{aligned}$$

Suppose assumptions (s), (p) with (p) (iii) replaced with (p) (iii)' and (b) hold. Then uniformly in  $P \in \mathcal{P}$ ,

$$\sqrt{n} \begin{pmatrix} \widehat{\mu}_{0|1}^L - \mu_{0|1}^L \\ \widehat{\mu}_{0|1}^U - \mu_{0|1}^U \end{pmatrix}' \Rightarrow N(0, \Sigma_{\mu|1}),$$

where  $\Sigma_{\mu|1} = E(Z'_{10|1} Z_{10|1})$  in which  $Z_{10|1} = (Z_{0|1}^L, Z_{0|1}^U)'$  with

$$\begin{aligned}
Z_{0|1}^L &= \int_0^{1-P_{00}} \frac{\nu_{V/W}}{g_{V/W|D}} \circ G_{V/W|D}^{-1}(u|0) du, \\
Z_{0|1}^U &= \int_{P_{00}}^1 \frac{\nu_{V/W}}{g_{V/W|D}} \circ G_{V/W|D}^{-1}(u|0) du.
\end{aligned}$$

Theorem 6.3 allows us to construct CSs for the counterfactual mean outcome  $\mu_{0|1}$  and the treatment effect for the treated  $\mu_{\Delta|1}$ , as  $\mu_{\Delta|1} = \mu_{1|1} - \mu_{0|1}$  and  $\widehat{\mu}_{1|1} = 1 - P_{01}$  is treated as a constant.

## 7 Conclusion

Combining tools and insights from the treatment effect and copula literatures, this paper has presented a novel approach to counterfactual analysis in EI models, in which aggregate and individual-level data must be combined in order to infer individual-level behavior. Under a “selection on observables” assumption familiar from the treatment effects literature, we establish partial identification results for the mean and other functionals of the counterfactual outcome distribution.

We provide estimators and derive inference tools by using empirical process methods combined with recent developments on inference for partially identified parameters. Inference

tools for distributional treatment effect parameters such as  $F_{\Delta}(\delta|D = 1)$  may be established by generalizing the technical tools in Fan and Song (2011). This is challenging and left for future research.

## A Appendix: Technical Proofs

The proofs of Theorems 6.1 and 6.3 are similar; they rely heavily on the lemma below which is adapted from the proof of Theorem 1 in Linton, Song, and Whang (2010). Closely related work include Andrews (1994), Chen, Linton, and van Keilegom (2003), and Linton, Maasoumi, and Whang (2005).

Let

$$X_i(\theta, \tau) = \varphi(Z_i; \theta, \tau),$$

where  $\varphi(\cdot; \theta, \tau)$  is a real valued function known up to the parameter  $(\theta, \tau) \in \Theta \times \mathcal{T}$  with  $\Theta$  a compact subset of a Euclidean space and  $\mathcal{T}$  an infinite dimensional space. For  $d = 1, 0$ , let  $\nu_n(\cdot; d)$  be the stochastic process on  $\mathcal{X}$  with

$$\nu_n(x; d) = \sqrt{n} \left[ n^{-1} \sum_{i=1}^n I \left\{ X_i(\hat{\theta}, \hat{\tau}) \leq x \right\} I \{D_i = d\} - E \left( I \{X_i(\theta_o, \tau_o) \leq x\} I \{D_i = d\} \right) \right],$$

where  $x \in \mathcal{X}$ ,  $(\theta_o, \tau_o) \in \Theta \times \mathcal{T}$ , and  $(\hat{\theta}, \hat{\tau})$  are consistent estimators of  $(\theta_o, \tau_o)$ .

Lemma A.1 below presents conditions under which the process  $\{\nu_n(\cdot; d)\}$  converges weakly to a Gaussian process.

Let  $B_{\Theta \times \mathcal{T}}(\delta) = \{(\theta, \tau) \in \Theta \times \mathcal{T} : \|\theta - \theta_o\| + \|\tau - \tau_o\|_\infty < \delta\}$  for  $\delta > 0$  and  $\mathcal{P}$  be the collection of all the potential distributions of  $(Z_i, D_i)$  that satisfy Assumptions 1-3 below.

**Assumption 1.** (i)  $\{Z_i, D_i\}_{i=1}^n$  is a random sample.

(ii)  $\log N(\varepsilon, \mathcal{T}, \|\cdot\|_\infty) \leq C\varepsilon^{-d}$  for some  $d \in (0, 1]$ .

(iii) Let

$$F_{X|D}(x; \theta, \tau | d) = \Pr(X_i(\theta, \tau) \leq x | D_i = d).$$

For some  $\delta > 0$ , there exists a functional  $\Gamma_{F,P}(x|d)[\theta - \theta_o, \tau - \tau_o]$  of  $(\theta - \theta_o, \tau - \tau_o)$ ,  $(\theta, \tau) \in B_{\Theta \times \mathcal{T}}(\delta)$  such that

$$\begin{aligned} & |F_{X|D}(x; \theta, \tau | d) - F_{X|D}(x; \theta_o, \tau_o | d) - \Gamma_{F,P}(x|d)[\theta - \theta_o, \tau - \tau_o]| \\ & \leq C_1 \|\theta - \theta_o\|^2 + C_2 \|\tau - \tau_o\|_\infty^2, \end{aligned}$$

with constants  $C_1$  and  $C_2$  that do not depend on  $P$ .

**Assumption 2.** (i)  $X_i(\theta_o, \tau_o)$  is a continuous random variable with a bounded support  $\mathcal{X}$ .

(ii) There exists  $\delta, C > 0$  and a subvector  $Z_1$  of  $Z$  such that: (a) the conditional density of  $Z$  given  $Z_1$  is bounded uniformly over  $(\theta, \tau) \in B_{\Theta \times \mathcal{T}}(\delta)$  and over  $P \in \mathcal{P}$ , (b) for each  $(\theta, \tau) \in B_{\Theta \times \mathcal{T}}(\delta)$  and  $(\theta', \tau') \in B_{\Theta \times \mathcal{T}}(\delta)$ ,  $\varphi(Z; \theta, \tau) - \varphi(Z; \theta', \tau')$  is measurable with respect to the  $\sigma$ -field of  $Z_1$ , and (c) for each  $(\theta_1, \tau_1) \in B_{\Theta \times \mathcal{T}}(\delta)$  and for each  $\delta > 0$ ,

$$\sup_{P \in \mathcal{P}} \sup_{z_1} E_P \left[ \sup_{(\theta_2, \tau_2) \in B_{\Theta \times \mathcal{T}}(\delta)} |\varphi(Z; \theta_1, \tau_1) - \varphi(Z; \theta_2, \tau_2)|^2 | Z_1 = z_1 \right] \leq C\delta^{2s},$$

for some  $s \in (d, 1]$  with  $d$  in Assumption 1 (ii), where the supremum over  $z_1$  runs in the support of  $Z_1$ .

**Assumption 3.** (i) For each  $\varepsilon > 0$ ,  $\sup_{P \in \mathcal{P}} P \left( \|\hat{\theta} - \theta_o\| + \|\hat{\tau} - \tau_o\|_\infty > \varepsilon \right) = o(1)$  and  $\sup_{P \in \mathcal{P}} P(\hat{\tau} \in \mathcal{T}) \rightarrow 1$  as  $n \rightarrow \infty$  such that  $\|\hat{\theta} - \theta_o\| = o_P(n^{-1/4})$  and  $\|\hat{\tau} - \tau_o\|_\infty = o_P(n^{-1/4})$ .

(ii) For each  $\varepsilon > 0$ ,

$$\sup_{P \in \mathcal{P}} P \left( \left| \sqrt{n} \Gamma_{F,P}(x|d) [\hat{\theta} - \theta_o, \hat{\tau} - \tau_o] - \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, d) \right| > \varepsilon \right) \rightarrow 0,$$

where  $\psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, d)$  satisfies that there exists  $\eta > 0$  such that for all  $x \in \mathcal{X}$ ,  $E_P[\psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, d)] = 0$  and

$$\sup_{P \in \mathcal{P}} E_P \left[ \sup_{x \in \mathcal{X}} |\psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, d)|^{2+\eta} \right] < \infty.$$

(iii) There exist constants  $C > 0$  and  $s_1 \in (d/2, 1]$  with  $d$  in Assumption 1 (ii) such that for each  $x_1 \in \mathcal{X}$  and for each  $\varepsilon > 0$ ,

$$E \left[ \sup_{x \in \mathcal{X}: |x - x_1| \leq \varepsilon} |\psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, d) - \psi_{x_1,F}(Z_i, D_i; \theta_o, \tau_o, d)|^2 \right] \leq C\varepsilon^{2s_1}.$$

Let  $\nu(\cdot; d)$  be a mean zero Gaussian process on  $\mathcal{X}$  with a covariance kernel given by

$$C(x_1, x_2; d) = \text{Cov}(V_i(x_1; \theta_o, \tau_o, d), V_i(x_2; \theta_o, \tau_o, d)),$$

where

$$V_i(x; \theta_o, \tau_o, d) = I\{X_i(\theta_o, \tau_o) \leq x\} I\{D_i = 1\} + \psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, d).$$

**Lemma A.1** *Suppose that Assumptions 1-3 hold. Then*

$$\nu_n(x; d) = \frac{1}{\sqrt{n}} \sum_{i=1}^n [V_i(x; \theta_o, \tau_o, d) - E(V_i(x; \theta_o, \tau_o, d))] + o_P(1)$$

*uniformly in  $x \in \mathcal{X}$  and  $P \in \mathcal{P}$  and hence  $\nu_n(\cdot; d)$  weakly converges to  $\nu(\cdot; d)$  uniformly in  $P \in \mathcal{P}$ .*

**Proof of Theorem 6.1.** We will show that for any constants  $c_1, c_2, c_3, c_4$ , the linear combination  $c_1 \widehat{\mu}_1^L + c_2 \widehat{\mu}_1^U + c_3 \widehat{\mu}_0^L + c_4 \widehat{\mu}_0^U$  is asymptotically normally distributed with variance  $(c_1, c_2, c_3, c_4) \Sigma_\mu (c_1, c_2, c_3, c_4)'$ .

Assumption (s) ensures that  $G_{W|D}(\cdot|1), G_{V|D}(\cdot|0)$  have compact supports and the corresponding pdfs are bounded away from zero on their supports. As a result, the map  $\phi_{F_1, F_0} : D(\mathcal{W}) \times D(\mathcal{V}) \rightarrow R$  defined as

$$\begin{aligned} \phi_{F_1, F_0} &= c_1 \int_0^{1-P_{01}} F_1^{-1}(u) du + c_2 \int_{P_{01}}^1 F_1^{-1}(u) du \\ &\quad + c_3 \int_0^{1-P_{00}} F_0^{-1}(u) du + c_4 \int_{P_{00}}^1 F_0^{-1}(u) du \end{aligned}$$

is Hadamard-differentiable at  $(G_{W|D}(\cdot|1), G_{V|D}(\cdot|0))$  tangentially to  $C(\mathcal{W}) \times C(\mathcal{V})$  with the derivative:

$$\begin{aligned} \phi'_{F_1, F_0}(\alpha_W, \alpha_V) &\mapsto c_1 \int_0^{1-P_{01}} \frac{\alpha_W}{g_{W|D}} \circ G_{W|D}^{-1}(u|1) du + c_2 \int_{P_{01}}^1 \frac{\alpha_W}{g_{W|D}} \circ G_{W|D}^{-1}(u|1) du \\ &\quad + c_3 \int_0^{1-P_{00}} \frac{\alpha_V}{g_{V|D}} \circ G_{V|D}^{-1}(u|0) du + c_4 \int_{P_{00}}^1 \frac{\alpha_V}{g_{V|D}} \circ G_{V|D}^{-1}(u|0) du, \end{aligned}$$

see van der Vaart and Wellner (1996).

We will complete the proof by establishing the weak convergence of the stochastic process:

$$\left\{ \sqrt{n} \left( \widehat{G}_{W|D}(w|1) - G_{W|D}(w|1), \widehat{G}_{V|D}(v|0) - G_{V|D}(v|0) \right)' : (w, v) \in \mathcal{W} \times \mathcal{V} \right\}$$

and invoking the Functional Delta method.

Let

$$\begin{aligned} \nu_{nG, W}(w) &= \sqrt{n} \left[ \widehat{G}_{W|D}(w|1) - G_{W|D}(w|1) \right], w \in \mathcal{W}, \\ \nu_{nG, V}(v) &= \sqrt{n} \left[ \widehat{G}_{V|D}(v|0) - G_{V|D}(v|0) \right], v \in \mathcal{V}. \end{aligned}$$

**Step 1.** We show:  $\nu_{nG,W}(w) = n^{-1/2} \sum_{j=1}^n V_{j,W}(w; \theta_o, \tau_o) + o_P(1)$  uniformly in  $w \in \mathcal{W}$  and  $P \in \mathcal{P}$ .

By the definition of  $\widehat{G}_{W|D}$ , we have:

$$\begin{aligned}
\nu_{nG,W}(w) &= \sqrt{n} \left[ 1 - \frac{n^{-1} \sum_{j=1}^n I \left\{ p \left( Z_j; \widehat{\beta}, \widehat{\tau} \right) \leq \widehat{p}_1/w \right\} I \{ D_j = 1 \}}{\widehat{p}_1} - G_{W|D}(w|1) \right] \\
&= - \frac{n^{-1/2} \sum_{j=1}^n I \left\{ p \left( Z_j; \widehat{\beta}, \widehat{\tau} \right) / \widehat{p}_1 \leq 1/w \right\} I \{ D_j = 1 \} - p_1 [1 - G_{W|D}(w|1)]}{\widehat{p}_1} \\
&\quad + \frac{\sqrt{n} (\widehat{p}_1 - p_1)}{\widehat{p}_1} [1 - G_{W|D}(w|1)] \\
&= - \frac{n^{-1/2} \sum_{j=1}^n I \left\{ p \left( Z_j; \widehat{\beta}, \widehat{\tau} \right) / \widehat{p}_1 \leq 1/w \right\} I \{ D_j = 1 \} - p_1 [1 - G_{W|D}(w|1)]}{p_1} \\
&\quad + \frac{\sqrt{n} (\widehat{p}_1 - p_1)}{p_1} [1 - G_{W|D}(w|1)] + o_P(1).
\end{aligned}$$

We apply Lemma A.1 to the first term on the right hand side of the last equation with  $X_j(\theta, \tau) = p(Z_j; \beta, \tau) / p_1$  and  $\theta = (p_1, \beta)$ . We verify Assumptions 1-3 in Lemma A.1 under Assumptions (s), (p) and (b).

Assumption 1 (i) and (ii) hold under Assumption (p) (i) and (ii). Now we verify Assumption 1 (iii). Note that for  $x = 1/w$ ,

$$\begin{aligned}
F_{X|D}(x; \theta, \tau|1) &= E_P [I \{ X_i(\theta, \tau) \leq x \} | D_i = 1] \\
&= E_P [I \{ p(Z_i; \beta, \tau) \leq p_1 x \} | D_i = 1] \\
&= F_{P|D}(p_1 x; \beta, \tau|1).
\end{aligned}$$

Let

$$\Gamma_{F,P}(x|1) [\theta - \theta_o, \tau - \tau_o] = x f_{P|D}(p_1 o x|1) (p_1 - p_1 o) + \Gamma_P(p_1 o x|1) [\beta - \beta_o, \tau - \tau_o],$$

where  $\Gamma_P(p_1 o x|1) [\beta - \beta_o, \tau - \tau_o]$  is defined in Assumption (p) (iii). Then by Assumption

(p) (iii), we conclude: for some  $\delta > 0$ ,  $(\theta, \tau) \in B_{\Theta \times \mathcal{T}}(\delta)$ ,

$$\begin{aligned}
& |F_{X|D}(x; \theta, \tau|1) - F_{X|D}(x; \theta_o, \tau_o|1) - \Gamma_{F,P}(x|1)[\theta - \theta_o, \tau - \tau_o]| \\
= & \left| \begin{array}{c} F_{P|D}(p_1x; \beta, \tau|1) - F_{P|D}(p_{1o}x|1) \\ -xf_{P|D}(p_{1o}x|1)(p_1 - p_{1o}) - \Gamma_P(p_{1o}x|1)[\beta - \beta_o, \tau - \tau_o] \end{array} \right| \\
= & \left| \begin{array}{c} F_{P|D}(p_1x; \beta, \tau|1) - F_{P|D}(p_1x|1) - \Gamma_P(p_1x|1)[\beta - \beta_o, \tau - \tau_o] \\ +\Gamma_P(p_1x|1)[\beta - \beta_o, \tau - \tau_o] - \Gamma_P(p_{1o}x|1)[\beta - \beta_o, \tau - \tau_o] \\ + [F_{P|D}(p_1x|1) - F_{P|D}(p_{1o}x|1) - xf_{P|D}(p_{1o}x|1)(p_1 - p_{1o})] \end{array} \right| \\
\leq & |F_{P|D}(p_1x; \beta, \tau|1) - F_{P|D}(p_1x|1) - \Gamma_P(p_1x|1)[\beta - \beta_o, \tau - \tau_o]| \\
& + |\Gamma_P(p_1x|1)[\beta - \beta_o, \tau - \tau_o] - \Gamma_P(p_{1o}x|1)[\beta - \beta_o, \tau - \tau_o]| \\
& + \frac{1}{2} \left| x^2 \frac{\partial^2}{\partial x^2} F_{P|D}(p_1^*x|1)(p_1 - p_{1o})^2 \right| \\
\leq & C_1 \|\theta - \theta_o\|^2 + C_2 \|\tau - \tau_o\|_\infty^2,
\end{aligned}$$

where  $p_1^*$  lies between  $p_{1o}$  and  $p_1$ .

Assumption 2 holds under Assumption (s) (i) and Assumption (p) (iv).

It remains to verify Assumption 3. Assumption 3(i) holds because of Assumption (b)(i). For Assumption 3(ii), we let

$$\psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, 1) = xf_{P|D}(p_{1o}x|1)(I\{D_i = 1\} - p_{1o}) + \psi_i(p_{1o}x; \beta_o, \tau_o, 1),$$

where  $\psi_i(p_{1o}x; \beta_o, \tau_o, 1)$  is defined in Assumption (b) (ii). Then by Assumption (b) (ii), we obtain:

$$\begin{aligned}
& \sup_{P \in \mathcal{P}} P \left( \left| \sqrt{n} \Gamma_{F,P}(x|1) [\hat{\theta} - \theta_o, \hat{\tau} - \tau_o] - \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, 1) \right| > \varepsilon \right) \\
= & \sup_{P \in \mathcal{P}} P \left( \left| \sqrt{n} \Gamma_P(p_{1o}x|1) [\hat{\beta} - \beta_o, \hat{\tau} - \tau_o] - \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_i(p_{1o}x; \beta_o, \tau_o, 1) \right| > \varepsilon \right) \rightarrow 0,
\end{aligned}$$

where  $E_P[\psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, 1)] = 0$  and

$$\begin{aligned}
& \sup_{P \in \mathcal{P}} E_P \left[ \sup_{x \in \mathcal{X}} |\psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, 1)|^{2+\eta} \right] \\
= & \sup_{P \in \mathcal{P}} E_P \left[ \sup_{x \in \mathcal{X}} |xf_{P|D}(p_{1o}x|1)(I\{D_i = 1\} - p_{1o}) + \psi_i(p_{1o}x; \beta_o, \tau_o, 1)|^{2+\eta} \right] \\
< & \infty
\end{aligned}$$

by Assumption (p)(iii) and Assumption (b)(ii). It remains to verify Assumption 3 (iii):

$$\begin{aligned}
& E \left[ \sup_{x \in \mathcal{X}: |x-x_1| \leq \varepsilon} |\psi_{x,F}(Z_i, D_i; \theta_o, \tau_o, 1) - \psi_{x_1,F}(Z_i, D_i; \theta_o, \tau_o, 1)|^2 \right] \\
&= E \left[ \sup_{x \in \mathcal{X}: |x-x_1| \leq \varepsilon} \left| \begin{array}{l} x f_{P|D}(p_{1o}x|1) (I\{D_i = 1\} - p_{1o}) + \psi_i(p_{1o}x; \beta_o, \tau_o, 1) \\ -x_1 f_{P|D}(p_{1o}x_1|1) (I\{D_i = 1\} - p_{1o}) - \psi_i(p_{1o}x_1; \beta_o, \tau_o, 1) \end{array} \right|^2 \right] \\
&\leq CE \left[ \sup_{x \in \mathcal{X}: |x-x_1| \leq \varepsilon} |x - x_1|^2 \right] \\
&\quad + CE \left[ \sup_{x \in \mathcal{X}: |x-x_1| \leq \varepsilon} |\psi_i(p_{1o}x; \beta_o, \tau_o, d) - \psi_i(p_{1o}x_1; \beta_o, \tau_o, d)|^2 \right] \\
&\leq C\varepsilon^2 + C\varepsilon^{2s_1}
\end{aligned}$$

by Assumption (b) (iii) and Assumption (p)(iii).

Using Lemma A.1, we now obtain:

$$\begin{aligned}
& \nu_{nG,W}(w) \\
&= -\frac{1}{p_{1o}} n^{-1/2} \sum_{j=1}^n [I\{p(Z_j; \beta_o)/p_{1o} \leq 1/w\} I\{D_j = 1\} - p_{1o} [1 - G_{W|D}(w|1)]] \\
&\quad - \frac{1}{w} f_{P|D}(p_{1o}/w; \beta_o, \tau_o|d) (\hat{p}_1 - p_{1o}) - \Gamma_P(p_{1o}/w|d) [\hat{\beta} - \beta_o, \hat{\tau} - \tau_o] \\
&\quad + \frac{1}{p_{1o}} n^{-1/2} \sum_{j=1}^n [I\{D_j = 1\} - p_{1o}] [1 - G_{W|D}(w|1)] + o_P(1) \\
&= -n^{-1/2} \sum_{j=1}^n \left\{ \begin{array}{l} \frac{1}{p_{1o}} [I\{p(Z_j; \beta_o, \tau_o)/p_{1o} \leq 1/w\} - [1 - G_{W|D}(w|1)]] I\{D_j = 1\} \\ \quad + \frac{1}{w} f_{P|D}(p_{1o}/w; \beta_o, \tau_o|d) [I\{D_j = 1\} - p_{1o}] \\ \quad + \psi_j(p_{1o}/w; \beta_o, \tau_o, 1) \end{array} \right\} \\
&\quad + o_P(1) \\
&= n^{-1/2} \sum_{j=1}^n V_{j,W}(w; \theta_o, \tau_o) + o_P(1).
\end{aligned}$$

**Step 2.** We show:  $\nu_{nG,V}(v) = n^{-1/2} \sum_{j=1}^n V_{j,V}(v; \theta_o, \tau_o) + o_P(1)$ .

Assumptions 1-3 in Lemma A.1 can be verified by following Step 1. So we just provide the main expressions. Note that  $\hat{G}_{V|D}(v|d) = \hat{F}_{P|D}\left(1 - \frac{1-\hat{p}_1}{v}|d\right)$  and

$$E_P \left[ I \left\{ p(Z_i; \beta, \tau) \leq 1 - \frac{1-p_1}{v} \right\} | D_i = 0 \right] = F_{P|D} \left( 1 - \frac{1-p_1}{v}; \beta, \tau | 0 \right).$$

We let

$$\begin{aligned} & \Gamma_{F,P}(x|d) [\theta - \theta_o, \tau - \tau_o] \\ &= \frac{1}{v} f_{P|D} \left( 1 - \frac{1 - p_{1o}}{v} | 0 \right) (p_1 - p_{1o}) + \Gamma_P \left( 1 - \frac{1 - p_{1o}}{v} | 0 \right) [\beta - \beta_o, \tau - \tau_o]. \end{aligned}$$

Thus,

$$\begin{aligned} & \nu_{nG,V}(v) \\ &= \sqrt{n} \left[ \widehat{G}_{V|D}(v|0) - G_{V|D}(v|0) \right] \\ &= \sqrt{n} \left[ \widehat{F}_{P|D} \left( 1 - \frac{1 - \widehat{p}_1}{v} | 0 \right) - G_{V|D}(v|0) \right] \\ &= n^{-1/2} \sum_{j=1}^n \left[ \frac{1}{\widehat{p}_0} I \left\{ p(Z_j; \beta_o, \tau_o) \leq 1 - \frac{1 - \widehat{p}_1}{v} \right\} I \{D_j = 0\} - G_{V|D}(v|0) \right] \\ &= n^{-1/2} \sum_{j=1}^n \left[ I \left\{ p(Z_j; \beta_o, \tau_o) \leq 1 - \frac{1 - \widehat{p}_1}{v} \right\} I \{D_j = 0\} - p_0 G_{V|D}(v|0) \right] \frac{1}{\widehat{p}_0} \\ & \quad + \frac{\sqrt{n}(p_0 - \widehat{p}_0)}{\widehat{p}_0} G_{V|D}(v|0) \\ &= n^{-1/2} \sum_{j=1}^n \left[ I \left\{ p(Z_j; \beta_o, \tau_o) \leq 1 - \frac{1 - \widehat{p}_1}{v} \right\} I \{D_j = 0\} - p_0 G_{V|D}(v|0) \right] \frac{1}{p_0} \\ & \quad + \frac{\sqrt{n}(p_0 - \widehat{p}_0)}{p_0} G_{V|D}(v|0) + o_P(1) \\ &= n^{-1/2} \sum_{j=1}^n \frac{1}{(1 - p_{1o})} \left[ I \left\{ p(Z_j; \beta_o, \tau_o) \leq 1 - \frac{1 - p_{1o}}{v} \right\} - G_{V|D}(v|0) \right] I \{D_j = 0\} \\ & \quad + n^{-1/2} \sum_{j=1}^n \frac{1}{(1 - p_{1o})} \psi_j \left( 1 - \frac{1 - p_{1o}}{v}; \beta_o, \tau_o, d \right) \\ & \quad + n^{-1/2} \sum_{j=1}^n \frac{1}{v(1 - p_{1o})} f_{P|D} \left( 1 - \frac{1 - p_{1o}}{v} | 0 \right) (I \{D_j = 1\} - p_{1o}) + o_P(1) \\ &= n^{-1/2} \sum_{j=1}^n V_{j,V}(v; \theta_o, \tau_o) + o_P(1). \end{aligned}$$

**Step 3.** Steps 1 and 2 imply: uniformly in  $P \in \mathcal{P}$ ,

$$\begin{aligned} & \left\{ \sqrt{n} \left( \widehat{G}_{W|D}(w|1) - G_{W|D}(w|1), \widehat{G}_{V|D}(v|0) - G_{V|D}(v|0) \right)' : (w, v) \in \mathcal{W} \times \mathcal{V} \right\} \\ \implies & \{ \nu_W(w|1), \nu_V(v|1) : (w, v) \in \mathcal{W} \times \mathcal{V} \}, \end{aligned}$$

where  $\{\nu_W(w|1), \nu_V(v|1) : (w, v) \in \mathcal{W} \times \mathcal{V}\}$  is a vector-valued Gaussian process on  $\mathcal{W} \times \mathcal{V}$  with zero mean and a covariance kernel given by  $C((w_1, v_1), (w_2, v_2))$ . Finally, we obtain: uniformly in  $P \in \mathcal{P}$ ,

$$\begin{aligned} & \sqrt{n} \left( c_1 \widehat{\mu}_1^L + c_2 \widehat{\mu}_1^U + c_3 \widehat{\mu}_0^L + c_4 \widehat{\mu}_0^U - [c_1 \mu_1^L + c_2 \mu_1^U + c_3 \mu_0^L + c_4 \mu_0^U] \right) \\ \Rightarrow & c_1 \int_0^{1-P_{01}} \frac{\nu_W}{g_{W|D}} \circ G_{W|D}^{-1}(u|1) du + c_2 \int_{P_{01}}^1 \frac{\nu_W}{g_{W|D}} \circ G_{W|D}^{-1}(u|1) du \\ & + c_3 \int_0^{1-P_{00}} \frac{\nu_V}{g_{V|D}} \circ G_{V|D}^{-1}(u|0) du + c_4 \int_{P_{00}}^1 \frac{\nu_V}{g_{V|D}} \circ G_{V|D}^{-1}(u|0) du. \end{aligned}$$

**Q.E.D.**

**Proof of Theorem 6.3.** We need to show that uniformly in  $P \in \mathcal{P}$ ,

$$\left[ c_1 \int_0^{1-P_{00}} \widehat{G}_{V/W|D}^{-1}(u|0) du + c_2 \int_{P_{00}}^1 \widehat{G}_{V/W|D}^{-1}(u|0) du \right]$$

is asymptotically normal for all constants  $c_1, c_2$ . It is sufficient to show that the process

$$\left\{ \sqrt{n} \left( \widehat{G}_{V/W|D}(a|0) - G_{V/W|D}(a|0) \right) : a \in \mathcal{A} \right\}$$

converges weakly to a Gaussian process uniformly in  $P \in \mathcal{P}$ .

**Step 1.** We show: uniformly in  $a \in \mathcal{A}$  and  $P \in \mathcal{P}$

$$\sqrt{n} \left( \widehat{G}_{V/W|D}(a|0) - G_{V/W|D}(a|0) \right) = n^{-1/2} \sum_{j=1}^n V_{j,V/W}(a; \theta_o, \tau_o) + o_P(1).$$

Let

$$X_i(\theta, \tau) = \frac{(1-p_1)p(Z_i; \beta, \tau)}{p_1[1-p(Z_i; \beta, \tau)]}.$$

Then

$$I \left\{ p(Z_i; \widehat{\beta}, \widehat{\tau}) \leq \frac{a \widehat{p}_1}{1 - (1-a) \widehat{p}_1} \right\} = I \left\{ X_i(\widehat{\theta}, \widehat{\tau}) \leq a \right\}.$$

Note that

$$\begin{aligned} & E_P [I \{X_i(\theta, \tau) \leq x\} | D_i = 0] \\ &= E_P \left[ I \left\{ p(Z_i; \beta, \tau) \leq \frac{p_1 x}{1 - (1-x)p_1} \right\} | D_i = 0 \right] \\ &= F_{P|D} \left( \frac{p_1 x}{1 - (1-x)p_1}; \beta, \tau | 0 \right). \end{aligned}$$

We have:

$$\begin{aligned} & \Gamma_{F,P}(x|d) [\theta - \theta_o, \tau - \tau_o] \\ = & \frac{x}{[1 - (1-x)p_{1o}]^2} f_{P|D} \left( \frac{p_{1o}x}{1 - (1-x)p_{1o}} | 0 \right) (p_1 - p_{1o}) \\ & + \Gamma_P \left( \frac{p_{1o}x}{1 - (1-x)p_{1o}} | 0 \right) [\beta - \beta_o, \tau - \tau_o]. \end{aligned}$$

Thus,

$$\begin{aligned} & \sqrt{n} \left( \widehat{G}_{V/W|D}(a|0) - G_{V/W|D}(a|0) \right) \\ = & \sqrt{n} \left( \widehat{F}_{P|D} \left( \frac{a\widehat{p}_1}{1 - (1-a)\widehat{p}_1} | 0 \right) - G_{V/W|D}(a|0) \right) \\ = & n^{-1/2} \sum_{j=1}^n \frac{1}{(1-p_{1o})} \left[ I \left\{ p(Z_j; \beta_o, \tau_o) \leq \frac{ap_{1o}}{1 - (1-a)p_{1o}} \right\} - G_{V^?W|D}(v|0) \right] I \{D_j = 0\} \\ & + n^{-1/2} \sum_{j=1}^n \frac{1}{(1-p_{1o})} \psi_j \left( \frac{p_{1o}a}{1 - (1-a)p_{1o}}; \beta_o, \tau_o, 0 \right) \\ & + n^{-1/2} \sum_{j=1}^n \frac{a}{(1-p_{1o}) [1 - (1-a)p_{1o}]^2} f_{P|D} \left( \frac{p_{1o}a}{1 - (1-a)p_{1o}} | 0 \right) (I \{D_j = 1\} - p_{1o}) \\ & + o_P(1) \\ = & n^{-1/2} \sum_{j=1}^n V_{j,V/W}(a; \theta_o, \tau_o) + o_P(1). \end{aligned}$$

**Step 2.** Step 1 implies:

$$\left\{ \sqrt{n} \left( \widehat{G}_{V/W|D}(a|0) - G_{V/W|D}(a|0) \right) : a \in \mathcal{A} \right\}$$

weakly converges to a Gaussian process  $\nu_{V/W}(\cdot)$  with zero mean and covariance kernel:

$$E(V_{j,V/W}(a_1; \theta_o, \tau_o) V_{j,V/W}(a_2; \theta_o, \tau_o))$$

By the Functional Delta method, we obtain:

$$\begin{aligned} & \sqrt{n} \left( \left[ c_1 \widehat{\mu}_{0|1}^L + c_2 \widehat{\mu}_{0|1}^U \right] - \left[ c_1 \mu_{0|1}^L + c_2 \mu_{0|1}^U \right] \right) \\ \Rightarrow & c_1 \int_0^{1-P_{00}} \frac{\nu_{V/W}}{g_{V/W|D}} \circ G_{V^?W|D}^{-1}(u|0) du + c_2 \int_{P_{00}}^1 \frac{\nu_{V/W}}{g_{V/W|D}} \circ G_{V^?W|D}^{-1}(u|0) du. \end{aligned}$$

**Q.E.D.**

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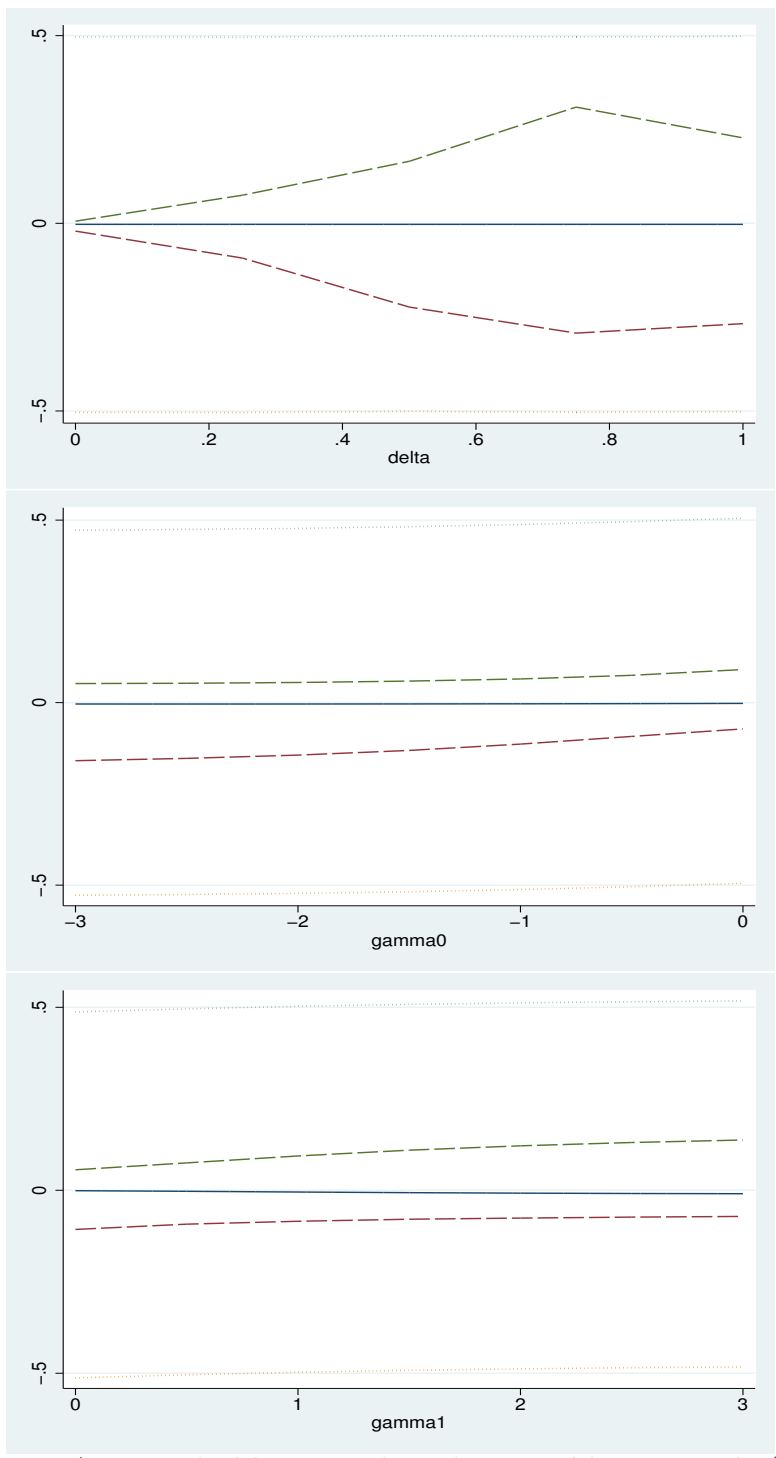
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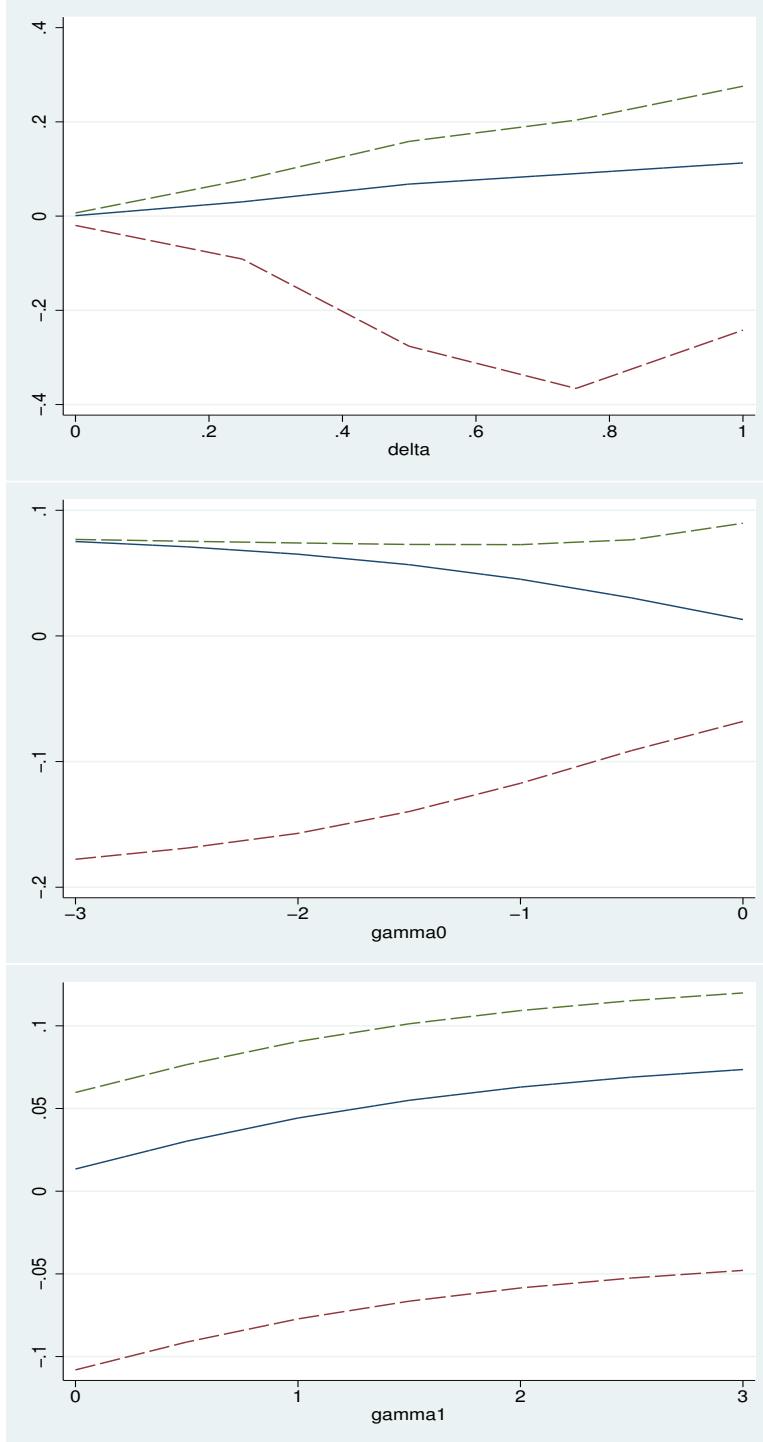
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Figure 1: Test Model: Effects of parameters on ATE bounds



Solid line: true ATE; Dashed lines: our bounds; Dotted lines: Manski (1990) bounds  
 Baseline parameter values:  $\delta = 0.25$ ;  $\gamma_0 = -0.5$ ;  $\gamma_1 = 0.5$

Figure 2: Test Model: Effects of parameters on ATT bounds



Solid line: true ATT; Dashed lines: our bounds  
Baseline parameter values:  $\delta = 0.25$ ;  $\gamma_0 = -0.5$ ;  $\gamma_1 = 0.5$

Figure 3: Bounds on ATE from US Election Example

For each state, left and right of rectangular box corresponds to upper and lower bound of estimated ATE, as reported in Table 3. The confidence sets are marked by the whiskers.

Y-axis: numbers correspond to states in Table 3.

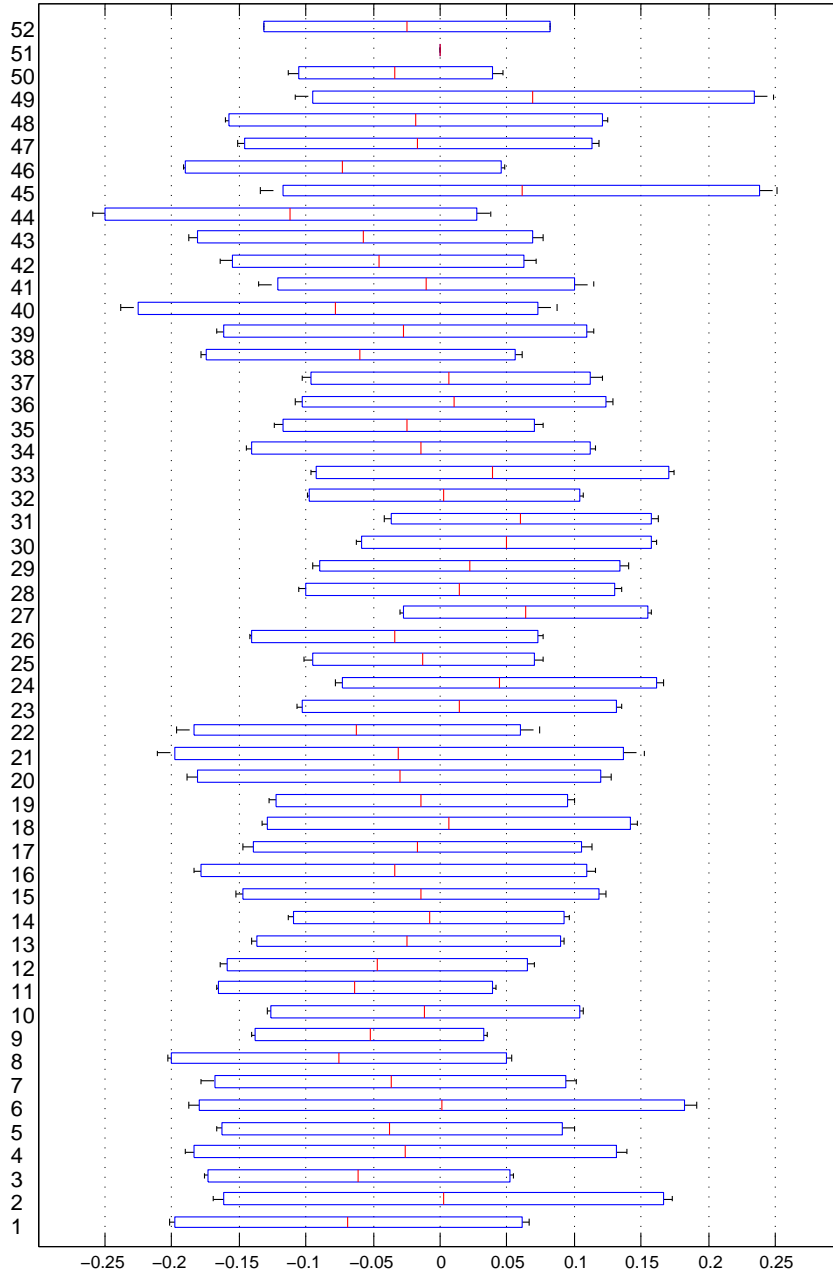


Figure 4: Bounds on ATT from US Election Example

For each state, top and bottom of rectangular box corresponds to upper and lower bound of estimated ATT, as reported in Table 3. The confidence sets are marked by the whiskers.

Y-axis: numbers correspond to states in Table 3.

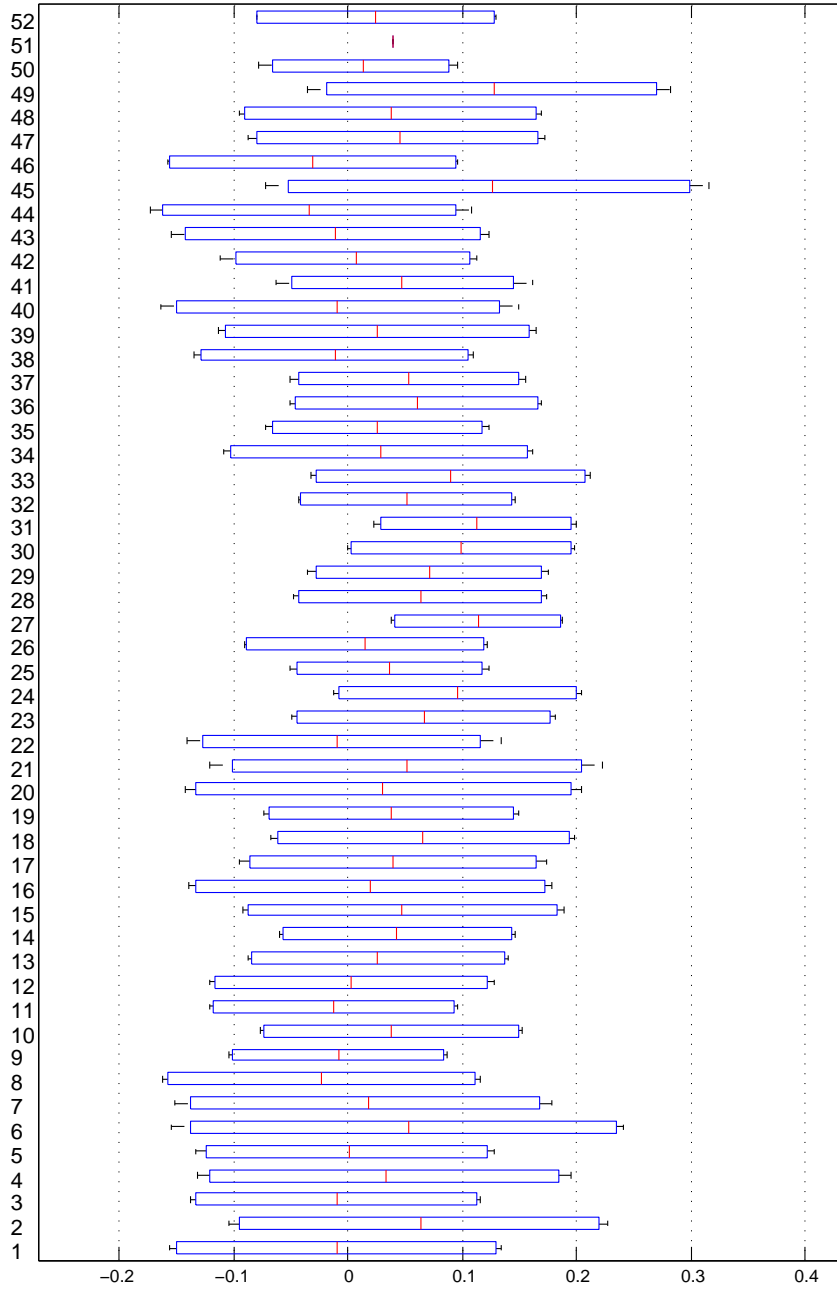


Table 1: Test Model: Small-Sample Performance

Fixed parameters:  $\delta = 0.25$ ;  $\gamma_0 = -0.5$ ;  $\gamma_1 = 0.5$

	# obs	[Low-bound, Up-bound]
<b>Average TE:</b>		
True		-0.0215
Our bounds	500	[-0.0852, 0.0849] [(0.0298), (0.0277)]
	1000	[-0.0855, 0.0857] [(0.0197), (0.0197)]
	2000	[-0.0844, 0.0848] [(0.0133), (0.0136)]
Manski bounds	500	[-0.5000, 0.5000] [(0.0100), (0.0100)]
	1000	[-0.4999, 0.5001] [(0.0072), (0.0072)]
	2000	[-0.4999, 0.5001] [(0.0050), (0.0050)]
<b>Average TT:</b>		
True		0.0135
Our bounds	500	[-0.0836, 0.0867] [(0.0290), (0.0272)]
	1000	[-0.0831, 0.0879] [(0.0201), (0.0200)]
	2000	[-0.0820, 0.0873] [(0.0133), (0.0140)]

Notes: True values computed by simulation. Propensity scores estimated using linear probability model. Reported bounds are averages over 1000 replications. Standard deviations (across replications) in parentheses.

Table 2: Summary statistics: variables included in propensity score

Variables		1980		2000	
		mean	std.dev.	mean	std.dev
LOGINC	Log household per-capita annual income	9.3457	0.8223	9.7317	0.8973
AGE		43.1512	17.9048	46.1245	17.6180
COLLEGE	=1 if at least some college, 0 otherwise	0.3185	0.4659	0.4252	0.4944
NONWHITE	=1 if nonwhite	0.1280	0.3343	0.2030	0.4022
UNEMPL	=1 if unemployed	0.0388	0.1932	0.0311	0.1735
EMPL <sup>a</sup>	=1 if employed	0.6061	0.4886	0.6268	0.4837
#obs		50000		50000	

<sup>a</sup>Omitted employment category: out of the labor force.

State <sup>a</sup>		Actual Repub. Voteshares			ATE		ATT	
		1980	2000	Diff.	lower bnd	upper bnd	lower bnd	upper bnd
1	CT	0.2937	0.2179	-0.0757	-0.1936	0.0554	-0.1441	0.1239
2	ME	0.2944	0.2927	-0.0017	-0.1547	0.1597	-0.0851	0.2124
3	MA	0.2470	0.1802	-0.0668	-0.1713	0.0490	-0.1289	0.1102
4	NH	0.3301	0.2925	-0.0376	-0.1760	0.1234	-0.1102	0.1753
5	RI	0.2181	0.1622	-0.0559	-0.1574	0.0813	-0.1142	0.1158
6	VT	0.2563	0.2582	0.0019	-0.1707	0.1725	-0.1219	0.2279
7	DE	0.2582	0.2312	-0.0270	-0.1573	0.0850	-0.1226	0.1577
8	NJ	0.2853	0.2020	-0.0834	-0.1974	0.0458	-0.1548	0.1083
9	NY	0.2238	0.1675	-0.0562	-0.1363	0.0310	-0.0985	0.0822
10	PA	0.2578	0.2433	-0.0145	-0.1234	0.1007	-0.0709	0.1464
11	IL	0.2870	0.2192	-0.0678	-0.1624	0.0361	-0.1151	0.0894
12	IN	0.3232	0.2752	-0.0480	-0.1544	0.0604	-0.1100	0.1163
13	MI	0.2940	0.2647	-0.0293	-0.1343	0.0854	-0.0814	0.1332
14	OH	0.2854	0.2768	-0.0086	-0.1066	0.0899	-0.0535	0.1397
15	WI	0.3230	0.3080	-0.0150	-0.1427	0.1137	-0.0820	0.1769
16	IA	0.3229	0.2887	-0.0343	-0.1731	0.1045	-0.1266	0.1657
17	KS	0.3283	0.3138	-0.0145	-0.1315	0.0986	-0.0764	0.1556
18	MN	0.2981	0.3031	0.0051	-0.1242	0.1376	-0.0568	0.1881
19	MO	0.3008	0.2839	-0.0168	-0.1165	0.0888	-0.0627	0.1380
20	NE	0.3717	0.3428	-0.0289	-0.1727	0.1117	-0.1247	0.1854
21	ND	0.4161	0.3636	-0.0526	-0.1851	0.1213	-0.0817	0.1856
22	SD	0.4074	0.3437	-0.0637	-0.1701	0.0447	-0.1146	0.0970
23	VA	0.2520	0.2668	0.0148	-0.0992	0.1280	-0.0408	0.1733
24	AL	0.2376	0.2822	0.0446	-0.0684	0.1568	-0.0023	0.1954
25	AR	0.2481	0.2362	-0.0119	-0.0880	0.0631	-0.0367	0.1104
26	FL	0.2702	0.2336	-0.0366	-0.1376	0.0704	-0.0856	0.1163
27	GA	0.1690	0.2329	0.0638	-0.0238	0.1522	0.0447	0.1837
28	LA	0.2724	0.2851	0.0127	-0.0956	0.1249	-0.0371	0.1634
29	MS	0.2565	0.2758	0.0193	-0.0832	0.1281	-0.0213	0.1644
30	NC	0.2142	0.2656	0.0514	-0.0544	0.1528	0.0075	0.1918
31	SC	0.1997	0.2599	0.0602	-0.0320	0.1530	0.0334	0.1907
32	TX	0.2481	0.2510	0.0029	-0.0948	0.1010	-0.0386	0.1414
33	KY	0.2451	0.2850	0.0400	-0.0886	0.1665	-0.0234	0.2038
34	MD	0.2209	0.2049	-0.0160	-0.1373	0.1079	-0.0973	0.1541
35	OK	0.3160	0.2899	-0.0261	-0.1112	0.0628	-0.0590	0.1103
36	TN	0.2374	0.2459	0.0085	-0.0989	0.1194	-0.0407	0.1615
37	WV	0.2393	0.2395	0.0001	-0.0899	0.1029	-0.0364	0.1421
38	AZ	0.2690	0.2044	-0.0647	-0.1702	0.0511	-0.1225	0.1010
39	CO	0.3079	0.2720	-0.0359	-0.1571	0.1032	-0.1007	0.1528
40	ID	0.4514	0.3601	-0.0913	-0.2133	0.0578	-0.1362	0.1174
41	MT	0.3699	0.3556	-0.0142	-0.1067	0.0863	-0.0345	0.1293
42	NV	0.2573	0.1984	-0.0589	-0.1462	0.0550	-0.0841	0.0990
43	NM	0.2793	0.2174	-0.0619	-0.1744	0.0608	-0.1304	0.1080
44	UT	0.4704	0.3354	-0.1350	-0.2414	0.0174	-0.1510	0.0829
45	WY	0.3347	0.4043	0.0696	-0.1012	0.2247	-0.0312	0.2838
46	CA	0.2578	0.1837	-0.0741	-0.1886	0.0441	-0.1536	0.0923
47	OR	0.2965	0.2749	-0.0216	-0.1407	0.1075	-0.0699	0.1607
48	WA	0.2849	0.2509	-0.0340	-0.1540	0.1170	-0.0848	0.1607
49	AK	0.3106	0.3808	0.0702	-0.0810	0.2196	0.0002	0.2560
50	HI	0.1870	0.1498	-0.0372	-0.0977	0.0309	-0.0549	0.0811
52	Natl	0.2671	0.2396	-0.0275	-0.1306	0.0812	-0.0785	0.1280

Table 3: Changes in Republican Voteshares in US Presidential Elections, 1980/2000

<sup>a</sup>State numbers correspond to y-axis labels in Figures 3 and 4.

	State	Bounds: marginal mean outcome				Bounds: counterfct mean	
		$\mu_0^L$	$\mu_0^U$	$\mu_1^L$	$\mu_1^U$	$\mu_{0 1}^L$	$\mu_{0 1}^U$
1	CT	0.1713	0.3010	0.1074	0.2267	0.0940	0.3621
2	ME	0.1678	0.3066	0.1520	0.3276	0.0803	0.3778
3	MA	0.1375	0.2496	0.0783	0.1865	0.0700	0.3091
4	NH	0.1889	0.3488	0.1728	0.3123	0.1172	0.4027
5	RI	0.1146	0.2157	0.0583	0.1959	0.0464	0.2764
6	VT	0.1136	0.2979	0.1272	0.2861	0.0303	0.3801
7	DE	0.1373	0.2849	0.1275	0.2223	0.0735	0.3538
8	NJ	0.1686	0.2925	0.0951	0.2144	0.0936	0.3567
9	NY	0.1328	0.2165	0.0802	0.1638	0.0853	0.2661
10	PA	0.1569	0.2566	0.1332	0.2576	0.0969	0.3142
11	IL	0.1857	0.2835	0.1211	0.2218	0.1298	0.3343
12	IN	0.2196	0.3257	0.1713	0.2800	0.1589	0.3852
13	MI	0.1910	0.2921	0.1578	0.2764	0.1315	0.3462
14	OH	0.1894	0.2797	0.1732	0.2794	0.1371	0.3303
15	WI	0.2034	0.3303	0.1876	0.3172	0.1310	0.3899
16	IA	0.2063	0.3362	0.1631	0.3108	0.1230	0.4153
17	KS	0.2226	0.3301	0.1986	0.3213	0.1582	0.3901
18	MN	0.1815	0.3040	0.1798	0.3192	0.1150	0.3599
19	MO	0.2032	0.2943	0.1778	0.2920	0.1459	0.3466
20	NE	0.2452	0.3896	0.2169	0.3569	0.1575	0.4675
21	ND	0.2799	0.4024	0.2173	0.4012	0.1779	0.4453
22	SD	0.3055	0.4046	0.2346	0.3502	0.2467	0.4583
23	VA	0.1445	0.2580	0.1588	0.2725	0.0935	0.3077
24	AL	0.1384	0.2349	0.1666	0.2952	0.0868	0.2845
25	AR	0.1623	0.2351	0.1472	0.2254	0.1258	0.2729
26	FL	0.1637	0.2745	0.1370	0.2341	0.1172	0.3192
27	GA	0.0807	0.1570	0.1332	0.2328	0.0492	0.1882
28	LA	0.1747	0.2697	0.1742	0.2996	0.1217	0.3222
29	MS	0.1616	0.2493	0.1661	0.2897	0.1114	0.2971
30	NC	0.1172	0.2130	0.1586	0.2700	0.0738	0.2581
31	SC	0.1071	0.1904	0.1584	0.2601	0.0692	0.2265
32	TX	0.1504	0.2474	0.1526	0.2514	0.1097	0.2896
33	KY	0.1399	0.2500	0.1614	0.3064	0.0812	0.3084
34	MD	0.1115	0.2358	0.0985	0.2194	0.0508	0.3022
35	OK	0.2271	0.3033	0.1921	0.2899	0.1796	0.3489
36	TN	0.1348	0.2381	0.1392	0.2543	0.0844	0.2866
37	WV	0.1477	0.2284	0.1386	0.2507	0.0973	0.2759
38	AZ	0.1498	0.2836	0.1134	0.2009	0.1034	0.3269
39	CO	0.1861	0.3174	0.1603	0.2893	0.1192	0.3727
40	ID	0.3220	0.4488	0.2355	0.3798	0.2426	0.4962
41	MT	0.2761	0.3527	0.2460	0.3625	0.2263	0.3901
42	NV	0.1336	0.2586	0.1125	0.1886	0.0994	0.2825
43	NM	0.1661	0.2917	0.1174	0.2269	0.1094	0.3478
44	UT	0.3273	0.4569	0.2156	0.3448	0.2525	0.4864
45	WY	0.2042	0.3586	0.2574	0.4289	0.1205	0.4354
46	CA	0.1459	0.2767	0.0881	0.1901	0.0914	0.3373
47	OR	0.1778	0.2976	0.1569	0.2853	0.1142	0.3448
48	WA	0.1572	0.2890	0.1351	0.2742	0.0902	0.3358
49	AK	0.1861	0.3264	0.2454	0.4057	0.1247	0.3806
50	HI	0.1010	0.1726	0.0749	0.1319	0.0687	0.2047
52	Natl	0.1643	0.2676	0.1370	0.2455	0.1115	0.3181

Table 4: Example: Bounds for Marginal and Counterfactual Outcome Distributions