The Monopolist's First-Order Condition

Using the second derivation, the monopoly solves the following problem: $\max_p pq(p) - C(q(p))$

As derived in class, at the optimal price p^*: $(p^* - MC(q(p^*))) = -\frac{q(p^*)}{q'(p^*)}$ or $\frac{p^* - MC(q(p^*))}{p^*} = -\frac{1}{\epsilon(p^*)}$. This is the inverse elasticity property.

What if p were such that $-1 < \epsilon(p) < 0$?

In this case, $-\frac{1}{\epsilon(p)} > 1$, and the only way that this can the first-order condition is if $p < 0$, which is clearly impossible. Therefore, the monopolist cannot be optimizing if one observes this.

To gain more insight into this requires math that is a bit more complicated. We must first work in terms of the *inverse demand function* $p(q)$, which gives the price p that a given quantity q can be sold for. A monopolist's revenue at quantity q is:

$$\text{Revenue}(q) \equiv R(q) = p(q)q \tag{1}$$

and marginal revenue is:

$$MR(q) = \frac{\partial R(q)}{\partial q} = p'(q)q + p(q). \tag{2}$$

Rewriting everything in terms of p, and using the relation that $p'(q) = 1/q'(p)$[1], we get

$$MR(q) \ = \frac{q(p)}{q'(p)} + p \ = \ \frac{p}{p}\frac{q(p)}{q'(p)} + p \ = \ p\left(\frac{1}{q'(p)}\frac{q(p)}{p} + 1\right) \ = \ p(\frac{1}{\epsilon(p)} + 1) \tag{3}$$

which is *negative* for prices where $-1 < \epsilon(p) < 0$. So another (more intuitive) way of saying that a monopolist never produces when $-1 < \epsilon < 0$ is that a monopolist never chooses a p (or equivalently $q(p)$) where its marginal revenue would be negative.

[1] This is because p and q are inverse functions. Let $f(p)$ be the demand function, and $g(q)$ be the inverse demand function, so that $f(g(q)) = q$. Differentiating both sides of this relation with respect to q, we get $f'(\cdot)g'(\cdot) = 1$, or $g' = \frac{1}{f'}$. Substituting in $f(p) = q(p)$, and $g(q) = p(q)$, we get $p'(q) = \frac{1}{q'(p)}$.