# A Dynamic Model of Legislative Bargaining

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PRELIMINARY AND INCOMPLETE

**Abstract**

We prove existence of stationary Markov perfect equilibria in an infinite-horizon model of legislative bargaining in which the policy outcome in one period determines the status quo in the next. We allow for a multidimensional policy space and arbitrary smooth stage utilities. We prove that all such equilibria are essentially in pure strategies and that proposal strategies are differentiable almost everywhere. The model is general enough to accommodate much of the institutional structure observed in real-world legislatures and parliaments.

## 1 Introduction

Political interaction in modern democracies qualifies as one of the most complex phenomena subjected to scientific inquiry. The need to accommodate this complexity in formal political theory models stems not only from the desire to sate our intellectual curiosity, but seems also essential for the analysis of the effects of public policy and the design of constitutions. In this spirit,

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we seek to develop a class of models of policy making that (i) accounts for the multidimensional nature of public policy, (ii) captures the continuing nature of policy over time, (iii) is rich enough to reflect institutional structure at a fine level of detail, and (iv) allows for the kinds of random shocks (e.g., on preferences and the social environment) to which political interaction is subjected over time. The political economy literature to date has had limited success in addressing these issues. At a formal level, the primary difficulty that arises is the existence of equilibria in which policy makers use relatively simple and intuitive strategies. Beyond that is the problem of characterizing equilibria, once they are known to exist, and finally there is the task of applying the model, i.e., developing useful special cases and, when the limits of formal analysis are reached, bringing numerical techniques to bear.

We lay the theoretical foundations for such future applications by establishing the existence of stationary Markov perfect equilibria in a class of models with the desiderata (i)–(iv) identified above, and we show that in every such model equilibria are essentially in pure strategies, and legislators' proposal strategies are differentiable almost everywhere. We consider an infinite-horizon model of legislative bargaining where each period begins with the random draw of a legislator, who proposes any feasible policy, which is then subject to a majority vote. In this respect, our protocol is familiar from the Baron-Ferejohn (1989) model of distributive bargaining in the political science literature. In their work, however, the game ends with the proposed allocation of surplus if a majority of legislators accept the proposal; otherwise, all agents receive a payoff of zero, and the bargaining game is repeated in the next period. Thus, that model is appropriate for examining policy choices across legislative sessions only if policies remain in place for a single session, with an exogenously fixed default outcome in every future session in which a new agreement is not reached. This is often the case, for example, in budgetary negotiations. The model is inadequate, however, for the analysis of continuing programs or legislation, where policy choices remain in place in the future, determining the status quo in future negotiations. In such environments, policy-makers must not only consider the impact of a policy proposal on the present, but also the future policies that would follow that choice.

We consider a fully dynamic model of legislative bargaining, in which every period begins with a status quo policy; the policy outcome in that period is the proposed policy if it garners a majority, the status quo otherwise;

and the status quo in the next period is determined by the outcome that prevails in the current period. Thus, the path of play in this game generates an infinite sequence of policies over time. The level of complexity of these models is significantly greater than models with a fixed default outcome, as proposal strategies must now depend on the status quo in a non-trivial way. It is therefore natural to focus on stationary Markov perfect equilibria, which due to their simplicity minimize the difficulty of strategic calculations and may possess a focal quality. From an econometric point of view, the Markov property of equilibrium strategies seems essential for estimation purposes. In seminal work on the endogenous status quo model, Baron (1996) considers a one-dimensional policy space with single-peaked stage utilities and proves that stationary equilibrium policy outcomes converge to the ideal point of the median voter over time. The model has been extended to special multidimensional settings by Kalandrakis (2004b,2005a), Fong (2005), Cho (2005), and Battaglini and Coate (2005), who give constructive proofs of equilibrium existence relying on the particular structure of their models.

Our model is distinguished from other work on endogenous status quo models in that we do not assume a specific set of policies or specific functional forms for legislators' utility functions. Instead, we allow the set of alternatives to be a very general subset of any finite-dimensional Euclidean space defined by smooth feasibility constraints. We assume smooth stage utility functions but do not impose any further conditions. Thus, we capture standard models with resource and consumption constraints, such as the classical spatial model of politics, economic environments, and distributive models in which a fixed surplus is allocated to the legislators. Furthermore, we incorporate uncertainty about future policy preferences and effects of policy. Specifically, we assume that at the end of each period, (i) next period's status quo is realized as the sum of the current period's policy outcome and a (possibly small) stochastic shock, and (ii) legislator preferences are subject to (possibly small) publicly observed stochastic shocks. These natural assumption smooth out the game sufficiently to allow us to deduce the existence of stationary equilibrium. We also derive several technical properties of equilibrium that will facilitate applications of the models. Our existence result is not covered in the abstract literature on stochastic games, because the transition probabilities of our game violates a standard continuity assumption used there.

In fact, we have thus far described a simplified version of our model, which we develop initially in Section 3. It is in this simplified framework where we introduce strategies and our concept of stationary legislative equilibrium, and where we state our first theorem on existence of equilibrium. In Section 4, we give an elaboration of the benchmark model that is able to accommodate much of the institutional structure observed in real-world legislatures and parliaments.

[More on the full model to be inserted here.]

Proofs are collected in the appendix.

## 2 Literature Review

Before turning to the analysis, we first give a more in-depth review of the literature on bargaining, as it relates to legislative modeling, and of the literature on existence of Markov perfect equilibrium in stochastic games.

**Bargaining Models** Most of the existing work on bargaining considers an infinite-horizon game where in each period one agent makes a proposal and that proposal is either accepted, in which case the game ends with the proposed outcome, or rejected, in which case bargaining continues for at least one more round. This literature begins with Rubinstein's (1982) work on two-person, alternating-offer bargaining, which is modified by Binmore (1987) to allow for a randomly determined proposer. This model was extended to cover legislative politics by Baron and Ferejohn (1989), who allow for an arbitrary number of legislators and assume a simple majority is required for a proposal to pass. As with Rubinstein's and Binmore's work, the subject of bargaining is the allocation of a fixed surplus, now interpreted as pork barrel spending.

A substantial literature cutting across economics and political science has grown from these papers. For example, Baron (1991) examines the case of a two-dimensional set of alternatives, three or four voters with quadratic preferences, and voting by majority rule. Merlo and Wilson (1995) prove uniqueness of stationary equilibrium, assuming unanimity rule and allowing the amount of the surplus to vary stochastically over time. Eraslan (2002) proves uniqueness of stationary equilibrium in the original Baron-Ferejohn model. Banks and Duggan (2000) prove existence and examine connections to the core of the cooperative voting game in a version of the model with general set of alternatives, preferences, and voting rule. Kalandrakis (2004c) gives a simplified proof of existence using a characterization of equilibrium in terms of the solutions to a finite number of equalities and inequalities. Kalandrakis (2006a) examines regularity of the general bargaining model for generic discount factors. While all of the previous work implicitly assumes that delay is bad for the agents, Banks and Duggan (2006) allow for an arbitrary status quo, re-establish results from the earlier framework, and provide a new analysis of the possibility of delay. Cho and Duggan (2003) prove uniqueness of stationary equilibrium in the one-dimensional model with quadratic utilities, and Cho and Duggan (2005) prove an asymptotic median voter theorem in the one-dimensional bargaining model without stationarity. This class of models has found numerous applications to legislative policy-making,[1] but while they capture some dynamic aspects of politics, but they uniformly assume that the game ends once a proposal is accepted.

A small literature considers the effects of endogenizing the status quo: each period begins with a status quo, then one agent makes a proposal and that proposal is either accepted, in which case it becomes the status quo for the next period, or rejected, in which case the current status quo remains in place. There are currently no general results for this model, though there are constructions of stationary equilibria in special cases. Baron (1996) analyzes the one-dimensional version of the model with single-peaked stage utilities. Kalandrakis (2004a,2005a) establishes existence and continuity properties for the constructed equilibrium strategies in the distributive model, obtains a fully strategic version of McKelvey's (1976,1979) dictatorial agenda setting in that setting, and studies the composition of equilibrium coalitions and the effect of risk-aversion on equilibrium.[2] Baron and Herron (2003) give a numerical calculation of equilibrium in a three-legislator, finite-horizon model. Fong (2005) considers a three-legislator model in which policies consist of locations in a two-dimensional space and allocations of surplus. Cho (2005) analyzes policy outcomes in a similar environment but with a stage game emulating aspects of parliamentary government. Similar in spirit to the above, Battaglini and Coate (2005) characterize stationary equilibria in a model of public good provision and taxation with identical legislators and a stock of

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[1] See, for example, Diermeier, Eraslan, and Merlo (2003), Jackson and Moselle (2002), Kalandrakis (2004a,2005b,2006b), McCarty (2000), and Merlo (1997).

[2] In contrast, Epple and Riordan (1987) prove folk theorem results in the distributive model.

public goods that evolves over time. All of the above analyses of stationary equilibria consist of explicitly constructing equilibrium strategies, which, given the dependence of proposals on the status quo, can be extremely complex.

A number of related papers depart in various ways from the above literature and our models. Penn (2005) considers a dynamic voting game with randomly generated policy proposals and probabilistic voting on these proposals. Lagunoff (2005a,b) considers a class of stochastic games that incorporate a social choice solution concept and analyzes endogenous political institutions. Finally, Gomez and Jehiel (2005) consider a class of stochastic games and characterize efficiency properties of equilibrium when players are patient. Unlike our model, they assume a finite number of states and transferable utility.

**Stochastic Games** Existence of stationary Markov perfect equilibrium is a central issue the literature on stochastic games, which analyzes dynamic games at a more abstract level. It is well-known that existence in games with finite state and action spaces follows from the straightforward application of Kakutani's fixed point theorem in finite dimensions (Rogers (1969) and Sobel (1971)). General results on existence have been elusive and have relied on the imposition of relatively special structure or departures from the concept of stationary equilibrium.[3] All of the known results rely on fairly strong assumptions on the transition probability. Letting $s$ denote a state and $a$ denote a profile of actions, the transition probability is a measurable mapping $\mu_t(\cdot|s, a)$ from state-action pairs to a probability measure on the set of states. Some assumptions used in the literature are, in increasing strength:

(A1) $\mu_t$ is strongly continuous in $a$,[4]

(A2) $\mu_t$ is norm-continuous in $a$

(A3) $\mu_t$ is norm-continuous in $a$ and absolutely continuous with respect to some fixed probability measure $\nu_t$

(A4) $\mu_t$ is norm-continuous in $a$ and absolutely continuous with respect to a fixed, non-atomic probability measure $\nu_t$

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[3] Dutta and Sundaram (1998) provide a lucid review of much of the literature on stochastic games and the problem of existence of Markov Equilibrium.

[4] That is, for each measurable set $Z$ of states, $\mu(Z|s, a)$ is continuous in $a$.

(A5) $\mu_t$ has a density $f(s'|s,a)$ with respect to Lebesgue measure that is continuous with respect to $a$.

It is well-known that even the weakest of the above assumptions, (A1), is inconsistent with deterministic transitions when action sets are uncountably infinite.

In finite-horizon stochastic games, Rieder (1979) (see also Chakrabarti (1999)) proves existence of Markov perfect equilibrium under (A1). By incorporating time in the state variable of a finite-horizon game, we may in fact view Rieder's equilibrium as stationary. Under strong continuity assumptions on the transition probability, akin to (A5), Amir (1996, 2002) and Curtat (1992) prove existence of stationary Markov perfect equilibria in games possessing strategic complementarities.[5] Other results have been obtained by weakening stationarity or considering weaker notions of equilibrium. Chackrabarti (1999) proves existence of (possibly non-stationary) Markov perfect equilibria in games satisfying (A3), and Mertens and Parthasarathy (1987, 1991) drop the assumption of absolute continuity and obtain existence of equilibria that are nearly Markovian.[6] Increasing (A3) to (A4), Chackrabarti (1999) proves existence of a stationary equilibrium, but now in semi-Markov perfect strategies.[7] Dutta and Sundaram (1998) give a simple proof of the existence of (possibly non-stationary) Markov perfect $\epsilon$-equilibria under (A1), whereas Nowak (1985) increases (A1) to (A4) and obtains a Markov perfect $\epsilon$-equilibrium in stationary strategies. Himmelberg, Parthasarathy, Raghavan, and van Vleck (1976) prove existence of $p$-equilibria assuming finite action sets.[8] Finally, Nowak and Raghavan (1992) prove existence of stationary Markov perfect equilibria with public randomization under (A4), and Duffie, Geanokoplos, Mas-Colell, and McLennan (1994) add mutual absolute continuity of transition probabilities and show that the equilibrium induces an ergodic process.

A first difficulty in applying existing results on stochastic games to leg-

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[5] Other work restricts the way in which players' actions affect each others' payoffs, e.g., Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), and Horst (2005).

[6] Players strategies in period $t$ can depend not only on the current state $s_t$ but the previous state $s_{t-1}$ as well.

[7] That is, players may condition on the current state and the state in the previous period, and the nature of that conditioning is constant over time.

[8] Here, $p$ is a probability measure on states, and a $p$-equilibrium is a strategy profile such that players optimize at all but perhaps a set of states with $p$-measure zero.

islative bargaining is that much of the above work uses weakenings of stationarity or the concept of equilibrium, whereas we seek truly stationary strategies such that legislators optimize at all states, without the availability of a public randomization device. A second difficulty stems from the deterministic element inherent in structure of legislative procedure, which when modelled naturally violates (A1) and the other stronger conditions used on transition probabilities. In describing legislative bargaining as a stochastic game, the state variable must include all relevant details of the game when legislators take actions, whether proposing policy or voting over proposals. Thus, the state must specify when the legislature is in a proposal stage or a voting stage. In a proposal stage, the state must also specify the proposer and status quo, and in a voting stage the stage must specify the proposed policy and the status quo. The transition from a voting stage to the subsequent proposal stage is not problematic, for action sets in the voting stage are finite, and the transition is trivially continuous following the votes of legislators. The problem is the fact that the proposer's action (the policy proposal) precisely determines the state in the subsequent voting stage, inevitably violating (A1). Adding noise to the model in the form of uncertainty about the status quo and the policy preferences of legislators in future session does not eliminate this problem, which we take to be an inherent feature of legislative policy-making, yet it allows us to prove the existence of stationary equilibria satisfying a number of desirable technical properties.

# 3 The Benchmark Model

We first present a simplified benchmark model that presents the most serious hurdles to establishing the existence of stationary Markov perfect equilibria and omits extra structure that, while important in applications, is not critical for the development of our analytical techniques.

**Framework** We posit a finite set $N$ of legislators, $i = 1, \ldots, n$, who must choose from a set $X \subseteq \Re^d$ of feasible policies. Legislative bargaining in each period $t = 1, 2, \ldots$ proceeds as follows. A status quo policy $q_t \in X$ and a vector $\theta = (\theta_1, \ldots, \theta_n) \in \Re^{nd}$ are taken as given. A legislator is drawn at random, with probabilities $p_1, \ldots, p_n$, to propose a policy $y \in X$. The legislators vote simultaneously to accept $y$ or reject it in favor of the status quo $q_t$. The proposal passes if it at least $\frac{n+1}{2}$ legislators vote to accept, and it fails otherwise. The policy for period $t$, denoted $x_t$, is $y$ if the proposal passes and is $q_t$ otherwise. Each legislator $j$ receives utility $\hat{u}_j(x_t, \theta_j)$, where $\theta_j \in \Re^d$ is the legislator's utility shock. Finally, the status quo $q_{t+1}$ for period $t+1$ is drawn from the density $g(\cdot|x_t)$, a new vector $\theta = (\theta_1, \ldots, \theta_n)$ of shocks is drawn from the density $f(\cdot)$ and publicly observed, and the above procedure is repeated in period $t+1$. Payoffs in the dynamic game are given by the expected discounted sum of stage utilities, as is standard, and we assume for notational simplicity that legislators share the discount factor $\delta \in [0, 1)$.

We impose a number of regularity conditions on the model. We assume that the set of feasible policies is cut out by a finite number of smooth functions $h_\ell \colon \Re^d \to \Re$, where $\ell$ varies over the index set $K = \{1, \ldots, k\}$, i.e.,

$$X \;\; = \;\; \{x \in \Re^d \mid h_\ell(x) \geq 0, \ell \in K,$$

and that this set is compact. For technical reasons, we also impose the weak condition that for all $x \in \Re^d$, $\{Dh_\ell(x) \mid \ell \in L(y)\}$ is linearly independent, where $L(y)$ is the subset of $K$ containing $\ell$ such that $h_\ell(x) = 0$. This allows us to capture, for example, standard models with resource and consumption constraints, such as the classical spatial model of politics, economic environments, and distributive models in which an amount of surplus is to be allocated among the legislators' districts.

We assume that the stage utilities take the form $\hat{u}_i(x, \theta_i) = u_i(x) + \theta_i \cdot x$, where $u_i \colon \Re^d \to \Re$ is a smooth function, capturing the standard assumption of strict quasi-concavity in the spatial modeling literature. Note that the utility shock enters stage utility in a linear fashion. In the special case of quadratic utility, i.e., $u_i(x) = -||x - \hat{x}_i||^2$ where $\hat{x}_i \in X$ is a fixed ideal point, the linear functional form is equivalent to assuming a noise term added to the ideal point of legislator $i$. To see this, note that

$$u_i(x) + \theta_i \cdot x \;\; = \;\; -(x - (\hat{x}_i + \frac{1}{2}\theta_i)) \cdot (x - (\hat{x}_i + \frac{1}{2}\theta_i)) + \theta_i \cdot \hat{x} + \frac{1}{4}\theta_i \cdot \theta_i,$$

which is just the sum of the constant term $\theta_i \cdot \hat{x} + \frac{1}{4}\theta_i \cdot \theta_i$ and the quadratic utility with ideal point $\hat{x}_i + \frac{1}{2}\theta_i$. In the special case of linear utility, the shock is simply a perturbation of the legislators' gradients. More generally, the shock is a simple way to introduce well-behaved shifts of the indifference curves of legislators.

We assume that each $\theta_i$ is distributed iid and has support in some (possibly small) open set $\Theta$, and we assume that for all $x$ and all $i$, the expectation $\int (u_i(x) + \theta_i \cdot x) f(\theta) d\theta$ is finite. It is therefore bounded as a function of $x$, and we let $c$ denote a bound over $x \in X$ and $i \in N$. We assume $g\colon X \times \Theta \rightarrow \Re$, with values $g(q|x)$, is measurable in $q$ and smooth in $x$, and that the support of the density $g(\cdot|x)$ lies in the set $X$ of feasible policies. Furthermore, we assume that the the partial derivatives of all orders of $g$ with respect to the coordinates of $x$ are uniformly bounded by some $b$.

Our approach to existence involves the addition of noise to policy outcomes and legislator utilities, but we emphasize that the status quo and the utility shocks at the beginning of a period $t$ are commonly known, and therefore, given the strategies of others, a proposer knows whether any given policy will pass or fail. Furthermore, once a vote is taken, the policy outcome is pinned down for period $t$: the legislators know, conditional on the outcome of voting, what the policy outcome in the current period will be, and a new status quo is drawn for period $t+1$ only after legislators receive their period $t$ utilities from outcome $x_t$. Thus, the addition of noise to the model amounts to the assumption that, while legislators are completely informed in the current period, there is at least some uncertainty about future policy preferences and the effects of policy. We view these as natural modeling assumptions. In any case, the supports of $f$ and $g(\cdot, x)$ can be assumed arbitrarily small, so that the element of noise in the model can be made innocuous.

**Strategies and Payoffs** A strategy in the game consists of two components telling us the proposals of legislators when recognized to propose and the votes of legislators after a proposal is made. While these choices can conceivably depend on histories arbitrarily, we seek subgame perfect equilibria in which legislators use simple strategies. We are therefore interested in pure, stationary strategies, which we denote $\sigma_i = (\pi_i, \alpha_i)$. Here, $\pi_i\colon X \times \Theta \rightarrow \Re^d$ is legislator $i$'s proposal strategy, where $\pi_i(q, \theta)$ is the policy proposed by $i$ given status quo $x$ and utility shocks $\theta$, and $\alpha_i\colon X \times X \times \Theta \rightarrow \{a, r\}$ is $i$'s voting strategy, where $\alpha_i(y, q, \theta)$ is $i$'s vote given proposal $y$, status quo $q$, and shocks $\theta$. We let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a stationary strategy profile. We may equivalently represent voting strategies by the set of proposals a legislator would vote for. We define this *acceptance set* for $i$ as $A_i(q, \theta; \sigma) = \{y \in X \mid \alpha_i(y, q, \theta) = a\}$. Letting $C$ denote a coalition of legislators, we then define

$$A_C(q,\theta;\sigma) \;=\; \bigcap_{i\in C} A_i(q,\theta;\sigma) \quad \text{and} \quad A(q,\theta;\sigma) \;=\; \bigcup_{C:\,\#C\geq\frac{n+1}{2}} A_C(q,\theta;\sigma)$$

as the coalitional acceptance set and social acceptance set, respectively. The latter consists of all policies that would pass, if proposed by legislator $i$.

Given strategies $\sigma$, we define legislator $i$'s induced preferences in the game by

$$U_i(y,\theta_i;\sigma) \;=\; (1-\delta)(u_i(y)+\theta_i\cdot y)+\delta v_i(y;\sigma),$$

where $v_i(x;\sigma)$ is $i$'s continuation value at the beginning of period $t+1$ from policy outcome $x$ in period $t$. We assume without loss of generality that when indifferent, legislators vote for the policy proposed, and this then allows us to focus on "no-delay" equilibria, in which no legislator ever proposes a policy that is rejected. (In lieu of that, the legislator can just as well propose the status quo.) For such equilibria, the continuation value $v_i$ satisfies

$$v_i(x;\sigma) \;=\; \int_q\int_\theta \sum_{j\in N} p_j U_i(\pi_j(q,\theta),\theta_i;\sigma)f(\theta)g(q|x)d\theta dq$$

for all policies $x$.

**Legislative Equilibrium** With this formalism established, we can now define a subset of stationary Markov perfect equilibria of special interest. Intuitively, we require that legislators always propose optimally and that they always vote in their best interest. It is well-known that the latter requirement is ambiguous in simultaneous voting games, as arbitrary outcomes can be supported by Nash equilibria in which no voter is pivotal. We follow the standard approach of refining the set of Nash equilibria in voting subgames by requiring that legislators delete votes that are dominated in the stage game. Thus, we say $\sigma$ is a *stationary legislative equilibrium* if

- for all shocks $\theta$, every status quo $q$, and every legislator $i$,

$$U_i(\pi_i(q,\theta),\theta_i;\sigma) \;=\; \sup\{U_i(y,\theta_i;\sigma) \mid y\in A(q,\theta;\sigma)\}.$$

- for all shocks $\theta$, every status quo $q$, every proposal $y$, and every legislator $i$,

$$\alpha_i(y, q, \theta_i) \quad = \quad \begin{cases} a & \text{if } U_i(y, \theta_i; \sigma) \geq U_i(q, \theta_i; \sigma) \\ r & \text{else.} \end{cases}$$

Note that we build in the feature that voters defer to the proposer when indifferent, and that we then without loss of generality restrict proposers to the social acceptance set.

The main result of this section is that there is a stationary legislative equilibrium satisfying a number of properties.

**Theorem 1** *There exists a stationary legislative equilibrium, $\sigma$, of the benchmark model possessing the following properties.*

1. *Continuation values are smooth: for every legislator $i$, $v_i(q; \sigma)$ is smooth as a function of $q$.*
2. *Coalitions are almost always minimum winning: for every status quo $q$, almost all shocks $\theta$, and every legislator $i$, if $\pi_i(q, \theta) \neq q$ and there exists $k \neq i$ such that $U_k(\pi_i(q, \theta), \theta_k; \sigma) = U_k(q, \theta_k; \sigma)$, then*

$$|\{j \notin N \setminus \{i\} \mid U_j(\pi_i(q, \theta), \theta_j) \geq U_j(q, \theta_j)\}| \quad = \quad \left\lceil \frac{n-1}{2} \right\rceil.$$

3. *Proposals are almost always strictly best: for every status quo $q$, almost all shocks $\theta$, every legislator $i$, and every $y \in A(q, \theta; \sigma)$ distinct from the proposal $\pi_i(q, \theta; \sigma)$, we have $U_i(\pi_i(q, \theta), \theta_i; \sigma) > U_i(y, \theta_i; \sigma)$.*
4. *The linear independence constraint qualification almost always holds for the proposer: for every status quo $q$, almost all shocks $\theta$, and every legislator $i$, if $\pi_i(q, \theta) \neq q$, then the collection*

$$\{Dh_\ell(\pi_i(q, \theta)), D_y U_j(\pi_i(q, \theta), \theta_i; \sigma) \mid \ell \in L, j \in C\}$$

*is linearly independent, where $L$ and $C$ represent the feasibility and voting constraints that bind at $\pi_i(q, \theta)$.*

5. *Proposal strategies are almost always differentiable: for almost all shocks $\theta$ and every legislator $i$, $\pi_i(q, \theta)$ is differentiable as a function of $(q, \theta)$.*

The fourth part of Theorem 1, which verifies that the linear independence constraint qualification (LICQ) almost always holds, is particularly important, as it implies a characterization of optimal proposals by means of the first order conditions for a constrained maximum. In particular, for every status quo $q$ and almost all shocks $\theta$, the proposal $\pi_i(q,\theta)$ is a critical point of the Lagrangian and the complementary slackness conditions hold: there exist $\lambda_\ell$, $\ell \in K$, and $\lambda_j$, $j \in N$ such that

$$D_y U_i(\pi_i(q,\theta),\theta_i;\sigma) + \sum_{\ell=1}^{k} \lambda_\ell D_y h_\ell(\pi_i(q,\theta)) + \sum_{j=1}^{n} \lambda_j D_y U_j(\pi_i(q,\theta),\theta_j;\sigma) = 0$$

$$\lambda_j \geq 0 \text{ and } \lambda_j(U_j(\pi_i(q,\theta),\theta_j;\sigma) - U_j(q,\theta_j;\sigma)) = 0,\ j \in N$$

$$\lambda_\ell \geq 0 \text{ and } \lambda_\ell h_\ell(\pi_i(q,\theta)) = 0,\ \ell \in K.$$

In fact, we show in the proof of Theorem 1 that the complementary slackness conditions hold strictly almost everywhere, i.e., $\lambda_j > 0$ for all $j$ such that $U_j(\pi_i(q,\theta),\theta_j;\sigma) = U_j(q,\theta_j;\sigma)$, and $\lambda_\ell > 0$ for all $\ell$ such that $h_\ell(\pi_i(q,\theta)) = 0$. This first order characterization of optimal proposals would seem necessary for a more detailed study of equilibria, either by analytical or numerical means.

We provide a sketch of the proof of Theorem 1, which is proved formally in the appendix. As expected, the proof proceeds by defining a suitable mapping, establishing the existence of a fixed point, and then verifying that it has the claimed properties. Let $C^\infty(\Re^d,\Re^n)$ denote the space of smooth, bounded mappings from $\Re^d$ to $\Re^n$ (endowed with the topology of $C^\infty$-uniform convergence on compacta, as described in Mas-Colell (1985)). We define $U_i(y,\theta;v)$ and $A_i(q,\theta;v)$, in the obvious way, as the induced utilities and acceptance sets when continuation values are given by $v$. We proceed in a number of steps

1. We note that for every status quo $q$ and almost all shocks $\theta$, the maximization problem

   $$\begin{array}{c} \max_y U_i(y,\theta_i;v) \\ \text{s.t. } y \in A(q,\theta;v) \end{array}$$

   has a unique solution, which we denote $\pi_i(q,\theta;v)$, almost everywhere. Elsewhere, we define the function $\pi_i(\cdot;v)$ by arbitrarily selecting from the set of maximizers.

2. We show that for every status quo $q$, almost all shocks $\theta$, and every legislator $i$, if $i$'s proposal is distinct from the status quo and makes any other legislator indifferent between accepting and rejecting, then the proposal garners a bare majority of votes, i.e., the coalition that forms contains no redundant legislators.

3. By an application of the transversality theorem, we show that for every status quo $q$, almost all shocks $\theta$, and every policy $y \in A(q, \theta; v)$ distinct from $q$, the LICQ holds. This means that if the vector of continuation value functions is perturbed slightly, the proposer can find proposals arbitrarily close to $y$ that continue to satisfy the voting constraints.

4. For every status quo, almost all shocks $\theta$, and every legislator $i$, LICQ implies that the necessary first and second order conditions for a constrained maximum are satisfied. By an application of the transversality theorem, we show that the complementary slackness conditions actually hold strictly, so that binding constraints are associated with positive multipliers.

5. We can then argue, by another application of the transversality theorem, that for every status quo and almost all shocks $\theta$, every legislator's proposal $\pi_i(q, \theta; v)$ is differentiable in $(q, \theta)$ whenever the legislator proposes the status quo.

6. We show by a maximum theorem-type argument that for every status quo $q$, almost all shocks $\theta$, and every legislator $i$, the optimal proposal $\pi_i(q, \theta; v)$ is continuous in $v$.

7. We define the subset $\mathcal{V} \subseteq C^\infty(\Re^d, \Re^n)$ to consist of all mappings $v$ that are bounded by $c$ and such that the partial derivatives of order $1, 2, \ldots$ share a particular bound. This set is nonempty, convex, and closed. We show that the set of $r$th order derivatives of functions in $\mathcal{V}$ is equicontinuous for each $r = 0, 1, 2, \ldots$, and it follows that $\mathcal{V}$ is compact.

8. We define the mapping $\Psi(v) = \hat{v}$, where

$$\hat{v}_i(x) = \int_q \int_\theta \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) g(q|x) d\theta dq. \qquad (1)$$

That is, $\Psi(v)$ is the vector of continuation value functions generated by optimal behavior when continuation values are given by $v$. By differentiating $\hat{v}_i$, we find that $\Psi(\mathcal{V}) \subseteq \mathcal{V}$.

9. Using continuity of the proposal strategies $\pi_i(q, \theta; v)$ almost everywhere, we prove that the mapping $\Psi$ is continuous.
10. Finally, we use standard fixed point arguments to show that the mapping $\Psi$ has a fixed point $v^*$.

From $v^*$, we define equilibrium proposal strategies $\pi_i(q, \theta; v^*)$ and voting strategies $A_i(q, \theta; v^*)$. It follows immediately from (1) and the properties of $g$ that the continuation values $v_i^*$ are smooth, and the other properties in Theorem 1 follow from our equilibrium construction.

The operator constructed in the proof of Theorem 1 can be used to give a somewhat more general characterization result: since every legislative equilibrium continuation value $v$ must be a fixed point of $\Psi$, it follows that the properties described in Theorem 1 are necessarily satisfied by all stationary legislative equilibria.

**Theorem 2** *Every stationary legislative equilibrium satisfies properties 1–5 of Theorem 1.*

With existence of a stationary legislative equilibrium $\sigma^*$ proved, it is of interest to consider the equilibrium dynamics of policy outcomes in the model. In so doing, we define the transition probability on policy outcomes by

$$P(x, Y) \;=\; \int_q \int_\theta \sum_{i \in N} p_i I_Y(\pi_i^*(q, \theta)) f(\theta) g(q|x) d\theta dq,$$

which is the probability, conditional on policy outcome $x$ this period, of a policy outcome in the set $Y \subseteq X$ next period. We define the associated Markov operator $T$ on the space of bounded, Borel measurable functions $\phi\colon X \to \Re$ by $T\phi(x) = \int \phi(z) P(x, dz)$. The adjoint $T^*$ operates on the Borel measures on $X$ and is defined by $T^*\mu(Y) = \int P(x, Y)\mu(dx)$. This describes, given a distribution of policy outcomes in the current period, the distribution of outcomes in the next period. The iterates of $T^*$, denoted $T^{*m}$, give the distribution of policy outcomes $m$ periods hence and are therefore key in describing the long run policy outcomes of the model.

It is straightforward to show that $T$ maps continuous functions to continuous functions and, therefore, satisfies the Feller property. Furthermore, it is tight, so that it admits an invariant distribution $\mu$ such that $\mu = T^*\mu$.

[More on uniqueness to be inserted here.]

Our equilibrium construction also informs us of subgame perfect equilibria (in stage-undominated voting strategies) in the finite-horizon version of our model. Let $v^0$ be the profile of zero functions. In the two-period version of the model, the continuation value from policy outcomes in the first period is given uniquely by $\Psi(v^0)$. This in turn generates essentially unique proposal strategies in the first period. By induction, the continuation value following the first period of the $T$-period version of the model is given by $\Psi(v^{T-2})$. Thus, our construction yields the (essentially) unique subgame perfect equilibrium of the finite-horizon model.

# 4 The Full Model

[Much more on the full model to be inserted here.]

# A Proofs of Theorems

**Proof of Theorem 1** Let $C_b^\infty(\Re^d, \Re^n)$ be the smooth, bounded functions from $\Re^d$ into $\Re^n$ with the topology of $C^\infty$-uniform convergence on compacta, so that a sequence $\{\phi^m\}$ of functions converges to $\phi$ if and only if for every compact set $Y \subseteq \Re^d$ and each $r = 0, 1, 2, \ldots$, all partial derivatives of order $r$ of $\phi^m - \phi$ converge uniformly to zero on $Y$. Given $v = (v_1, \ldots, v_n) \in C_b^\infty(\Re^d, \Re^n)$, define the induced utility

$$U_i(y, \theta_i; v) = (1 - \delta)(u_i(y) + \theta_i \cdot y) + \delta v_i(y),$$

where future payoffs are assumed to be generated by $v$, and define the associated acceptance sets

$$A_i(q, \theta; v) = \{y \in X \mid U_i(y, \theta_i; v) \geq U_i(q, \theta_i; v)\}.$$

By Mas-Colell's (1985) Theorem K.1.2, the function $U_i$ is jointly continuous in $(y, \theta_i, v)$, and it follows that the correspondence $A_i$ has closed graph in $(q, \theta, v)$.[9]

Steps 1–6 take a profile $v = (v_1, \ldots, v_n) \in C_b^\infty(\Re^d, \Re^n)$ as given.

1. *Selection of maximizers.* For each $i$, $q$, and $\theta$, $U_i(y, \theta; v)$ is continuous in $y$ and $A(q, \theta; v)$ is compact, and therefore the maximization problem

$$\begin{array}{c} \max_y U_i(y,\theta;v) \\ \text{s.t. } y \in A(q,\theta;v) \end{array}$$

admits at least one solution. By Aliprantis and Border's (1999) Theorem 17.18, the correspondence of maximizers admits a measurable selection, which we denote $\pi_i(\cdot; v) \colon X \times \Theta \to X$. Let

$$A_{-i}(q, \theta_{-i}; v) = \bigcup \left\{ A_C(q, \theta; v) \mid C \subseteq N, \#C \geq \frac{n-1}{2}, i \notin C \right\}$$

denote the set of proposals that will pass assuming $i$ votes to accept, and note that $\pi_i(q, \theta; v)$ also solves

$$\begin{array}{c} \max_y U_i(y,\theta_i;v) \\ \text{s.t. } y \in A_{-i}(q,\theta_{-i};v). \end{array}$$

Then, by Mas-Colell's (1985) Theorem I.3.1, for all $q$, all $\theta_{-i}$, and almost all $\theta_i$, $\pi_i(q, \theta; v)$ is uniquely defined. Let $\Theta_i(q; v)$ be the measure zero set of vectors $\theta$ such that the maximizer $\pi_i(q, \theta; v)$ is not unique.

2. *Claim: For all $q$, all $i$, and almost all $\theta$, if $\pi_i(q, \theta; v) \neq q$, and if there exists $k \neq i$ such that $U_k(\pi_i(q, \theta; v), \theta_k; v) = U_k(q, \theta_k; v)$, then*

$$|\{j \in N \setminus \{i\} \mid U_j(\pi_i(q, \theta; v), \theta_j) \geq U_j(q, \theta_j)\}| = \left\lceil \frac{n-1}{2} \right\rceil .$$

Fix $q$ and $i$ arbitrarily. For each $C$ with $i \notin C$, define $\pi_i^C(q, \theta; v)$ as an arbitrary solution to

$$\max_{y \in A_C(q,\theta;v)} U_i(y, \theta_i; v).$$

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[9] Note that linear independence of $\{Dh_\ell(y) \mid \ell \in L(y)\}$ implies that $X$ is contained in the closure of its interior, as assumed by Mas-Colell.

By Theorem I.3.1 of Mas-Colell's (1985), for all $\theta_{-i}$, this solution is uniquely defined outside a measure zero set of $\theta_i$'s, so let $\tilde{\Theta}_i^C(\theta_{-i}, q; v)$ denote the measure zero set of $\theta_i$ where $i$'s optimal proposal to $C$ is not unique, given $\theta_{-i}$. Let $\tilde{\Theta}_i(\theta_{-i}, q; v)$ denote the union of these sets, and let

$$\tilde{\Theta}_i(q; v) = \{\theta \mid \theta_i \in \tilde{\Theta}_i(\theta_{-i}, q; v)\},$$

which is measure zero in $\Theta$. Now take two distinct majority coalitions $C$ and $C'$, and take any $\theta$. Suppose that $\pi_i^C(q, \theta; v) = \pi_i^{C'}(q, \theta; v) \neq q$ and that there exists $j \in C \setminus C'$ such that $U_j(\pi_i^C(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v)$. This equality defines a hyperplane in $\Theta$, which is lower-dimensional. If $\theta \notin \tilde{\Theta}(q; v)$, then proposals are uniquely determined, so we may define

$$\Theta_i^*(q; v) = \left\{ \theta \notin \tilde{\Theta}_i(q; v) \;\middle|\; \begin{array}{c} \text{there exist } C, C' \text{ and } j \in C \setminus C' \text{ such} \\ \text{that } \pi_i^C(q, \theta; v) = \pi_i^{C'}(q, \theta; v) \neq q \\ \text{and } U_j(\pi_i^C(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v) \end{array} \right\},$$

which is measure zero by Fubini's theorem (see Aliprantis and Border's (1999) Theorem 11.26). By Step 1, outside the measure zero set $\Theta_i(q; v)$, player $i$ has a unique global maximizer $\pi_i(q, \theta; v)$. Consider $\theta \in \Theta \setminus (\Theta_i^*(q; v) \cup \Theta_i(q; v))$ and suppose that $\pi_i(q, \theta; v) \neq q$ and $U_k(\pi_i(q, \theta; v), \theta_k; v) = U_k(q, \theta_k; v)$ for some $k \neq i$. Now let

$$C = \{j \in N \setminus \{i\} \mid U_j(\pi_i(q, \theta; v), \theta_j; v) \geq U_j(q, \theta_j; v)\},$$

and suppose, to obtain a contradiction, that $|C| \geq \left\lceil \frac{n+1}{2} \right\rceil$. Let $C' = C \setminus \{k\}$, and note that $\pi^C(q, \theta; v) = \pi^{C'}(q, \theta; v) = \pi_i(q, \theta; v) \neq q$. This contradicts $U_k(\pi_i(q, \theta; v), \theta_k; v) = U_k(q, \theta_k; v)$, since $\theta \notin \Theta_i^*(q; v)$. Thus, the conclusion of the claim holds outside the set of measure zero $\Theta_i^*(q; v) \cup \Theta_i(q; v)$.

We now give notation used in Steps 3–5. For arbitrary subsets $L \subseteq K$ and $C \subseteq N$, define the functions $h \colon \Re^d \times \Theta \to \Re^{|L|}$ by $h(y, \theta) = (h_\ell(y))_{\ell \in L}$ and $U \colon \Re^d \times \Theta \to \Re^{|C|}$ by $U(y, \theta; v) = (U_i(y, \theta_i; v) - U_i(q, \theta_i; v))_{i \in C}$. Define the mapping $F^{L,C} \colon (\Re^d \setminus \{q\}) \times \Theta \to \Re^{|L|+|C|}$ by

$$F^{L,C}(y, \theta; v) = \begin{pmatrix} h(y, \theta) \\ U(y, \theta; v) \end{pmatrix},$$

where here (and whenever relevant) we view vectors as column matrices, making $F^{L,C}(y, \theta; v)$ a $(|L| + |C|) \times 1$ matrix.

3. *Claim: for all $q$, almost all $\theta$, and all $y \in A(q,\theta;v) \setminus \{q\}$, LICQ is satisfied at $y$.* Fix $q$ arbitrarily, and consider any $L \subseteq K$ and $C \subseteq N$. Then $DF^{L,C}(y,\theta;v)$ is the $(|L|+|C|) \times (d+nd)$ matrix

$$\left( \begin{array}{ccc} Dh(y) & 0 & 0 \\ D_yU(y,\theta;v) & (1-\delta)(y-q)^T \otimes I_{|C|} & 0 \end{array} \right),$$

where $\otimes$ denotes Kronecker product. For all $(y,\theta)$ such that $F^{L,C}(y,\theta;v)=0$, it follows that $L$ is contained in the binding feasibility constraints at $y$, i.e., $L \subseteq L(y)$, and therefore rows $1,\ldots,|L|$ are linearly independent by assumption. Since $y \neq q$, rows $|L|+1,\ldots,|L|+|C|$ are linearly independent, and we see that $DF^{L,C}(y,\theta;v)$ has full row rank. We conclude that $F^{L,C}$ is transversal to $\{0\}$. For each $\theta$, define $F^{L,C}_\theta \colon \Re^d \setminus \{q\} \to \Re^{|L|+|C|}$ by $F^{L,C}_\theta(y;v) = F^{L,C}(y,\theta;v)$. By an application of the transversality theorem (see Mas-Colell's (1985) Theorem I.2.2), it follows that for almost all $\theta$, $F^{L,C}_\theta \pitchfork \{0\}$, i.e., 0 is a regular value of $F^{L,C}_\theta$. Let $\Theta^{L,C}(q;v)$ be the measure zero set of $\theta$'s where this does not hold. Let $\Theta(q;v)$ be the finite union of these, which has measure zero. Now take any $\theta \notin \Theta(q;v)$ and any $y \in A(q,\theta;v) \setminus \{q\}$, and let $L$ and $C$ represent the constraints satisfied with equality at $y$. Then $y \in (F^{L,C}_\theta)^{-1}(\{0\})$, so $DF^{L,C}_\theta(y)$ has full rank, i.e.,

$$\{Dh_\ell(y), D_yU_i(y,\theta_i) \mid \ell \in L, i \in C\}$$

is linearly independent, fulfilling LICQ.

Before we move to Step 4, we develop some necessary notation. For any legislator $i$ and any $L \subseteq K$ and $C \subseteq N$, define the function $\mathcal{L}^{L,C}_i \colon \Re^d \times \Re^{|L|+|C|} \to \Re$ by

$$\mathcal{L}^{L,C}_i(y,\lambda,\theta;v) = U_i(y,\theta_i;v) + \sum_{\ell \in L} \lambda_\ell h_\ell(y) + \sum_{j \in C} \lambda_j (U_j(y,\theta_j;v) - U_j(q,\theta_j;v)).$$

Further, define $G^{L,C}_i \colon (\Re^d \setminus \{q\}) \times \Re^{|L|+|C|} \times \Theta \to \Re^d \times \Re^{|L|+|C|}$ by

$$G^{L,C}_i(y,\lambda,\theta;v) = \left( \begin{array}{c} D_y\mathcal{L}^{L,C}_i(y,\lambda,\theta;v) \\ F^{L,C}(y,\theta;v) \end{array} \right).$$

where $F^{L,C}(y,\theta;v)$ is defined prior to Step 3.

4. *Claim: for all $q$, all $i$, almost all $\theta$, all $L \subseteq K$, all $C \subseteq N$, and every $(y,\lambda)$ such that $G^{L,C}_i(y,\lambda,\theta;v)=0$, we have $\lambda_m \neq 0$ for all $m \in$*

$L \cup C$ *and* $D_{(y,\lambda)} G_i^{L,C}(y,\lambda,\theta;v)$ *is non-singular.* Fix $q$ and $i$ arbitrarily, and consider any $L \subseteq K$ and $C \subseteq N$. For $m \in C \cup L$, define the mapping $G_i^{L,C,m} \colon (\Re^d \setminus \{q\}) \times \Re^{|L|+|C|} \times \Theta \to \Re^d \times \Re^{|L|+|C|} \times \Re$ by

$$G_i^{L,C,m}(y,\lambda,\theta;v) = \begin{pmatrix} G_i^{L,C}(y,\lambda,\theta;v) \\ \lambda_m \end{pmatrix}.$$

Then the derivative $DG_i^{L,C,m}(y,\lambda,\theta;v)$ is the $(d+|L|+|C|+1) \times (d+|L|+|C|+nd)$ matrix

$$\begin{pmatrix} D_{yy}\mathcal{L}^{L,C}(y,\lambda,\theta;v) & D_y F^{L,C}(y,\theta;v)^T & (1-\delta)I_d & ((1-\delta)\lambda_j I_d)_{j \in C} \\ Dh(y) & 0 & 0 & 0 \\ D_y U(y,\theta;v) & 0 & 0 & (1-\delta)(y-q)^T \otimes I_{|C|} \\ 0 & 0 \cdots 1 \cdots 0 & 0 & 0 \end{pmatrix}.$$

(We omit zero-columns corresponding to derivatives with respect to $\theta_j$, $j \notin C \cup \{i\}$.) For all $(y,\lambda,\theta)$ such that $G_i^{L,C,m}(y,\lambda,\theta;v) = 0$, this derivative evidently has maximal rank. Then, by the transversality theorem, for almost all $\theta \in \Theta$, the mapping $G_{i,\theta}^{L,C,m} \colon (\Re^d \setminus \{q\}) \times \Re^{|L|+|C|} \to \Re^d \times \Re^{|L|+|C|} \times \Re$ defined by $G_{i,\theta}^{L,C,m}(y,\lambda;v) = G_i^{L,C,m}(y,\lambda,\theta;v)$ is transversal to $\{0\}$. Since the dimension of the domain of $G_{i,\theta}^{L,C,m}$ is smaller than that of the range, the preimage theorem (see Mas-Colell's (1985) Theorem H.2.2) implies that $(G_\theta^{L,C,m})^{-1}(\{0\})$ is empty for almost all $\theta \in \Theta$, i.e., outside a set $\hat{\Theta}_i^{L,C,m}(q;v)$ of measure zero, $G_i^{L,C,m}(y,\lambda,\theta;v) \neq 0$. Furthermore, defining $G_{i,\theta}^{L,C} \colon (\Re^d \setminus \{q\}) \times \Re^{|L|+|C|} \to \Re^d \times \Re^{|L|+|C|}$ by $G_{i,\theta}^{L,C}(y,\lambda;v) = G_i^{L,C}(y,\lambda,\theta;v)$, an application of the transversality theorem to the mapping $G_i^{L,C}$ ensures that outside a set $\Theta^{L,C}(q;v)$ of measure zero, $G_{i,\theta}^{L,C}(y,\lambda;v)$ is also transversal to $\{0\}$. Repeating the above arguments for all $L, C$ and all $m \in L \cup C$, we conclude that outside a set $\hat{\Theta}_i(q;v)$ of measure zero in $\Theta$, every solution $(y,\lambda)$ to $G_i^{L,C}(y,\lambda,\theta;v) = 0$ for any $L$ and $C$ is such that $\lambda_m \neq 0$ for all $m \in L \cup C$ and $D_{(y,\lambda)} G_i^{L,C}(y,\lambda,\theta;v)$ is non-singular.

5. *Claim: for all* $q$*, all* $i$*, and almost all* $\theta$*, if* $\pi_i(q,\theta;v) \neq q$*, then it is differentiable in* $(q,\theta)$*.* Fix $q$ and $i$ arbitrarily. Borrowing from Steps 1–4, consider $\theta \in \Theta \setminus (\Theta_i(q;v) \cup \Theta(q;v) \cup \hat{\Theta}(q;v) \cup \Theta_i^*(q;v))$, and assume $\pi_i(q,\theta;v) \neq q$. Since $\theta \notin \Theta_i(q;v)$, $\pi_i(q,\theta;v)$ is a unique global maximizer by Step 1. Suppose the binding constraints are given by $L$ and $C$, which may be empty. By Step 2, there exists $C' \subseteq N$ such that $\pi_i(q,\theta;v)$ is the solution

to

$$\max_{y \in A_{C'}(q,\theta;v)} U_i(y, \theta_i; v), \tag{2}$$

where $C \subseteq C'$, $i \notin C'$, and $|C'| = \lceil \frac{n-1}{2} \rceil$. Thus, borrowing from Step 2, $\pi_i(q, \theta; v) = \pi_i^{C'}(q, \theta; v)$. We claim that $\pi_i^{C'}(\cdot; v)$ is continuous at $(q, \theta)$. To see this, consider any sequence $\{(q^m, \theta^m)\}$ converging to $(q, \theta)$, and let $y$ be any accumulation point of $\{\pi_i^{C'}(q^m, \theta^m; v)\}$. By closed graph of $A$, we have $y \in A(q, \theta; v)$, and uniqueness of a global maximizer implies $\pi_i(q, \theta; v) = y$. Since $X$ is compact, this proves the claim. We now claim that there is an open set around $(q, \theta)$ such that $i$'s optimal proposal solves (2) over that set. Indeed, suppose the claim is false. Then there is a sequence $\{(q^m, \theta^m)\}$ converging to $(q, \theta)$ and a sequence $\{C^m\}$ of coalitions $C^m \neq C'$, with $i \notin C^m$ and $|C^m| \geq \lceil \frac{n-1}{2} \rceil$, such that $U_i(\pi_i^{C^m}(q^m, \theta^m; v), \theta_i^m; v) > U_i(\pi_i^{C'}(q^m, \theta^m; v), \theta_i^m; v)$ for all $m$. By finiteness of $N$ and compactness of $X$, we may suppose that $C^m = C''$ for all $m$ and $\pi_i^{C''}(q^m, \theta^m; v) \to y$ for some $y \in X$. By closed graph of $A_{C''}$, we then have $y \in A(q, \theta; v)$, and continuity implies $U_i(y, \theta_i; v) \geq U_i(\pi_i^{C'}(q, \theta; v), \theta_i; v)$. By uniqueness of the global maximizer, we then have $y = \pi_i^{C'}(q, \theta; v)$. Then continuity implies that

$$C'' \subseteq \{j \in N \setminus \{i\} \mid U_j(\pi_i^{C'}(q, \theta; v), \theta_j; v) \geq U_j(q, \theta_j; v)\}.$$

If $C \neq \emptyset$, then, by Step 2, $C' = C''$, a contradiction. If $C = \emptyset$, then for all $j \neq i$, we have $U_j(\pi_i^{C'}(q, \theta; v), \theta_j; v) \neq U_j(q, \theta_j; v)$, and by continuity there are open sets $Y$ around $\pi_i^{C'}(q, \theta; v)$ and $Z$ around $(q, \theta)$ such that for all $\hat{y} \in Y$, all $(\hat{q}, \hat{\theta}) \in Z$, and all $j \neq i$, we have $U_j(\hat{y}, \hat{\theta}_j; v) > U_j(\hat{q}, \hat{\theta}_j; v)$ if and only if $U_j(\pi_i^{C'}(q, \theta; v), \theta_j; v) > U_j(q, \theta_j; v)$. This implies that for all $(\hat{q}, \hat{\theta}) \in Z$, we have $Y \subseteq A_{C'}(\hat{q}, \hat{\theta}; v) \subseteq A(\hat{q}, \hat{\theta}; v)$. But for high enough $m$, we have $\pi_i^{C''}(q^m, \theta^m; v) \in Y$ and $(q^m, \theta^m) \in Z$, and then $U_i(\pi_i^{C''}(q^m, \theta^m; v), \theta_i^m; v) \leq U_i(\pi_i^{C'}(q^m, \theta^m; v), \theta_i^m; v)$, a contradiction that proves the claim. Now, as a consequence, it suffices for Step 5 to show differentiability of the solution $\pi_i^{C'}(q, \theta; v)$ to (2). By Theorems 2 and 3 in Fiacco and McCormick (1990), since LICQ is satisfied at $\pi_i^{C'}(q, \theta; v)$ by Step 3, the first and second order necessary conditions must hold, i.e.,

(i) $G_\theta^{L,C}(\pi_i^{C'}(q, \theta; v), \lambda; v) = 0$,

(ii) for all $m \in L \cup C$, $\lambda_m \geq 0$,

(iii) for all $z$ such that $D_y F^{L,C}(\pi_i^{C'}(q,\theta;v),\theta;v)z = 0$, we have $z^T D_{yy}\mathcal{L}_i^{L,C}(\pi_i^{C'}(q,\theta;v),\lambda,\theta;v)z \leq 0$.

By Step 4, we have $\lambda_m \neq 0$, and then (ii) implies that $\lambda_m > 0$ for all $m \in L \cup C$. By Step 4, $D_{(y,\lambda)}G_i^{L,C}(\pi_i^{C'}(q,\theta;v),\lambda,\theta;v)$ is non-singular, and therefore the second order necessary condition implies that the second order sufficient condition is satisfied at $\pi_i^{C'}(q,\theta;v)$, i.e., for all $z \neq 0$ such that $D_y F^{L,C}(\pi_i^{C'}(q,\theta;v),\theta;v)z = 0$, $z^T D_{yy}\mathcal{L}_i^{L,C}\left(\pi_i^{C'}(q,\theta;v),\lambda,\theta\right)z < 0$. For if, instead, $z^T D_{yy}\mathcal{L}_i^{L,C}\left(\pi_i^{C'}(q,\theta;v),\lambda,\theta\right)z = 0$ for some $z \neq 0$ such that $D_y F^{L,C}(\pi_i^{C'}(q,\theta;v),\theta)z = 0$, then

$$D_{(y,\lambda)}G_i^{L,C}(\pi_i^{C'}(q,\theta;v),\lambda,\theta;v)\begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is impossible since $D_{(y,\lambda)}G_\theta^{L,C}(\pi_i^{C'}(q,\theta;v),\lambda,\theta;v)$ is invertible and $z \neq 0$. Thus, for almost all $\theta$, $\pi_i^{C'}(q,\theta;v) \neq q$ satisfies all the conditions of Theorem 6 of Fiacco and McCormick (1990), and we conclude that it is differentiable in an open set around $q,\theta$.

6. *Claim: for all $q$, all $i$, all $\{v^m\}$ converging to $v$, and almost all $\theta$, we have $\pi_i(q,\theta;v^m) \to \pi_i(q,\theta;v)$.* Fix $q$ and $i$ arbitrarily, and consider any sequence $\{v^m\}$ converging to $v$. Borrowing from Steps 1 and 3, let

$$\Theta_i(q,\{v^m\}) \;\; = \;\; \Theta_i(q;v) \cup \Theta(q;v) \cup \bigcup_{m=1}^{\infty} \Theta(q;v^m),$$

which has measure zero. Take any $\theta \notin \Theta_i(q,\{v^m\})$, and suppose $\{\pi_i(q,\theta;v^m)\}$ does not converge to $\pi_i(q,\theta;v)$. Since $X$ is compact, we may go to a subsequence such that $\pi_i(q,\theta;v^m) \to y \neq \pi_i(q,\theta;v)$ for some $y \in X$. By continuity, $A(q,\theta;v)$ has closed graph, so we know that $y \in A(q,\theta;v)$. Since $\pi_i(q,\theta;v)$ is uniquely defined, by $\theta \notin \Theta_i(q;v)$, and is distinct from $y$, we have $U_i(\pi_i(q,\theta;v),\theta_i;v) > U_i(y,\theta_i;v)$. Case 1: $\pi_i(q,\theta;v) \neq q$. Since $\pi_i(q,\theta;v) \in A(q,\theta;v) \setminus \{q\}$, there exists some majority coalition $C$ such that $\pi_i(q,\theta;v) \in A_C(q,\theta;v) \setminus \{q\}$. By Step 3, we know that LICQ is satisfied at $\pi_i(q,\theta;v)$, so there exists a direction $s$ such that $Dh_\ell(y)\cdot s > 0$ and $DU_i(q,\theta_i;v)\cdot s > 0$ for all binding $\ell$ and $i \in C$. Choose $\epsilon > 0$, small enough that $\pi_i(q,\theta;v) + \epsilon s \in A_C(q,\theta;v) \subseteq A(q,\theta;v)$ and $U_i(\pi_i(q,\theta;v) + \epsilon s,\theta_i;v) > U_i(y,\theta_i;v)$. Since no constraints are binding at $\pi_i(q,\theta;v) + \epsilon s$ for $v$, we have $\pi_i(q,\theta;v) + \epsilon s \in$

$A_C(q, \theta; v^m)$ for large enough $m$. By continuity, we have $U_i(\pi_i(q, \theta; v) + s, \theta_i; v^m) \to U_i(\pi_i(q, \theta; v) + s, \theta_i; v)$ and $U_i(\pi_i(q, \theta; v^m), \theta_i; v^m) \to U_i(y, \theta_i; v)$, but then $U_i(\pi_i(q, \theta; v) + s, \theta_i; v^m) > U_i(\pi_i(q, \theta; v^m), \theta_i; v^m)$ for large enough $m$, contradicting optimality of $\pi_i(q, \theta_i; v^m)$. Case 2: $\pi_i(q, \theta; v) = q$. Then $U_i(q, \theta_i; v) > U_i(y, \theta_i; v)$, and, with the fact that $q$ is always a feasible choice, continuity again leads to a contradiction with optimality of $\pi_i(q, \theta; v^m)$.

7. *The domain of continuation values.* Define $\mathcal{V}$ to consist of the functions $v \in C_b^\infty(\Re^d, \Re^n)$ that are bounded by $c$ and such that for all $i$ and all partial derivatives of order $1, 2, \ldots$ of $v_i$ are bounded by $\lambda bc$, where $\lambda$ is the Lebesgue measure of $X$. Clearly, $\mathcal{V}$ is a nonempty, convex, closed, and bounded subset of $C_b^\infty(\Re^d, \Re^n)$. For each $i$ and each $v \in \mathcal{V}$, let $w_i(x; v)$ be an arbitrary partial derivative of order $r = 0, 1, 2, \ldots$ of $v_i$. For every compact $Y \subseteq \Re^d$, the set of functions

$$\{(w_1(\cdot; v)|_Y, \ldots, w_n(\cdot; v))|_Y \mid v \in \mathcal{V}\}$$

is equicontinuous. Indeed, given $\epsilon > 0$, set $\delta = \epsilon/\lambda bc\sqrt{nd}$. Suppose $x, y \in Y$ satisfy $||x - y|| < \delta$. Since the partial derivatives of $\phi(\cdot; v)$ are bounded by $\lambda bc$, we have $|w_i(x; v) - w_i(y; v)| \leq ||x - y||\lambda bc\sqrt{d} < \epsilon/\sqrt{n}$. Therefore, by Mas-Colell's (1985) Theorem K.2.2, $\mathcal{V}$ is compact.

8. *The fixed point mapping.* We define the mapping $\Psi\colon \mathcal{V} \to \mathcal{V}$ as follows. Let $\Psi(v) = \hat{v}$, where

$$\hat{v}_i(x) = \int_q \int_\theta \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) g(q|x) d\theta dq.$$

Let $\phi(q, x)$ be an arbitrary partial derivative of order $r = 0, 1, 2, \ldots$ of $g(q|x)$ with respect to the coordinates of $x$, so that by Mas-Colell's (1985) Theorem E.5.2, the corresponding partial derivative of $\hat{v}_i(x)$ is

$$w_i(x; v) = \int_q \int_\theta \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) \phi(q, x) d\theta dq.$$

Note that

$$\begin{aligned} |w_i(x; v)| &\leq \int_q \left( \int_\theta \sum_{j \in N} p_j |U_i(\pi_j(q, \theta; v), \theta_i; v)| f(\theta) d\theta \right) |\phi(q|x)| dq \\ &\leq \int_q c |\phi(q|x)| dq. \end{aligned}$$

If $r = 0$, so that $\phi = g$, then the latter integral is less than or equal to $c$; if $r \geq 1$, then it is less than or equal to $\lambda bc$. Thus, $\Psi(\mathcal{V}) \subseteq \mathcal{V}$.

9. *Continuity of* $\Psi$. Let $\{v^m\}$ be a sequence in $\mathcal{V}$ such that $v^m \to v$. We need to show that $\Psi(v^m) \to \Psi(v)$. Let $\phi(q, x)$ be an arbitrary partial derivative of order $r = 0, 1, 2, \ldots$ of $g(q|x)$ with respect to the coordinates of $x$, and define $w_i(x; v)$ as in Step 8. We must prove that $w_i(\cdot; v^m) \to w_i(\cdot; v)$ uniformly, i.e, for all $\epsilon > 0$, there exists $m'$ such that for all $m \geq m'$ and all $x \in X$, we have $|w_i(x; v^m) - w_i(x; v)| < \epsilon$. If this does not hold, then there exists a subsequence $\{v^m\}$, still indexed by $m$, and a corresponding sequence $\{x^m\}$ in $X$ such that for all $m$, $|w_i(x^m; v^m) - w_i(x^m; v)| \geq \epsilon$. By compactness of $X$, we may further assume that $x^m \to x$ for some $x \in X$. Note that

$$w_i(x^m; v^m) = \int_{q,\theta} \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v^m), \theta_i; v^m) f(\theta) \phi(q, x^m) d(q, \theta).$$

Let

$$Z = \{(q, \theta) \in X \times \Theta \mid \theta \in \Theta(q, \{v^m\})\}$$

denote the set of $(q, \theta)$ pairs from Step 6 such that $\{\pi(q, \theta; v^m)\}$ may not converge to $\pi_i(q, \theta; v)$, and note that this set has Lebesgue measure zero in $X \times \Theta$. Take any $(q, \theta) \notin Z$. Then $\pi_j(q, \theta; v^m) \to \pi_j(q, \theta; v)$ for all $j$, and joint continuity of $U_i$ implies that

$$U_i(\pi_j(q, \theta; v^m), \theta_i; v^m) \phi(q, x^m) \to U_i(\pi_j(q, \theta; v), \theta_i; v) \phi(q, x).$$

Therefore, by Lebesgue's dominated convergence theorem (see Aliprantis and Border's (1999) Theorem 11.20), we have $w_i(x^m; v^m) \to w_i(x; v)$, a contradiction. We conclude that $\Psi$ is continuous.

10. *Existence of equilibrium.* By Mas-Colell's (1985) Theorem K.1.1, $C^\infty(\Re^d, \Re^n)$ is metrizable and therefore Hausdorff, and it is also locally convex. The space $\mathcal{V}$ inherits these properties, and therefore the Brouwer-Schauder-Tychonoff theorem (see Aliprantis and Border's (1999) Corollary 16.52) implies that $\Psi$ has a fixed point, $v^* \in \Psi(v^*)$. It is clear from our construction that the strategy profile $\sigma^*$ defined by $(\pi_i^*, A_i^*) = (\pi_i(\cdot; v^*), A_i(\cdot; v^*))$ for all $i$ is a stationary legislative equilibrium that satisfies properties 1–5 of the theorem.

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