Experiments on Decisions Under Uncertainty: A Theoretical Framework

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ABSTRACT. The analysis of lab data entails a joint test of the underlying theory and of subjects' conjectures regarding the experimental design itself, how subjects frame the experiment. We provide a theoretical framework for analyzing the impacts of such conjectures or frames. We use experiments of decision making under uncertainty as a case study. Absent restrictions on subjects' framing of the experiment, we show that any behavior is consistent with standard updating ("anything goes"), including that suggestive of anomalies such as over-confidence, excess belief stickiness, etc. When the experimental protocol restricts subjects' conjectures (plausibly, by generating information during the experiment), standard updating has non-trivial testable implications. Such "transparent" protocols restrict action reversals that Bayesian subjects exhibit when they are provided with additional information. In the extreme case in which the amount of information revealed is conjectured to be independent of the underlying realized uncertainty, Bayesian updating is tantamount to dynamic consistency.

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1. OVERVIEW

1.1 INTRODUCTION

Experiments studying behavior under uncertainty (be it regarding some underlying state, such as income in savings problems, or regarding others' behavior, as is the case in practically all strategic interactions) usually consist of three stages. First, the uncertainty is realized; Second, subjects are provided with partial information about that realization; Third, subjects choose an action. When analyzing data of such experiments, one essentially tests joint hypotheses regarding *responses* to the experimental design (namely, the realization of uncertainty and the information generation procedure) and subjects' *beliefs* about the design itself. The focus of this paper is the analysis of the potential consequences of such joint tests. We allow subjects to hold arbitrary conjectures about the experimental design and identify links between classes of conjectures subjects hold and the testable implications of theoretical predictions in the lab.

We concentrate on a general case study of laboratory decision making under uncertainty. Most experiments in this class involve some form of updating. Consequently, a natural first step in the theoretical analysis of such experiments pertains to experiments having to do with elicitation of beliefs, which are the experiments we study in this paper.

We consider experiments in which payoffs depend on some unknown state. After the state is realized, a subject is provided with some information regarding the state in the form of a sequence of potentially informative signals. Ultimately, the subject chooses one of several alternatives. Many extant experiments fall under this category (particularly when thinking of other subjects' actions as part of the state). For instance, consumption-saving experiments, individual experimentation and learning experiments, herding, and sequential voting experiments all naturally fall under this rubric (see, for example, Kagel and Roth, 1997 for an overview of experiments in different fields).

1.2 A MOTIVATING EXAMPLE

Before describing our formal results, consider the following example, providing a simple caricature of the structure of experiments of decision making under uncertainty.

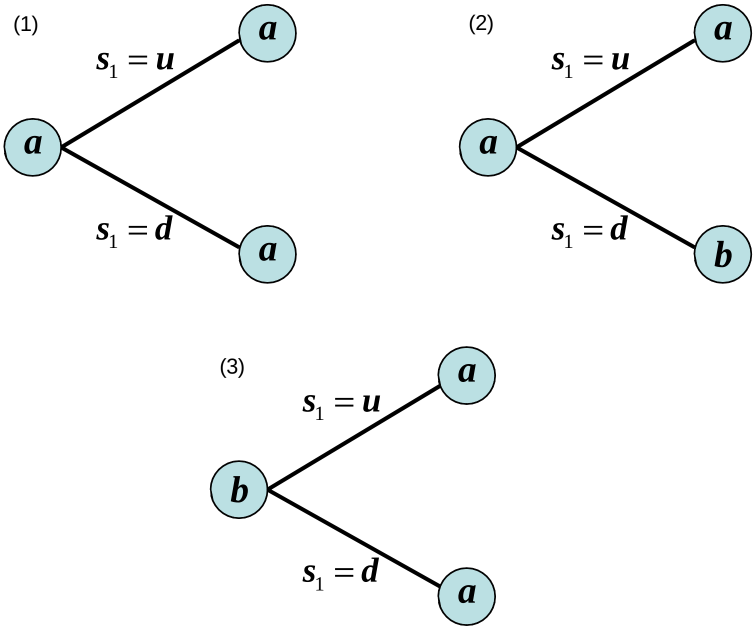

Figure 1: Simple Reversal

There are two states of nature, a and b, realized at the outset of the experiment. The subject's goal is to report which of the states is most likely given different amounts of information.[1] For simplicity, assume that initially, she is given no information. Then she receives some information, say in the form of a signal taking the values u or d. Figure 1 captures, up to relabeling, the three types of responses one could observe in such an experiment. The actions at the roots of each tree correspond to the reports absent any signals, while the other two nodes contain the subject's responses following the appropriate signals.

In panel (1), the subject does not change her mind regardless of which signal she observes. This is consistent with Bayesian updating. For instance, the subject may have a strong prior favoring the state a and view the signals as uninformative. In panel (2), the subject changes her mind when the signal is d. This can also be explained with Bayesian updating by assuming, e.g., that the prior for a is $p > 1/2$ while the signal s_1 follows the following distribution:

$$\mathbb{P}(s_1 = u \mid a) = \mathbb{P}(s_1 = d \mid b) = q \quad \text{and} \quad \mathbb{P}(s_1 = d \mid a) = \mathbb{P}(s_1 = u \mid b) = 1 - q,$$

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[1] For instance, she may be paid a fixed amount only when her response matches the realized state.

where $q > p$. Thus, upon observing the signal u, the subject views a as more likely and upon observing the signal d, the subject views the state b as more likely.[2]

Panel (3) describes a situation in which regardless of what the signal is, the subject changes her view of what is the most likely state. These observations cannot be explained with simple Bayesian updating. Indeed, if the subject updates in a standard manner, absent any information, she must put some weight on the realization of s_1 being u and complementary weight on the realization of s_1 being d. In particular, her belief that the state is a when she does not observe a signal must be a convex combination of her beliefs following the realization of the signal. Thus, unless the subject is indifferent, a belief that puts weight lower than 1/2 on the state a must be a convex combination of two beliefs that put weight greater than 1/2 on the state a, which is not possible.[3]

In fact, many experiments that are used to illustrate updating anomalies exhibit observations of the form of Panel (3) – Tversky and Shafir's (1992) disjunction effect experiments fall into this format[4], as do an assortment of experiments on over-confidence, belief polarization, etc. (for an overview, see, e.g., Camerer, Loewenstein, and Rabin, 2006 or Nisbett and Ross, 1980).

To summarize, even without placing any restrictions on the subject's belief about the initial distribution of states and the signal generation process, Bayesian updating has clear testable implications. Panels (1) and (2) can be explained, while Panel (3) cannot.

In the current paper we also entertain the idea that the subject conjectures the *amount* of information revealed to her by the experimenter is a function of the realized uncertainty: the state and the signals. Then, Panel (3) can easily be explained with Bayesian updating with such richer conjectures. For instance, the subject could conjecture that a-priori the states are equally

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[2] In fact, one can come up with many belief structures for the subject that would be consistent with the observed reports and Bayesian updating for either Panel (1) or Panel (2).

[3] This is reminiscent of a simple violation of Savage's (1954) "Sure Thing Principle." We elaborate on the comparison below.

[4] For example, Tversky and Shafir (1992) asked students whether they would be interested in a vacation package that would take place after the results of an exam would be revealed. 32% of the subjects were interested, 7% of the subjects were not interested, and the remaining 61% were willing to pay \$5 to have the option to cancel the vacation after realizing whether they had passed or failed. Nonetheless, when asked whether they would buy the same vacation package *after* discovering they had failed or passed the exam, 57% or 54%, resepectively, of the subjects reported affirmatively.

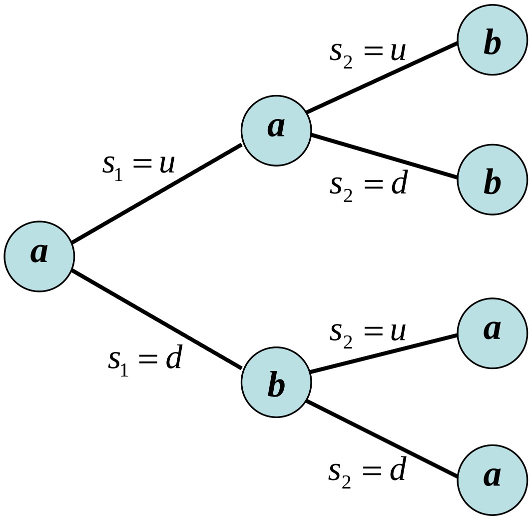

Figure 2: Reversal Not Explained by a Partially Restricted Conjectured Experiment

likely, but that when the state is b (or when the appropriate response is b), the experimenter does not reveal any information to her, while when the state is a, the experimenter reveals either signal with equal probability. In particular, behavior that seems to suggest anomalous updating procedures may be attributed to the subject's framing of the experiment instead.

Nonetheless, note that in explaining Panel (3), we only needed the subject to conjecture the amount of information depends on the realized *state* (and not necessarily on the realized signals). Such restricted conjectures are relevant for experiments of decisions under uncertainty in which there is transparency regarding how signals are generated (for example, signals may be generated during the experiment, *after* their number has been determined, as is common practice in, e.g., voting experiments). Interestingly, they do entail testable implications.

Consider Figure 2 that illustrates reports from an experiment as above in which there are two signal realizations, each taking the value of u or d. Note that the sub-tree corresponding to $s_1 = u$ is of the form appearing in Panel (3) of Figure 1. In particular, in order to explain the reversal following either signal s_2, we need to assume the subject conjectures that the mere observation of two signals is more likely when the state is b. However, switch to the other sub-tree corresponding to $s_1 = d$. Here, we see a similar reversal, but now the subject reports the state a regardless of s_2. The only way to explain this would be with a conjecture that makes the observation of two signals more likely when the state is a. It is therefore impossible to explain these observations with such restricted conjectures. Thus, transparency of the signal generation procedure assures that Bayesian updating has clear testable implications, even without restricting the subject's conjecture on the distribution of the underlying state, signal generation process, *and* the experimenter's strategy of information revelation (that may depend on the realized state).

This example highlights two important methodological messages of the paper. First, when no restrictions are placed on subjects' framing of the experiment, any behavior is consistent with standard updating ("anything goes"). Second, when the experimental protocol restricts subjects' conjectures (say, by generating information during the experiment), standard updating entails non-trivial testable implications. Namely, such protocols restrict the type of action reversals that Bayesian subjects may exhibit when they are provided with additional information.

1.3 DESCRIPTION OF RESULTS

In general, the design of the experiment dictates how much information, i.e., the number of signals, the subjects observe prior to making choices. Subjects may hold a wide range of beliefs regarding the link between the amount of information they observe and the underlying uncertainty. We use the term *conjectured experiment* to denote the subject's beliefs about the design of the experiment. We study the impacts of two types of conjectured experiments: one in which the amount of information is perceived independent of any realized uncertainty and another in which the amount of information can be correlated (in an arbitrary manner) with the realized uncertainty.

It is important to emphasize that in experiments of decision making under uncertainty, the amount of information per-se is usually independent of the actual state and the signals' realizations. There are many reasons why a subject's conjecture about the experiment may not coincide with the actual experimental design (say, due to the subject's misunderstanding of the notion of independence, the subject's theory about the experimenter's goals and selective choice of information, and so on; See Friedman and Sunder, 1994, and Kagel and Roth, 1997). We stress that subjects' conjectures do not necessarily reflect suspicions regarding how the experiment is carried out.[5]

Assume subjects report to the experimenter the state they view as most likely.[6] Suppose first that subjects' conjectured experiments are restricted so that the *amount* of information revealed to them is assumed independent of both the realized state and the realized signals. In that case, there is a natural restriction on subjects' reports for them to be consistent with Bayesian updating that generalizes the insight from our introductory example. Namely, start with a sequence of signals $\mathbf{s}$ and suppose that no matter what additional signal the subject observes, she chooses the alternative a. If she is Bayesian, when observing only the sequence $\mathbf{s}$, she must put a probability on each continuation signal sequence, and realize that for each of these, a is the most likely alternative. Consequently, it must be that a is the most likely alternative when observing the original sequence $\mathbf{s}$. Thus, if the subject is Bayesian, it is necessary for her to choose the action a for signal sequence $\mathbf{s}$ whenever all continuation signal sequences lead her to choose a. This, in fact, is the spirit of the "Sure Thing Principle" a-la Savage (1954) and dynamic consistency. In Theorem 1 we illustrate that the converse also holds and that this is, in fact, a necessary and sufficient condition for Bayesian updating when subjects hold restricted conjectures.[7]

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[5] Indeed, most economic experiments are preceded by a set of instructions, and the prevailing norm in the profession banning deception would presumably trickle down to subjects.

[6] This is a natural starting point for our analysis since looking at reports of the most likely state requires the subject to go through very simple calculations and makes no behavioral assumptions on the subject's decision making, such as maximization of expected utility. Furthermore, the entire analysis here is based on ordinal utilities (monotonic transformation of a subject's utility will not change her optimal response). In contrast, in order to elicit the full beliefs a subject holds, choosing the proper utility would require certain behavioral assumptions, e.g., a quadratic scoring rule combined with maximization of expected utility.

[7] It is important to note that the experimental observations are mappings from sequences of signals. Thus, they consist a subset of elements considered by the Savage (1954) type of analysis. In particular, the sufficiency of the condition cannot be deduced from the work on foundations of expected utility. We return to this point in Section 3.

The other polar case of subjects' conjectures occurs when no restrictions are put and subjects' conjectures are allowed to entail correlations between the realization of all uncertainty (on the state and signals) and the number of signals observed. In such a setting, subjects are free to *frame* the experiment however way they like. In that case, as illustrated in the example, even when all continuations of a particular signal sequence $\mathbf{s}$ of length $n+1$ lead to a report of one alternative a, the subject may be a perfect Bayesian and still report an alternative different than a when observing $\mathbf{s}$, say b. Indeed, the subject may hold a conjecture suggesting that when the state is a, observing $n+1$ signals is very likely, whereas when the state is b, observing only n signals is very likely. In fact, when subjects' conjectures are unrestricted, Theorem 2 illustrates an "anything goes" result, in which *any* responses of subjects can be explained as arising from a particular conjectured experiment and Bayesian updating of the released information. In other words, in that case the theoretical predictions have no bite once jointly tested with the conjectured experiment.

From a positive and prescriptive point of view, certain experimental protocols are likely to place restrictions on the conjectures subjects hold. Indeed, in numerous experiments, subjects themselves generate the signals. For instance, in many voting experiments, the number of signals is determined at the outset, but subjects produce their own signals, e.g., by drawing a ball from a physical urn (the composition of which represents the underlying state), during the experiment.[8] In terms of the subjects' conjectures, such designs are likely to entail independence between the realized signals and the number of signals provided.

In Section 5 we study the intermediate case in which subjects' conjectures allow for correlations between the amount of information revealed and realized state, but the amount of information is independent of the signal realizations. Such partially restricted conjectures allow subjects to believe that a particular volume of information is associated with a particular state realization. Therefore, simple reversals as above are consistent with Bayesian updating (using the conjecture that associates $n+1$ signals with the state a and n signals with the state b). Nonetheless, not everything can be explained. Generalizing the example corresponding to

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[8] See Palfrey (2006) for a description of various experiments on voting behavior.

Figure 2, suppose that for all continuations of a different sequence $\mathbf{s}'$ of length $n+1$ the subject reports the state b. Since $n+1$ signals are supportive of the state a, this suggests that the subject's posteriors are more supportive of the state b than any of the posteriors corresponding to the continuations of the sequence $\mathbf{s}$. Therefore, the subject must report b when observing only $\mathbf{s}'$. Figure 2 illustrated a violation of this restriction.

Theorem 3 provides a characterization of the class of reversals that violate Bayesian updating when subjects hold partially restricted conjectures. Much as in the example, the reversals that are consistent with Bayesian updating are roughly described by going "in the same direction" (or tending to a specific report) for a particular length of signal sequence. The theorem provides the tools for deducing departures from Bayesian updating when the signal generation process is transparent in the lab.

The underlying assumption in our baseline analysis is that there is a natural sequencing of signals. This is why the experimental design was captured by the *number* of signals reported to the subject. This assumption is applicable in many situations (indeed, any context in which signals are tied with time), and eases the presentation. Nonetheless, in certain environments there is no natural ordering of signals and a general conjectured experiment pertains to the *dimensions* of information that are reported (or which elements of the set of signals are reported) and their correlation with the underlying uncertainty. In Section 6 we analyze such environments.

Our "anything goes" result still holds when signals are unordered and conjectured experiments are unrestricted.

However, the implications imposed by the natural analogue of restricted conjectured experiments do not carry through. In fact, we identify stronger observational restrictions that are in the spirit of "Dutch books." Experimental observations can be explained if and only if, after observing the subjects' responses, the experimenter cannot design a sequence of bets that would lead the agent to lose money for sure.

Our analysis is very closely related to the unformalized notion of *experimenter demand*, the idea that the design itself makes subjects (consciously or subconsciously) frame the problem at hand in a particular way that makes them believe certain responses are more appropriate than others (see, e.g., Friedman and Sunder, 1994 and Kagel and Roth, 1997). Experimenter demand type of arguments broadly take one of two forms. The first suggests that the way the experimenter phrases problems indicates something to the subject about the realized uncertainty and the correct response.[9] The second refers to the subject trying to "help" or "satisfy" the experimenter by aiming at the answers the experimenter is seeking. Our approach provides a first step in formalizing the former manifestation of experimenter demand, by contemplating a large class of conjectures, or ways to *frame* the experimental design. It is a necessary step for future models of the second demand manifestation as well, that potentially requires more behavioral and strategic assumptions on both experimenter and subject.

Specifically, the idea that individuals may exhibit a variety of non-standard behaviors in the presence of uncertainty, such as excess stickiness of beliefs, over-confidence, etc., has received much attention in both psychology and economics (see, for example, Part 1 in Brocas and Carrillo, 2004, Kahneman, Slovic, and Tversky, 1982, and Nisbett and Ross, 1980).[10] The economics literature has taken two approaches for utilizing these observations:

1. Suggest alternative models of belief updating to that prescribed by Bayes' rule (e.g., Rabin and Schrag, 1999 and Compte and Postlewaite, 2004);
2. Modify the subjects' utilities to account for arguments going beyond pure monetary rewards and directly depending on held beliefs (e.g., Benabou and Tirole, 2002, Part 3 in Brocas and Carrillo, 2003, Koszegi, 2006, Yariv, 2005, and references therein).

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[9] For instance, one prominent example is that from Tversky and Kahneman (1983), in which subjects were told that "Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations." and were then asked to determine which is more likely?

1. Linda is a bank teller.
2. Linda is a bank teller and is active in the feminist movement.

85% of those asked chose the second option.

This was taken as evidence for the failure of conjunction of probabilities. However, a different interpretation is that the mere fact the experimenter chooses to expose Linda's participation in demonstrations, suggest that 2 is the appropriate answer.

[10] In fact, some work suggests that certain biases in probability assessments are associated with mental health, see Taylor and Brown, 1988.

Note that our approach is very different in that we fix utilities and the belief updating algorithm, but allow for a wide range of theories subjects hold on the design of the experiment itself.[11]

1.4 Paper Structure

The paper is organized as follows. In the following section we spell out the model. Section 3 analyzes the benchmark model in which conjectures are constrained to entail independence between the amount of information observed and any realized uncertainty. Section 4 then presents our "anything goes" result when no restrictions are imposed on subjects' conjectures. The intermediate case, in which the amount of information observed is perceived independent of the signal realizations, is investigated in Section 5. Finally, the case in which there is no natural order for the arrival of information is examined in Section 6. Section 7 concludes. Technical proofs are relegated to the Appendix.

2. Description of the Model

2.1 Setup

Let A be a finite set of *alternatives* and S a finite set of *signals*. Let N be the number of available signals and $\mathbf{N} = \{0, 1, \ldots, N\}$. We denote by $S^{\leqslant N}$ the set of all *instances*, i.e., sequences of elements of S of length no greater than N, including the empty sequence e:

$$S^{\leqslant N} = \bigcup_{0 \leq n \leq N} S^n.$$

Experimental observations are summarized by a mapping $\sigma : S^{\leqslant N} \to A$. For every signal sequence $\mathbf{s}$, $\sigma(\mathbf{s})$ is the subject's report of the most probable alternative given $\mathbf{s}$.[12]

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[11] There is an old and wide literature going back to Savage (1954) and Anscombe and Aumann (1963) that considers the question of when experimental observations match some form of optimization (allowing utilities to be arbitrary) *together with* standard probabilistic assessments/updating, essentially restricting the conjectured experiment to coincide with the actual design (for recent references, see Green and Park, 1996).

[12] Our primitive, the experimental observations, or the mapping from all possible signal collections to alternatives chosen, is reminiscent of what is observed in experiments utilizing the *strategy method*, an experimental procedure dating back to Selten (1967) under which subjects report *contingent choices*, thereby eliciting their preferred alternatives for any observed realization of uncertainty in the lab.

For $x = (s_1, \ldots, s_N) \in S^N$ and $1 \leq n \leq N$, let $x|_n$ be the *truncation* of x to the first n signals: $x|_n = (s_1, \ldots, s_n)$.

If $\mathbf{s} = (s_1, \ldots, s_n)$ is an instance and $s \in S$ we denote by $\mathbf{s}\hat{\ }s$ the *concatenation* of $\mathbf{s}$ with s: $\mathbf{s}\hat{\ }s = (s_1, \ldots, s_n, s)$.

The set $S^{\leqslant N}$ of finite signal sequences has a natural rooted tree structure: The *root* is the empty sequence, the *depth* $\mathrm{d}(\mathbf{s})$ of an instance $\mathbf{s} = (s_1, \ldots, s_n)$ is $\mathrm{d}(\mathbf{s}) = n$ and the *children* of $\mathbf{s}$ are instances of the form $\mathbf{s}\hat{\ }s$ for $s \in S$. The *n-th layer* of the tree is the set of nodes of depth n. Technically, all our results and proofs depend only on the rooted tree structure of the set of instances. In particular, the results remain true if the number of available signals is infinite.

2.2 CONJECTURED EXPERIMENT

We consider subjects who hold beliefs regarding the experimental procedures, namely about the connection between the signals they observe and the underlying realized alternative (and their preferred action). We call such a belief a *conjectured experiment*, defined formally as:

Definition 1. *[Conjectured Experiment] A* conjectured experiment *is given by a triplet* $(\alpha, \tau, \zeta = \{\zeta_n\}_{1 \leq n \leq N})$ *of random variables over some probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *with values in* $A, \mathbf{N}, S^N$ *respectively*

The random variable α denotes the conjecture about the set of alternatives A, τ stands for the length of the observed signal sequence that, therefore, takes values in $\mathbf{N}$, and ζ captures the random variables corresponding to the realization of any signal sequence.[13]

Our results inspect the impacts of different restrictions on the conjectured experiment (that are consequences of either the transparency of the experimental design, or the subjective interpretations of subjects).

In Definition 1 we pose no restrictions on the dependence between the variables α, τ, and ζ. To emphasize this fact we will often refer to such a conjectured experiment as an *unrestricted conjectured experiment.*

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[13] Note that the set A of alternatives, the number $\mathbf{N}$ of signals, and the set S from which signals are drawn are fixed. The only element of the experimental design about which the subject conjectures regards the *distribution* over A, N, and S^N.

In particular, the opposite polar case to that of an unrestricted conjectured experiment is a *restricted conjectured experiment* in which the subject believes the number of signals she sees is uncorrelated with neither the realized alternative nor with the realization of the signals themselves. This is the case, for example, when the subject believes that the experimenter does not know both the realized alternative and realized signals when determining how many signals to provide. Formally,

Definition 2. *[Restricted Conjectured Experiment] A* restricted conjectured experiment *is a conjectured experiment* (α, τ, ζ) *such that* τ *is independent of the pair* (α, ζ)*.*

We are interested in conjectured experiments that explain a subject's behavior as arising from simple Bayesian updating given the number of signals available to her and their realizations. That is,

Definition 3. *[Explaining Observations] A conjectured experiment* $(\alpha, \tau, \zeta = \{\zeta_n\}_{1 \leq n \leq N})$ explains *the experimental observations* σ *if:*

1. *for every* $n \in \mathbf{N}$ *and every* $s_1, \ldots, s_n \in S$,

$$\mathbb{P}(\tau = n,\ \zeta_i = s_i \textit{ for } 1 \leq i \leq n) > 0.$$

2. *For every instance* $\mathbf{s} = (s_1, \ldots, s_n)$,

$$\sigma(\mathbf{s}) = \arg\max_{a \in A} \mathbb{P}\left(\alpha = a \mid \tau = n, \zeta_i = s_i \textit{ for } 1 \leq i \leq n\right). \tag{1}$$

Here and elsewhere, when we use the arg max notation, we implicitly assert the uniqueness of the maximizer.

The first condition in the definition requires that every finite sequence of signals the subject is faced with is indeed conceivable (has positive probability) under her conjectured experiment.

The second condition requires that under the conjectured experiment, the subject (Bayesian) updates on which alternative is most likely, conditioning on the signals she receives, and chooses that alternative.

3. Restricted Conjectured Experiments

We start by analyzing the case in which subjects' conjectures are restricted. That is, subjects believe that the *amount* of information they observe is independent of the realized alternative and the realized signals. In that case, conjectured experiments can be written in a simplified manner:

Remark 1. *[Restricted Conjectures – Simplified Notation] If τ is independent of the pair (ζ, α) then (1) becomes*

$$\sigma(\mathbf{s}) = \arg\max_{a \in A} \mathbb{P}\left(\alpha = a \mid \zeta_i = s_i \text{ for } 1 \le i \le n\right). \tag{2}$$

Thus, a restricted conjectured experiment that explains the experimental observations is identified by a pair (α, ζ) of random variables with values in A, S^N such that (2) is satisfied.

Let $\mathbf{s}$ be an instance and suppose that no matter what additional signal $s \in S$ is observed, the subject reports the alternative a^* to be the most likely. That is, $\sigma(\mathbf{s}\hat{}s) = a^*$ for every $s \in S$. If the subject is updating using Bayes rule, with a restricted conjecture, she cannot deduce anything regarding the realized state by the sheer volume of signals she observes. Thus, her assessment of each realized alternative under $\mathbf{s}$ is a convex combination of the corresponding assessments over all continuations $\mathbf{s}\hat{}s$. In particular, the most likely alternative must be a^*. This suggests a clear necessary requirement for observations to be explained. Theorem 1 illustrates that this requirement is, in fact, also sufficient. That is,

Theorem 1. *[Restricted Conjectured Experiments] The experimental observations σ can be explained by a restricted conjectured experiment if and only if the following condition is satisfied:*
Let $\mathbf{s}$ be an instance. If, for some $a^* \in A$ one has $\sigma(\mathbf{s}\hat{}s) = a^*$ for every $s \in S$ then $\sigma(\mathbf{s}) = a^*$.

The condition in Theorem 1 is reminiscent of the "Sure Thing Principle" and the notion of dynamic consistency. Indeed, dynamic consistency suggests that if the decision maker knows that given additional information tomorrow she will prefer action a to b regardless of what that information will be, then she should prefer a to b today (see, e.g., Ghirardato, 2002, and references therein, and Savage, 1954). Note, however, that while the condition in the theorem is expressed solely in terms of the maximal element within the agent's conditional beliefs, dynamic consistency is a property of a preference relation over contingent plans. Moreover, our primitive σ determines only the subject's action conditioned on a certain class of events, i.e., events that are given by the assignment of values to certain sets of signals. The condition of the theorem is therefore weaker than dynamic consistency: if the order induced by observations σ over actions can be extended to an order over contingent plans that satisfies dynamic consistency then σ must satisfy the condition. The Theorem implies that the converse is also true: If σ satisfies the condition then it can be explained by Bayesian updating, and therefore it can be extended to an order over contingent plans that satisfies dynamic consistency. In a more general setup, in which signals are unordered, dynamic consistency is a strictly stronger condition (see example 4 below).

The proof of the Theorem is instructive for some of the future analysis in the paper and will soon be spelled out. For the reader who prefers to skip the details, we stress that illustrating the sufficiency of the requirement is done in two steps. First, we consider an auxiliary (and hypothetical) experiment in which subjects would report full posteriors over states for every instance (generating a mapping from $S^{\leqslant N}$ to $\Delta(A)$) and identify the responses consistent with Bayes rule. Second, for any experimental observations satisfying the requirement of Theorem 1, we construct recursively a set of posteriors satisfying these identified restrictions and consistent with the experimental observations.

We now turn to the formal proof. The first step, captured by the following lemma, follows Shmaya and Yariv (2008), and addresses the question of which assignments of posterior distributions over states of nature can be explained.[14]

Lemma 1. *[Explainable Posteriors – Restricted] For every instance* $\mathbf{s} \in S^{\leqslant N}$*, let* $p_{\mathbf{s}} \in \Delta(A)$*. Then there exist random variables* $\{\alpha, \zeta\}$ *over some probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *with values in* A, S^N *respectively such that:*

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[14] See Theorem 1 and the last paragraph of Section 6 in Shmaya and Yariv (2008). Note that Theorem 1 in Shmaya and Yariv (2008) is formulated in terms of the joint distribution of α and ζ.

1. $\mathbb{P}(\zeta = x) > 0$ *for every* $x \in S^N$*; and*

2. For every instance $\mathbf{s} = (s_1, \ldots, s_n)$ *and every* $a \in A$

$$\mathbb{P}\left\{\alpha = a \mid \zeta_i = s_i \text{ for } 1 \leq i \leq n\right\} = p_{\mathbf{s}}[a] \tag{3}$$

if and only if, for every $n < N$ *and every* $\mathbf{s} = (s_1, \ldots, s_n)$ *one has*[15]

$$p_{\mathbf{s}} \in ri(Conv\{p_{\mathbf{s}\hat{}s} | s \in S\}). \tag{4}$$

As a technical note, the relative interior ri appearing in condition (4) assures that there is positive weight on any instance observed, which will be useful for constructing conjectured experiments that entail this type of restriction. Simply requiring $p_{\mathbf{s}} \in$Conv$\{p_{\mathbf{s}\hat{}s} | s \in S\}$ would correspond to consistent posterior reports, but ones that may place zero probability on certain instances.

We are now ready to present the proof of Theorem 1.

Proof of Theorem 1. It follows directly from Lemma 1 that the experimental observations $\sigma : S^{\leqslant N} \to A$ that can be explained by restricted conjectured experiments must satisfy the condition in Theorem 1. Indeed, let (α, τ, ζ) be a restricted conjectured experiment that explains σ, and let

$$p_{\mathbf{s}}[a] = \mathbb{P}\left(\alpha = a \mid \zeta_i = s_i \text{ for } 1 \leq i \leq n\right).$$

From (2), $\sigma(\mathbf{s}) = \arg\max_a p_{\mathbf{s}}[a]$, and from Proposition 1 we get that $p_{\mathbf{s}}$ satisfies (4). Assume that for some instance $\mathbf{s}$ and some $a^* \in A$ one has $\sigma(\mathbf{s}\hat{}s) = a^*$ for every $s \in S$. Then it follows that $\arg\max_a p_{\mathbf{s}\hat{}s}[a] = a^*$ for every s. Therefore $\sigma(\mathbf{s}) = \arg\max_a p_{\mathbf{s}}[a] = a^*$, where the last equality follows from the previous equality and (4).

---

[15] Recall that the *relative interior* ri(C) of a convex set C in finite dimensional real vector space is the interior of C with respect to the smallest affine space that contains C. If F is a finite set of points then ri(Conv$F$) is given by the set of all convex combinations of elements of F with strictly positive coefficients:

$$\text{ri}(\text{Conv}F) = \left\{\sum_{v \in F} \lambda_v v | \lambda_v > 0 \ \forall v \in F \text{ and } \sum_{v \in F} \lambda_v = 1\right\}.$$

To prove the other direction, we start with experimental observations that satisfy the condition of Theorem 1 and construct posterior distributions $p_{\mathbf{s}}$ for every instance $\mathbf{s}$ such that (4) is satisfied and, in addition, $\sigma(\mathbf{s}) = \arg\max_a p_{\mathbf{s}}[a]$ for every $\mathbf{s}$. We will use the following additional lemma. Let $\Delta^*(A) = \Big\{p \in \Delta(A) | p[a] < \frac{1}{|A|-1} \text{ for every } a \in A\Big\}$.

Lemma 2. *[One Period Construction] Let* $p \in \Delta^*(A)$ *and suppose* $a_1, \ldots, a_m \in A$ *are such that either* $a_i \neq a_j$ *for some* $1 \leq i, j \leq m$ *or* $a_1 = \cdots = a_m = \arg\max p$*. Then there exists* $p_1, \ldots, p_m \in \Delta^*(A)$ *such that* $p \in ri(Conv\{p_1, \ldots, p_m\})$ *and* $\arg\max p_i = a_i$.

The proof of the Lemma 2 is technical and is deferred to the Appendix. It follows from the lemma that we can inductively assign posterior probabilities $p_{\mathbf{s}} \in \Delta^*(A)$ for every instance $\mathbf{s}$ such that (4) is satisfied and, in addition,

$$\arg\max_a p_{\mathbf{s}}[a] = \sigma(a) \tag{5}$$

for every instance $\mathbf{s}$ (indeed, assuming we already defined $p_{\mathbf{s}}$, we define $p_{\mathbf{s}\hat{}S}$ using the lemma, with $m = |S|$). By Lemma 1 there exists random variables α, ζ over some probability space with values in A, S^N, respectively, such that ζ has full support and (3) is satisfied. Let τ be some random variable with values in $\mathbf{N}$, independent of (α, ζ) whose distribution has full support. Then for every $a \in A$ and $\mathbf{s} = (s_1, \ldots, s_n) \in S^{\leqslant N}$,

$$\mathbb{P}(\alpha = a \mid \tau = n,\ \zeta_1 = s_1, \ldots, \zeta_n = s_n) = \mathbb{P}(\alpha = a \mid \zeta_1 = s_1, \ldots, \zeta_n = s_n) = p_{\mathbf{s}}[a].$$

By (5) and the last equation it follows that (α, ζ, τ) explains σ, as desired. ■

Since the proof that the condition in Theorem 1 is necessary was based solely on (2), it bears on a larger set of conjectures that allow for some correlation between the volume of information and the realization of signals. Let us say that a conjectured experiment (α, τ, ζ) is *adapted* if τ is measurable with respect to $\zeta_1, \ldots, \zeta_n$, i.e., if

$$\mathbb{P}(\tau = n | \alpha = a,\ \zeta_1 = s_1, \ldots, \zeta_N = s_N) = \mathbb{P}(\tau = n | \zeta_1 = s_1, \ldots, \zeta_n = s_n),$$

for every $n \in \mathbf{N}$, $a \in A$ and $s_1, \ldots, s_N \in S$ (see page 113 in Prokhorov and Shirayev, 1997 for a general definition of adaptable stopping times). This means that when the experimenter decides whether to continue providing signals after stage n, he bases his decision upon the already released signals $s_1, \ldots, s_n$ (which were observed by the subject). Importantly, every restricted conjectured experiment is adapted.

It is easy to verify that (2) is still valid for adapted conjectured experiments. Therefore, even though the set of adapted conjectured experiments is larger than the set of restricted conjectured experiments, both sets of conjectured experiments have the same explanatory power and we get the following corollary:

Corollary 1. *[Adapted Conjectured Experiments] If experimental observations can be explained by an adapted conjectured experiment then they can also be explained by a restricted conjectured experiment.*

In particular, the Corollary implies that allowing subjects to conjecture that the experimenter provides information using a sequential procedure, one in which the experimenter decides whether or not to provide an additional signal depending on the signals that have been unraveled thus far, leads to the same testable implications described in Theorem 1.

We now turn to analyze the testable implications derived when conjectured experiments are fully unrestricted.

4. UNRESTRICTED CONJECTURED EXPERIMENTS

When conjectured experiments are unrestricted, the amount of information observed can be perceived as correlated with the realization of uncertainty and the requirement appearing in Theorem 1 is too strong, as the following simple example illustrates:

Example 1. *[Explanatory Power of Unrestricted Conjectures] Assume that* $N = 1$, $S = \{u, d\}, A = \{a, b\}$, *and consider the experimental observations* σ *given by:*

$$\sigma(e) = a$$

$$\sigma(u) = \sigma(d) = b,$$

as depicted in tree form in Figure 1, Panel (3). Note that the condition of Theorem 1 is not satisfied, and therefore σ cannot be explained by a restricted conjectured experiment. However, σ can be explained by an unrestricted conjectured experiment (α, τ, ζ) satisfying that τ and ζ are independent conditional on the realized alternative α and, in addition, characterized by the following:

$$\begin{aligned}
&\mathbb{P}(\alpha = \mathrm{a}) = \mathbb{P}(\alpha = b) = \frac{1}{2}\\
&\mathbb{P}(\tau = 0|\alpha = \mathrm{a}) = 1 - \mathbb{P}(\tau = 1|\alpha = \mathrm{a}) = 0.9\\
&\mathbb{P}(\tau = 0|\alpha = b) = 1 - \mathbb{P}(\tau = 1|\alpha = b) = 0.1\\
&\mathbb{P}(\zeta = u \mid \alpha = \tilde{a}) = \mathbb{P}(\zeta = d \mid \alpha = \tilde{a}) = 0.5 \quad \textit{for every } \tilde{a} \in \{\mathrm{a}, b\}.
\end{aligned}$$

As explained in the paper's introductory example, the conjectured experiment is such that when the realized alternative is $\alpha = \mathrm{a}$, the subject perceives receiving no signals as extremely likely, whereas when the realized alternative is $\alpha = \mathrm{b}$, the subject perceives receiving a signal as very likely. However, the determination of the number of signals observed is done independently of the generation of the signals themselves. In fact, the signals in and of themselves are uninformative.

It turns out that *any* experimental observations can be explained with an unrestricted conjectured experiment, formally captured in the following "*anything goes*" result:

Theorem 2. *[Unrestricted Conjectured Experiments] For every $\sigma : S^{\leqslant N} \to A$ the experimental observations σ admit an explanation by an unrestricted conjectured experiment.*

It follows from the theorem that when subjects' conjectures are unconstrained, anomalous updating behavior is indistinguishable from particular framing of the experimental design itself.

The proof of Theorem 2, as the proof of Theorem 1, follows two steps. We first show that any set of posteriors can be explained with an unrestricted conjectured experiment. This is the crucial step in the proof, since it is then immediate to choose a sequence of posteriors that is consistent with the observations, and therefore explain the observations with an unrestricted conjecture.

Formally, the analogue of Lemma 1 is the following:

Lemma 3. *[Explainable Posteriors – Unrestricted] Let* $p_{\mathbf{s}} \in \Delta(A)$ *be an assignment of probability distribution over* A *for every instance* $\mathbf{s} \in S^{\leqslant N}$*. Then there exist random variables* (α, τ, ζ) *over some probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *with values in* $A, \mathbf{N}, S^N$ *respectively such that*

1. *for every* $n \in \mathbf{N}$ *and every* $s_1, \ldots, s_n \in S$*,*

$$\mathbb{P}(\tau = n,\ \zeta_i = s_i \text{ for } 1 \leq i \leq n) > 0;$$

2. *For every instance* $\mathbf{s} = (s_1, \ldots, s_n)$ *and every* $a \in A$

$$\mathbb{P}\left(\alpha = a \mid \tau = n,\ \zeta_i = s_i \text{ for } 1 \leq i \leq n\right) = p_{\mathbf{s}}[a]. \tag{6}$$

The Lemma is conceptually important in that it highlights how robust the message of Theorem 2 is to what is elicited by the experimenter when subjects' conjectures are unrestricted. Indeed, even if for each instance the full belief were elicited (instead of the most likely state), Lemma 3 assures that an analogous "anything goes" result would still hold and any experimental observations (now entailing posterior beliefs for all instances) could be explained with an unrestricted conjecture.

The proof of Lemma 3 is at the root of our "anything goes" result. Intuitively, an unrestricted conjectured experiment ultimately corresponds to a joint distribution μ over $A \times \mathbf{N} \times S^N$. For any assignment of posteriors $\{p_{\mathbf{s}}\}_{\mathbf{s} \in S^{\leqslant N}}$, $p_{\mathbf{s}} \in \Delta(A)$, pick an arbitrary distribution ν over $\mathbf{N} \times S^N$ and define μ as:

$$\mu(k, n, s_1, ..., s_N) = \nu\left(n, s_1, ..., s_N\right) p_{(s_1,...,s_N)}\left[a\right].$$

Then, the conditional distribution of μ given the number of signals n and full realization $(s_1, ..., s_N)$ is $p_{(s_1,...,s_N)}$. It follows that the conditional distribution of μ given n and $(s_1, ..., s_n)$ is $p_{\mathbf{s}}$, where $\mathbf{s} = (s_1, ..., s_n)$, as required. Formally,

Proof of Lemma 3. Choose arbitrarily a distribution ν over $\mathbf{N} \times S^N$ with full support. Let α, τ, ζ be random variables over some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $A, \mathbf{N}, S^N$ such that:

1. The joint distribution of τ and ζ is ν.
2. The conditional distribution of α given $\tau = n$ and $\zeta = x$ is $p_{\mathbf{s}}$ where $\mathbf{s} = x|_n$:

$$\mathbb{P}\left(\alpha = a | \tau = n,\ \zeta = x\right) = p_{\mathbf{s}}[a] \tag{7}$$

We proceed to prove that (6) is satisfied for every instance $\mathbf{s} = (s_1, \ldots, s_n)$. The event $\{\tau = n,\ \zeta_i = s_i \text{ for every } 1 \leq i \leq n\}$ is the disjoint union of the events $\{\tau = n,\ \zeta = x\}$, ranging over all $x \in S^N$ such that $x|_n = \mathbf{s}$. Therefore, by the law of total probability,

$$\begin{aligned}&\mathbb{P}\left(\alpha = a \mid \tau = n,\ \zeta_i = s_i\ \forall 1 \leq i \leq n\right) = \\ &\qquad \sum_{x \in S^N \text{ and } x|_n = \mathbf{s}} P(\zeta = x | \tau = n,\ \zeta_i = s_i\ \forall 1 \leq i \leq n)\mathbb{P}\left(\alpha = a | \tau = n,\ \zeta = x\right) = p_{\mathbf{s}}[a],\end{aligned}$$

where the last equality follows from (7). ■

The proof of Theorem 2 now follows immediately:

Proof of Theorem 2. Given experimental observations $\sigma : S^{\leqslant N} \to A$ we choose arbitrarily, for every instance $\mathbf{s} \in S^{\leqslant N}$, a probability distribution $p_{\mathbf{s}} \in \Delta(A)$ such that $\arg\max_a p_{\mathbf{s}}[a] = \sigma(\mathbf{s})$. The corresponding random variables α, τ, and ζ, whose existence is asserted in Lemma 3, explain σ. ■

To summarize our results up to now, we have characterized the testable implications of Bayesian updating for two polar cases pertaining to subjects' freedom regarding the framing of the experimental design. When conjectures are fully restricted, standard updating is equivalent to a type of dynamic consistency condition. When conjectures are unrestricted, standard updating has no testable implications. In what follows, we analyze testable implications derived from intermediate restrictions.

5. Partially Restricted Conjectured Experiments

In this section, we consider intermediate restrictions that may correspond to experimental designs in which signals are determined in the lab in front of the subjects (and the number of signals is a design choice). Indeed, many of the recent social learning and voting experiments follow protocols of this nature (in these experiments two uncertain alternatives are often manifested in two jars that contain a majority of either, say, red or blue, balls. Subjects, not knowing which jar had been selected can then draw a *pre-determined* number of balls, often with replacement, from the chosen jar and observe their color).

Under such designs, it is natural that the subject believes the number of signals she receives is uncorrelated with the *realizations* of the signals (but may be correlated with the realized alternative). We formalize such conjectures as follows:

Definition 4. *[Partially Restricted Conjectured Experiment] A* partially restricted conjectured experiment *is a conjectured experiment* (α, τ, ζ) *such that* τ *and* ζ *are conditionally independent given* α*, i.e.,*

$$\mathbb{P}(\tau = n, \zeta = x|\alpha = a) = \mathbb{P}(\tau = n|\alpha = a) \cdot \mathbb{P}(\zeta = x|\alpha = a),$$

for every $n \in \mathbf{N}$, $x \in S^N$ *and* $a \in A$*.*

For the sake of presentational simplicity, we assume hereafter a binary state space, $A = \{\mathrm{a}, \mathrm{b}\}$.

Note that the conjecture proposed in Example 1 are partially restricted and so that example illustrates that the explanatory power of partially restricted conjectured experiments is larger than that of restricted conjectured experiments. However, not all experimental observations can be explained by a partially restricted conjectured experiment, as illustrated by the following example, formalizing the example we discussed in our introduction:

Example 2. *[Unexplainable Reversals] Assume that* $N = 2$*,* $S = \{u, d\}$ *and consider the experimental observations* σ *depicted in Figure 2. Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be the signal sequences* (u) *and* (d) *respectively (so* $d(\mathbf{s}) = d(\mathbf{t}) = 1$*). On the one hand, controlling for any learning from the* *sheer amount of information released,* $\mathbf{s}$ *is more supportive of the alternative* a *than* $\mathbf{t}$ *(from the subject's point of view) since* $\sigma(\mathbf{s}) = a$ *and* $\sigma(\mathbf{t}) = b$*. On the other hand, the same can be said in reverse, since conditioning on depth 2 of the tree,* $\sigma(\mathbf{s}\hat{\ }s) = b$ *and* $\sigma(\mathbf{t}\hat{\ }t) = a$ *for every* $s, t \in \{u, d\}$*. This inconsistency implies that the experimental observations cannot be explained by a partially restricted conjectured experiment.*

Examples 1 and 2 illustrate that the explanatory power of partially restricted conjectured experiments is strictly between that corresponding to unrestricted and restricted conjectured experiments.

Our characterization of the class of observations that are explainable with partially restricted conjectures entails ruling out a generalized set of reversals of the type appearing in Example 2. Roughly speaking, reversals that are consistent with Bayesian updating are described by "tending to a specific report" for a particular length of signal sequence.

Formally, fix experimental observations σ. For a pair of instances $\mathbf{s}, \mathbf{t}$ of the same depth we define recursively what we mean by "the conditional probability of state 'a' given $\mathbf{s}$ is behaviorally revealed to be higher than the conditional probability of state 'a' given $\mathbf{t}$," where the conditioning is on the depth or amount of information revealed. We say shortly that $\mathbf{s}$ is *revealed higher* than $\mathbf{t}$.[16]

Definition 5. *[Revealed Higher Relation] Let* $\sigma : S^{\leqslant N} \to \{a, b\}$ *be experimental observations. For a pair of instances* $\mathbf{s}, \mathbf{t} \in S^{\leqslant N}$ *of the same depth the relation '*$\mathbf{s}$ *is revealed higher than* $\mathbf{t}$ *under* σ*' is recursively defined using the following rules:*

1. *If* $\sigma(\mathbf{s}) = a$ *and* $\sigma(\mathbf{t}) = b$ *then* $\mathbf{s}$ *is revealed higher than* $\mathbf{t}$*.*
2. *If* $\mathbf{s}\hat{\ }s$ *is revealed higher than* $\mathbf{t}\hat{\ }t$ *for every* $s, t \in S$ *then* $\mathbf{s}$ *is revealed higher than* $\mathbf{t}$*.*

The following lemma illustrates the implication of one instance being revealed higher than another in terms of probabilistic assessments.

---

[16] Note that "revealed higher" need not be a complete relation, nor is it necessarily anti-symmetric.

Lemma 4. *[Revealed Higher – Probabilities] Let* $\sigma : S^{\leqslant N} \to \{\mathrm{a}, b\}$ *be experimental observations, and let* (α, τ, ζ) *be a partially restricted conjectured experiment that explains* σ*. If* $\mathbf{s} = (s_1, \ldots, s_n)$ *and* $\mathbf{t} = (t_1, \ldots, t_n) \in S^{\leqslant N}$ *are a pair of instances such* $\mathbf{s}$ *is revealed higher than* $\mathbf{t}$ *under* σ*, then*

$$\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = s_1, \ldots, \zeta_n = s_n) > \mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = t_1, \ldots, \zeta_n = t_n).$$

Lemma 4 assures that if the experimental observations can be explained by a partially restricted conjectured experiment, it cannot be the case that there are reversals of the form $\mathbf{s}$ is revealed higher than $\mathbf{t}$ and $\mathbf{t}$ is revealed higher than $\mathbf{s}$ for some instances $\mathbf{s}$ and $\mathbf{t}$. As it turns out, the reverse is also true. Indeed, the following theorem characterizes the set of experimental observations that can be explained by a partially restricted conjectured experiment.

Theorem 3. *[Partially Restricted Conjectured Experiments] The experimental observations* $\sigma : S^{\leqslant N} \to \{\mathrm{a}, b\}$ *can be explained by a partially restricted conjectured experiment if and only if the order revealed higher is anti-symmetric. That is, there exists no pair* $\mathbf{s}, \mathbf{t}$ *of instances such that* $\mathbf{s}$ *is revealed higher than* $\mathbf{t}$ *and* $\mathbf{t}$ *is revealed higher than* $\mathbf{s}$.

The formal proof of the sufficiency of the condition is intricate and appears in the Appendix.

In order to provide the reader with some intuition, we provide here a sketch of the proof. Assume that the experimental observations $\sigma : S^{\leqslant N} \to \{\mathrm{a}, \mathrm{b}\}$ satisfy the condition of the theorem. We construct the partially restricted conjectured experiment (α, τ, ζ) that explains σ in two step. First, we construct random variables (α, ζ) such that

$$\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = s_1, \ldots, \zeta_n = s_n) > \mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = t_1, \ldots, \zeta_n = t_n).$$

for every pair $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{t} = (t_1, \ldots, t_n)$ of instances of the same length such that $\sigma(\mathbf{s}) = \mathrm{a}$ and $\sigma(\mathbf{t}) = \mathrm{b}$. In the second step, we add a random variable τ that is independent of ζ given α and, for every n, we choose the probabilities $\mathbb{P}(\alpha = \mathrm{a} | \tau = n)$ such that

$$\mathbb{P}(\alpha = \mathrm{a} | \tau = n,\ \zeta_1 = s_1, \ldots, \zeta_n = s_n) > 1/2$$

for all instances $\mathbf{s} = (s_1, \ldots, s_n)$ such that $\sigma(\mathbf{s}) = \mathrm{a}$ and

$$\mathbb{P}(\alpha = \mathrm{a} | \tau = n,\ \zeta_1 = t_1, \ldots, \zeta_n = t_n) < 1/2$$

for all instances $\mathbf{t} = (t_1, \ldots, t_n)$ such that $\sigma(\mathbf{t}) = \mathrm{b}$.

In words, we first construct conjectures that are consistent within each layer of the tree of observations (that is, for a *fixed* number of signals). We then construct the correlation between the number of signals observed and underlying state so that the assessments *across* layers are consistent.

The main difficulty is in the first stage. The proof makes use of the tree structure over the set of instances. For every instance $\mathbf{s} = (s_1, \ldots, s_n) \in S^{\leqslant \mathbf{N}}$ we assign a number $p_{\mathbf{s}} \in [0, 1]$ such that $p_{\mathbf{s}} \in \mathrm{ri}(\mathrm{Conv}\{p_{\mathbf{s}\hat{\ }s} | s \in S\})$ and $p_{\mathbf{s}} > p_{\mathbf{t}}$ whenever $\mathbf{s}, \mathbf{t}$ are two instances of the same length and $\mathbf{s}$ is revealed higher than $\mathbf{t}$. Then, by Lemma 1 we can construct random variables α, ζ such that

$$p_{\mathbf{s}} = \mathbb{P}(\alpha = \mathrm{a} | \tau = n,\ \zeta_1 = s_1, \ldots, \zeta_n = s_n)$$

for every instance $\mathbf{s} = (s_1, \ldots, s_n)$.

We assign the numbers $p_{\mathbf{s}}$ to nodes $\mathbf{s}$ by induction over the depth of $\mathbf{s}$, working from the root of the tree to the leafs. Assume that we have already defined the values $p_{\mathbf{s}}$ for nodes of depth $n - 1$ and consider now the set X of nodes of depth n. The relation revealed higher induces a partial order over X, which we denote by $\leq$. In addition, there is a natural partition $\mathcal{A}$ over the set X, whose atoms are the set of instances with common source in the tree. Thus, the atoms of $\mathcal{A}$ correspond to nodes of depth $n - 1$. Since we have already defined $p_{\mathbf{s}}$ over these nodes we need to 'lift' p to a function over X, that will be monotone with respect to $\leq$. Much of the technical aspects of the proof are dedicated to showing that this is indeed possible, using the fact that the order "revealed higher" is an interval order.

We note that Theorem 3 gives rise to a simple algorithm for checking whether experimental observations in a finite tree can be explained by a partially restricted conjectured experiment: Go over all the layers of the tree of instances, from layer of depth $d = N$ to layer of depth $d = 0$

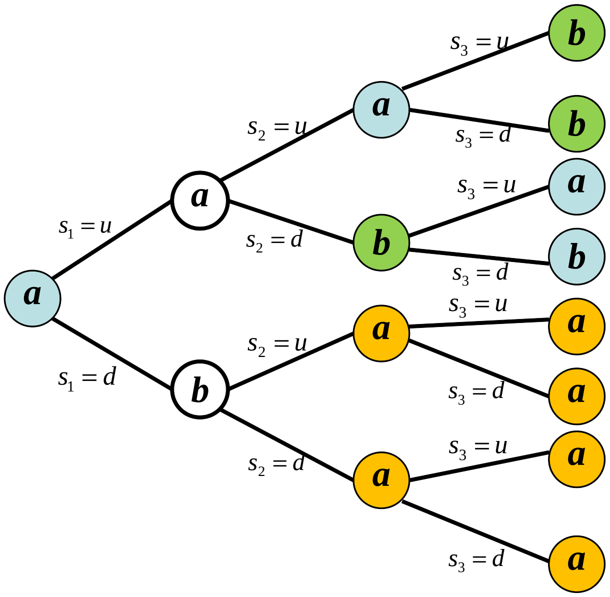

Figure 3: Algorithmically Checking Consistency

(i.e., from the leafs to the root of the corresponding tree). For every layer d, construct the relation "$\mathbf{s}$ is higher than $\mathbf{t}$" for instance $\mathbf{s}, \mathbf{t}$ of that layer (using Definition 5 and the already constructed relation over layer $d + 1$) and check that the condition is satisfied over nodes at that layer. Using this algorithm, the reader can verify that the experimental observation in the following example cannot be explained by a partially restricted conjectured experiment.

Example 3. *[Explainability Algorithm] Consider the Example depicted in Figure 3 (notation following that of our previous examples). Note that* (u) *is revealed higher than* (d). *However, algorithmically proceeding from the leafs to layer 1 we see that, from layer 3,* (d,u) *and* (d,d) *are revealed higher than* (u,u), *and from layer 2,* (d,u) *and* (d,d) *are also revealed higher than* (u, d), *which therefore implies that* (d) *is revealed higher than* (u). *In particular, the condition of Theorem 3 is not satisfied and these observations cannot be explained by a partially restricted conjectured experiment.*

Example 3 also demonstrates that lack of "simple reversals" of the type of Example 2 is not sufficient for experimental observations to admit an explanation by a partially restricted conjectured experiment.

6. Unordered Dimensions

Up to now, we have assumed the different dimensions of information have a natural ordering (so that the natural parameter to consider was the *number* of signals provided). In this section we extend our analysis to contexts in which the dimensions do not have a natural ordering. In particular, conjectured experiments specify the statistical dependence of the *dimensions* of information provided and the underlying states and realized signals.

We now require notation in which there is no natural order between dimensions, and the experimenter decides which dimensions to reveal, and not only how many of them. Formally, let A, S, N be, as in our original setting, the set of alternatives (finite, but arbitrary), the set of possible realizations of signals, and the number of dimensions. An *instance* is now given by a pair (D, δ), where $D \subseteq \{1, \dots, N\}$ and $\delta : D \to S$. The interpretation is that the subject observes the realization of the signals pertaining only to dimensions in D. Let $\mathcal{O}$ be the set of instances. As before, experimental observations are summarized by a mapping $\sigma : \mathcal{O} \to A$.

A *conjectured experiment* is given by a triplet (α, τ, ζ) with values in $A, \mathcal{P}(N)$, and S^N respectively, where $\mathcal{P}(N)$ is the set of subsets of $\{1, \dots, N\}$. As before, a conjectured experiment is restricted if τ is independent of (α, ζ). A conjectured experiment *explains* the experimental observations σ if for every instance (D, δ) one has

$$\mathbb{P}(\tau = D, \quad \zeta_i = \delta(d) \text{ for every } d \in D) > 0 \quad \text{and} \tag{8}$$

$$\sigma(\mathbf{s}) = \arg\max_a \mathbb{P}(\alpha = a | \tau = D, \quad \zeta_d = \delta(d) \text{ for every } d \in D) \tag{9}$$

The “anything goes” result captured in Theorem 2, and its proof, are valid, *mutatis mutandis*, in the unordered model: for every $\sigma : \mathcal{O} \to A$, the experimental observations summarized by σ can be explained with an unrestricted conjectured experiment.

In the rest of this section, we focus on the case of restricted conjectured experiments.

Say that an instance $\mathbf{s}' = (D', \delta')$ *extends* an instance $\mathbf{s} = (D, \delta)$ if $D \subseteq D'$ and $\delta'(d) = \delta(d)$ for every $d \in D$. The condition in Theorem 1 can be adapted to a necessary condition for existence of an explanation by a restricted conjectured experiment in the unordered model. Namely, suppose the experimental observations are given by $\sigma : \mathcal{O} \to A$. If σ can be explained by a restricted conjectured experiment then for any instance $\mathbf{s} = (D, \delta)$, if for some $a^* \in A$, $\sigma(\mathbf{s}') = a^*$ for every instance $\mathbf{s}' = (D', \delta')$ that extends $\mathbf{s}$, then $\sigma(\mathbf{s}) = a^*$.

The proof that the above condition is necessary to the existence of an explanation by a restricted conjectured experiment follows that shown in the ordered model. However, we will soon see that in the unordered model, this condition is not sufficient. First, we require some additional definitions.

Let $\sigma : \mathcal{O} \to A$ be experimental observations. Let us say that an instance $\mathbf{s} = (D, \delta)$ *agrees* with a realization $(s_1, \ldots, s_N)$ of the signals if $\delta(d) = s_d$ for every $d \in D$. A *positive bet* is given by a triplet $(z, \mathbf{s}, c)$ such that z is a positive real number, $\mathbf{s} = (D, \delta)$ is an instance, and $c \in A$ is such that $c \neq \sigma(\mathbf{s})$. A bet of this form is activated when $\mathbf{s}$ agrees with the realization of the signals and provides the subject z if the state of nature is $\sigma(\mathbf{s})$ and $-z$ if the state of nature is c. Since $\sigma(\mathbf{s})$ is the subjectively most probable state of nature given $\mathbf{s}$, the subject's subjective expected payoff from the bet is strictly positive. For a bet $\beta = (z, \mathbf{s}, c)$, we denote by $p(\beta, \tilde{a}, s_1, \ldots, s_N)$ the payoff under β if the state of nature is $\tilde{a}$ and the realization of the signals is $s_1, \ldots, s_N$. Thus,

$$p(\beta, \tilde{a}, s_1, \ldots, s_N) = \begin{cases} z & \text{if } \mathbf{s} \text{ agrees with } (s_1, \ldots, s_N) \text{ and } \tilde{a} = \sigma(\mathbf{s}) \\ -z & \text{if } \mathbf{s} \text{ agrees with } (s_1, \ldots, s_N) \text{ and } \tilde{a} = c \\ 0 & \text{otherwise.} \end{cases}$$

A *Dutch book* is given by a non-empty set B of positive bets such that

$$\sum_{\beta \in B} p(\beta, \tilde{a}, s_1, \ldots, s_N) \leq 0$$

for every $\tilde{a} \in A$ and every $s_1, \ldots, s_N \in S$. Thus, if the subject accepts all the bets in B then her final payoff would be non-positive for every realization of the state of nature and the signals.

If the subject had accurate beliefs regarding the underlying process, she would never accept a

Dutch book. It follows intuitively that if the experimental observations can be explained with a restricted conjectured experiment, the subject would never accept a Dutch book. This already suggests additional restrictions that are required for explaining observations with restricted conjectured experiments when dimensions are unordered, as the following example illustrates.

Example 4. *[Dutch Book] Let* $A = \{a, b\}$*,* $S = \{1, 2, 3, 4\}$*, and* $N = 2$*. Let* ζ_1, ζ_2 *be the signals. Assume that the experimental observations are given as follows: if both signals are provided on both dimensions and* $\zeta_1 = i, \zeta_2 = j$*, then the subject chooses the* (i, j) *entry in the following matrix:*

$$\begin{pmatrix} \mathrm{a} & \mathrm{a} & b & \mathrm{a} \\ \mathrm{a} & \mathrm{a} & \mathrm{a} & b \\ b & \mathrm{a} & b & b \\ \mathrm{a} & b & b & b \end{pmatrix}.$$

If only ζ_1 *is given then the agent chooses b if* $\zeta_1 \in \{1, 2\}$ *and a if* $\zeta_1 \in \{3, 4\}$*.*
If only ζ_2 *is given then the agent chooses b if* $\zeta_2 \in \{1, 2\}$ *and a if* $\zeta_2 \in \{3, 4\}$*. If no realization is given then the subject chooses a. Note that these experimental observations satisfy the generalized version of the condition appearing in Theorem 2.*

Now, for every instance $\mathbf{s} = (D, \delta)$ *such that* $|D| = 2$ *and either* $\rho(D) \subseteq \{1, 2\}$ *or* $\rho(D) \subseteq \{3, 4\}$*, let* $\beta_{\mathbf{s}}$ *be the positive bet that is activated if the realization of the signals agree with* $\mathbf{s}$ *and gives* $+2$ *if the state of nature is* $\sigma(\mathbf{s})$ *and* -2 *otherwise. For every instance* $\mathbf{s} = (D, \delta)$ *such that* $|D| = 1$ *let* $\beta_{\mathbf{s}}$ *be the positive bet that is activated if the realization of the signals agree with* $\mathbf{s}$ *and gives* $+1$ *if the state of nature is* $\sigma(\mathbf{s})$ *and* -1 *otherwise. It is easy to verify that the set* $\{\beta_{\mathbf{s}}\}_{\mathbf{s}}$ *constitutes a Dutch book. Consequently, these experimental observations cannot be explained with a restricted conjectured experiment.*

In fact, the set of experimental observations that can be explained with a restricted conjectured experiment is characterized by the absence of Dutch books, as formalized in the following theorem:

Theorem 4. *[Unordered Dimensions] Experimental observations admit an explanation by a restricted conjectured experiment if and only if they do not entail a Dutch book.*

The proof appears in the Appendix. Again, at the root of the proof is the construction of a system of beliefs that are consistent with both Bayesian updating and the experimental observations.[17]

7. Conclusions

This paper provides a theoretical framework for analyzing experimental data accounting for subjects' conjectures regarding the experimental design itself. When subjects' conjectures are unrestricted, we illustrated an "anything goes" result: any experimental observations can be explained with standard updating and the appropriate choice of a conjectured experiment (in fact, generically, multiple conjectured experiments would explain the observations). When subjects' conjectures are restricted, in terms of the perceived correlation between the amount of information revealed to them in the lab and the underlying realized uncertainty, our results provide a full characterization of the testable implications standard updating entails.

To the extent that experimental transparency (say, regarding the way information is generated in the lab) yields more restricted conjectures, the overwhelming message from our results is that transparency is crucial in allowing for meaningful testable implications of theoretical predictions pertaining to decision making under uncertainty.

The class of experiments our analysis encompasses is rather general. Indeed, the uncertainty we model can be a metaphor for uncertainty regarding an underlying payoff-relevant parameter (such as whether a defendant is guilty or innocent in a jury setup) or regarding other agents' choices (in experiments entailing interactions between several agents, even when information is complete), as the state of nature can reflect uncertainty about others' strategies.

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A. APPENDIX

A.1. Restricted Conjectured Experiments.

Proof of Lemma 2. If $a_1 = \cdots = a_m = \arg\max p$ then we can choose $p_i = p$ for $1 \leq i \leq m$.

Assume now that $a_1 \neq a_2$. We choose $p_3, \ldots, p_m \in \Delta^*(A)$ arbitrarily such that $\arg\max_a p_i[a] = a_i$. Let $\varepsilon > 0$ be sufficiently small such that $(m-2)\varepsilon < 1$ and $p' \in \Delta^*(A)$, where

$$p' = \frac{1}{1-(m-2)\varepsilon}\left(p - \varepsilon(p_3 + \cdots + p_m)\right). \tag{10}$$

The existence of such ε follows from the fact that $\Delta^*(A)$ is an open set and the right hand side of (10) converges to $p \in \Delta^*(A)$ as ε goes to 0.

We now define p_1 and p_2. Choose q such that

$$\max_a p[a] < q < \frac{1}{|A|-1}, \tag{11}$$

and let

$$r = 1 - \sum_{a \neq a_1, a_2} p'[a] - q.$$

Note that from the fact that $p'[a] < \frac{1}{|A|-1}$ for every $a \in A$ and $q < \frac{1}{|A|-1}$ it follows that $r > 0$. Moreover, since $q > p'[a_1]$ it follows that

$$r = 1 - \sum_{a \neq a_1, a_2} p'[a] - q < 1 - \sum_{a \neq a_2} p'[a] = p'[a_2] < q < \frac{1}{|A|-1}. \tag{12}$$

For $i = 1, 2$ and $j = 3 - i$ let

$$p_i[a] = p'[a] \text{ for } a \neq a_1, a_2 \tag{13}$$

$$p_i[a_i] = q \tag{14}$$

$$p_i[a_j] = r. \tag{15}$$

Then it follows from (12) that $p_i \in \Delta^*(A)$ and $\arg\max_a p_i[a] = a_i$.

In addition, it follows from (11) and (12) that $r < p'[a_2] < q$. Therefore,

$$p'[a_2] = \lambda r + (1 - \lambda) q \tag{16}$$

for some $0 < \lambda < 1$. We claim that

$$p' = \lambda p_1 + (1 - \lambda) p_2 \tag{17}$$

$$\Leftrightarrow \quad p'[a] = \lambda p_1[a] + (1 - \lambda) p_2[a] \text{ for every } a \in A. \tag{18}$$

Indeed, for $a \neq a_1, a_2$ the equality follows from the fact that by (13) $p_1[a] = p_2[a] = p'[a]$. For $a = a_2$ the equality follows from (14),(15), and (16). For $a = a_1$ the equality follows from the equality in all other coordinates and the fact that both sides of (18) sum to 1 (over $a \in A$). Finally, it follows from (10) and (17) that

$$p = (1 - (m - 2)\varepsilon)\, p' + \varepsilon(p_3 + \cdots + p_m) = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \cdots + \lambda_m p_m,$$

where

$$\lambda_1 = (1 - (m - 2)\varepsilon) \cdot \lambda,$$

$$\lambda_2 = (1 - (m - 2)\varepsilon) \cdot (1 - \lambda), \text{ and.}$$

$$\lambda_3 = \cdots = \lambda_m = \varepsilon.$$

Therefore, $p \in \mathrm{ri}(\mathrm{Conv}\{p_1, \ldots, p_m\})$, as desired. ■

A.2. Partially Restricted Conjectured Experiments.

A.2.A PROOF OF LEMMA 4

The proof follows several additional lemmas.

Lemma 5. *[Conditioning on Independent Events] Let X, Y, Z be events in some probability space such that Y, Z are independent given the partition (X, X^c). Then*

$$\mathbb{P}(X|Y,Z) = \rho\left(\frac{\mathbb{P}(Y|X^c)}{\mathbb{P}(Y|X)}, \mathbb{P}(X|Z)\right),$$

where

$$\rho(r,q) = \frac{q}{q + r \cdot (1-q)}. \tag{19}$$

Proof. One has

$$\mathbb{P}(X|Y,Z) = \frac{\mathbb{P}(X,Y|Z)}{\mathbb{P}(Y|Z)} = \frac{\mathbb{P}(X|Z)\mathbb{P}(Y|X,Z)}{\mathbb{P}(X|Z)\mathbb{P}(Y|X,Z) + \mathbb{P}(X^c|Z)\mathbb{P}(Y|X^c,Z)} =$$
$$\frac{\mathbb{P}(X|Z)\mathbb{P}(Y|X)}{\mathbb{P}(X|Z)\mathbb{P}(Y|X) + \mathbb{P}(X^c|Z)\mathbb{P}(Y|X^c)} = \rho\left(\frac{\mathbb{P}(Y|X^c)}{\mathbb{P}(Y|X)}, \mathbb{P}(X|Z)\right),$$

where we used the fact that $\mathbb{P}(Y|X,Z) = \mathbb{P}(Y|X)$ (respectively, $\mathbb{P}(Y|X^c,Z) = \mathbb{P}(Y|X^c)$), which follows from Y and Z being independent given X (respectively, X^c). ■

Lemma 6. *[Ranking of Conditional Probabilities] Let X, Y, Z_1, Z_2 be events in some probability space such that*

1. *Y and Z_1 are independent given the partition (X, X^c).*

2. *Y and Z_2 are independent given the partition (X, X^c).*

If $\mathbb{P}(X|Y,Z_1) > \mathbb{P}(X|Y,Z_2)$ then $\mathbb{P}(X|Z_1) > \mathbb{P}(X|Z_2)$.

Proof. Let $r = \mathbb{P}(Y|X^c)/\mathbb{P}(Y|X)$ and $q_i = \mathbb{P}(X|Z_i)$ for $i = 1, 2$. By the previous lemma $\mathbb{P}(X|Y,Z_i) = \rho(r, q_i)$ where ρ is given by (19). The assertion follows from the fact that $\rho(r,q)$ is monotone in q. ■

Proof of Lemma 4. Following the structure of Definition 5, We prove the lemma by induction on the number of remaining layers in the tree describing the experimental observations. Assume first that $\sigma(\mathbf{s}) = \mathrm{a}$ and $\sigma(\mathbf{t}) = \mathrm{b}$. Then by Definition 3, since (α, τ, ζ) explains σ it follows that

$$\mathbb{P}(\alpha = \mathrm{a} | \tau = n,\ \zeta_1 = s_1, \ldots, \zeta_s = s_n) > 1/2 > \mathbb{P}(\alpha = \mathrm{a} | \tau = n,\ \zeta_1 = t_1, \ldots, \zeta_n = t_n).$$

Applying Lemma 6 we get

$$\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = s_1, \ldots, \zeta_n = s_n) > \mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = t_1, \ldots, \zeta_n = t_n),$$

as desired. In particular, this also provides the first induction step pertaining to instances of length N (with no remaining signals that can be observed).

Assume now that $\mathbf{s}\hat{}s$ is revealed higher than $\mathbf{t}\hat{}t$ for $s, t \in S$. By the induction hypothesis, it follows that

$$\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = s_1, \ldots, \zeta_n = s_n, \zeta_{n+1} = s) > \mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = t_1, \ldots, \zeta_n = t_n, \zeta_{n+1} = t), \tag{20}$$

for every $s, t \in S$. From Lemma 1 it follows that

$$\begin{aligned} &\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = s_1, \ldots, \zeta_n = s_n) \\ &\qquad \in \mathrm{Conv}\{\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = s_1, \ldots, \zeta_n = s_n, \zeta_{n+1} = s) | s \in S\} \text{ and} \\ &\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = t_1, \ldots, \zeta_n = t_n) \\ &\qquad \in \mathrm{Conv}\{\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = t_1, \ldots, \zeta_n = t_n, \zeta_{n+1} = t) | t \in S\}. \end{aligned} \tag{21}$$

From (20) and (21) we get

$$\mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = s_1, \ldots, \zeta_n = s_n) > \mathbb{P}(\alpha = \mathrm{a} | \zeta_1 = t_1, \ldots, \zeta_n = t_n),$$

as desired. ■

A.2.B PRELIMINARIES

Before turning to the proof of Theorem 3, we require some results on interval orders, which we now present.

We use the standard terminology of a *partial order* $\leq$ over a set X being a reflexive, transitive, and antisymmetric relation. A function $f : X \to \mathbb{R}$ is called *strictly monotone* if $f(x) < f(y)$ whenever $x < y$ for every $x, y \in X$. If $\leq, \leq'$ are partial orders over X, we say that $\leq'$ is an *extension of* $\leq$ if $x \leq' y$ whenever $x \leq y$. If $U, V \subseteq X$ then we write $U < V$ if $x < y$ for every $x \in U$ and $y \in V$. If $x \in X$ and $V \subseteq X$ we write $x < V$ when $\{x\} < V$.

A partial order $\leq$ is called a *linear order* if, for every $x, y \in X$, either $x \leq y$ or $y \leq x$.

A partial order $\leq$ over a finite set X is called *interval order* (see Fishburn, 1985) if there is an assignment of closed real intervals $I_x = [l(x), r(x)]$ (where $l(x), r(x)$ are real numbers and $l(x) \leq r(x)$) to the elements x of X such that $x \leq y$ if and only if I_y is to the right of I_x (i.e. $r(x) \leq l(y)$). Such an assignment is called an *interval representation* of $\leq$. If $\leq$ is an interval order then it admits an interval representation such that I_x and I_y have no common endpoints for any distinct x and y in X. Note that if all the intervals I_x in a representation of $\leq$ are distinct and degenerated then $\leq$ is a linear order, and every linear order admits such a representation.

A partial order $\leq$ over a set X is called a *ranking* if its elements can be partitioned into *ranks* $X_1, \ldots, X_m$ such that two elements are incomparable if and only if they belong to the same rank. Every ranking is, in particular, an interval order.

A *partition* of a set X is a collection $\mathcal{A}$ of non-empty mutually disjoint subsets of X such that $X = \bigcup_{U \in \mathcal{A}} U$. Elements of $\mathcal{A}$ are called *atoms*.

Lemma 7. *[Representation] Let $(X, \leq)$ be a finite set equipped with an interval order, $\mathcal{A}$ a partition of X, and $F : \mathcal{A} \to (0,1)$ a one-to-one real-valued function such that $0 < F(U) < 1$ for every atom U of $\mathcal{A}$. Assume that the following condition is satisfied*

$$\text{If } V, U \text{ are atoms of } \mathcal{A} \text{ and } V < U \text{ then } F(V) < F(U). \tag{22}$$

Then there exists a strictly monotone one-to-one function $f : X \to \mathbb{R}$ such that

$$\min\{f(x) | x \in U\} \leq F(U) \leq \max\{f(x) | x \in U\} \tag{23}$$

for every atom U of $\mathcal{A}$, and the inequalities in (23) are strict whenever $|U| > 1$.

Proof. We first prove the lemma under the stronger assumption that $\leq$ is a linear order over X. Under this assumption, we can assume without loss of generality that $X = \{1, \ldots, n\}$ with the standard order over numbers. For every $r \in \mathbb{R}$ let

$$U(r) = \min\{F(U)|U \text{ is an atom of } \mathcal{A}, F(U) > r\},$$

where we define the minimum of over the empty set $\varnothing$ to be 1. From the definition of $U(r)$ it follows that

$$r < U(r) \text{ and} \tag{24}$$

$$\text{if } r < F(U) \text{ then } U(r) \leq F(U), \tag{25}$$

for every $r \in \mathbb{R}$ and every atom U of $\mathcal{A}$.

For $x \in X$, let $\pi(x)$ be the atom of $\mathcal{A}$ that contains x. Call elements $x, y \in X$ *siblings* if $\pi(x) = \pi(y)$.

We now define $f : X \to \mathbb{R}$ inductively so that f is strictly monotone and the following condition is satisfied for every $x \in X$ and every atom U of $\mathcal{A}$:

$$\text{If } x < U \text{ the } f(x) < F(U) \tag{26}$$

Let $f(0) = 0$. Let $z \geq 1$ and suppose we have already defined $f(1) < \cdots < f(z-1)$, such that (26) is satisfied for $x = 1, \ldots, z-1$.

Case 1. *[$|\pi(z)| = 1$] Choose $f(z) = F(\pi(z))$. Note that in this case $z - 1 < \{z\} = \pi(z)$ and therefore, $f(z-1) < F(\pi(z)) = f(z)$, where the first inequality follows from (26) for $x = z - 1$ and $U = \{z\}$. In addition, if U is an atom of $\mathcal{A}$ and $z < U$ then $\pi(z) = z < U$ and therefore $F(\pi(z)) < F(U)$ by (22). Thus (26) is satisfied for $x = z$.*

Case 2. *[$|\pi(z)| > 1$ and $z = \max \pi(z)$] Let $r = f(z-1) \vee F(\pi(z))$. We choose $f(z)$ such that $r < f(z) < U(r)$. In particular $f(z-1) < f(z)$. Let U be an atom of $\mathcal{A}$ such that $z < U$.*

Then, in particular, $z-1<U$ *and therefore* $f(z-1)<F(U)$ *by (26) with* $x=z-1$*. Also, since* $z=\max\pi(z)$ *it follows that* $\pi(z)<U$ *and, therefore,* $F(\pi(z))<F(U)$ *by (22). This implies that* $r<F(U)$ *and so* $f(z)<U(r)\leq F(U)$*, where the second inequality follows from (25). Thus, (26) is satisfied for* $x=z$*.*

Case 3. *[*$|\pi(z)>1|$ *and* $z<\max\pi(z)$*] Choose* $f(z)$ *so that* $f(z-1)<f(z)<U(f(z-1))$*. Let* U *be an atom of* $\mathcal{A}$ *such that* $z<U$*. Then,* $z-1<U$ *and therefore* $f(z-1)<F(U)$ *by 26 with* $x=z-1$*. Hence,* $f(z)<U(f(z-1))\leq F(U)$*, where the inequality follows from (25).*

We now claim that the function f defined above satisfies (23). Indeed, let U be an atom of $\mathcal{A}$. Suppose first that $|U|=1$, so that $U=\{z^*\}$ for some $z^*\in X$. Then $\pi(z^*)=U$ and therefore $f(z^*)=F(U)$ by Case 1 in the construction of f. In particular, (23) is satisfied with equalities. Suppose now that $|U|>1$ and let $z_{\max}$ and $z_{\min}$ be the maximal and minimal elements of U. Thus, $\pi(z_{\min})=\pi(z_{\max})=U$. From Case 2 above, $F(U)<f(z_{\max})$. In addition, $z_{\min}-1<U$ so that $f(z_{\min}-1)<F(U)$ by (26), and therefore $f(z_{\min})<U(f(z_{\min}-1))\leq F(U)$, where the first inequality follows from Case 3 in the construction of f and the second inequality follows from (25). In particular, (23) is satisfied with strict inequalities. The proof of the lemma for linear orders is now complete.

We now turn to environments in which the order relation $\leq$ over X is an interval order. We will show that $\leq$ can be extended to a linear order $\leq'$ over X that satisfies (22). Then, by the previous argument, there exists a ($\leq'$-strictly monotone and in particular) $\leq$-strictly monotone function f that satisfies (23). Let $I_x=[l(x),r(x)]$ be a representation of $\leq$ and assume without loss of generality that I_x and I_y have no common endpoints whenever $x,y\in X$ and $x\neq y$.

For an atom U of $\mathcal{A}$ let $L(U)=\min\{l(x)|x\in U\}$ and $R(U)=\max\{r(x)|x\in U\}$. Fix $x^*\in X$ such that I_{x^*} is not degenerate. We will find $p^*\in I_{x^*}$ such that the extension $\leq'$ that is induced by the interval representation I'_x given by

$$I'_x=\begin{cases}[p^*,p^*], & \text{if } x=x^*\\ I_x, & \text{otherwise}\end{cases}$$

satisfies (22). If $r(x^*) < R(\pi(x^*))$ we choose $p^* = l(x^*)$. Then I' and I induce the same order over atoms of $\mathcal{A}$ and therefore (22) holds. If $r(x^*) = R(\pi(x^*))$ we choose $p^* = r(x^*)$ and if $r(X^*) = R(\pi(x^*))$ and $l(x^*) = L(\pi(x^*))$ we proceed as follows. Let

$$\begin{aligned} p_{\max} &= \min\{R(V) | V \in \mathcal{A}, F(\pi(x^*)) \leq F(V)\} \text{ and} \\ p_{\min} &= \max\{L(W) | W \in \mathcal{A}, F(W) \leq F(\pi(x^*))\}, \end{aligned} \tag{27}$$

where the minimum and maximum over the empty set $\varnothing$ are taken as 1 and 0 respectively. We claim that $p_{\min} < p_{\max}$. Indeed, let $V, W \in \mathcal{A}$ such that $F(W) \leq F(\pi(x^*)) \leq F(V)$. We distinguish between two cases:

Case 4. *[$V = W$] Since F is one-to-one it follows that $W = \pi(x^*) = V$. Since I_{x^*} is not degenerate it follows that*

$$L(W) \leq l(x^*) < r(x^*) \leq R(V).$$

Case 5. *[$V \neq W$] By (22) there exists $y \in V$ and $z \in W$ such that $y \not< z$ (i.e., it is not the case that $y < z$.) Since $V \neq W$ it follows that $y \neq z$. Since I_x is a representation of $\leq$ it follows that $l(z) < r(y)$. In particular, $L(W) < R(V)$.*

Thus, $L(W) < R(V)$ in both cases. As V, W were arbitrary, it follows that $p_{\min} < p_{\max}$ as desired. Let p^* be chosen so that $p_{\min} < p^* < p_{\max}$ and $I_x \neq [p^*, p^*]$ for every $x \in X$. We claim that $\leq'$ satisfies (22). For atoms U, V of $\mathcal{A}$ which are different from $\pi(x^*)$ $\leq'$ satisfies (22) because $\leq$ does. Assume now that $U = \pi(x^*)$, $V \in \mathcal{A}$, and $V <' U$. We have to show that $F(V) < F(U)$. Indeed, if $F(U) \leq F(V)$ then $p_{\max} \leq R(V)$ by (27), which leads to contradiction, since $R(V) \leq p^* < p_{\max}$, where the first inequality follows from the definition of $\leq'$. By a similar argument, (22) is satisfied when $U = \pi(x^*)$, $V \in \mathcal{A}$ and $U <' V$.

We showed that if $\leq$ is an interval order over X with representation I_x that satisfies (22) and $x^* \in X$ then $\leq$ can be extended to an interval order over X with representation I'_x that satisfies (22) such that $I'_x \subseteq I_x$ for every $x \in X$ and I_{x^*} is degenerate. Going over all the elements of X, we get an extension $\leq'$ of $\leq$ which satisfies (22) such that all the intervals in the representation $\leq'$ are degenerate. Therefore. $\leq'$ is a linear order. ■

A.2.c PROOF OF THEOREM 3

Let $\sigma : S^{\leqslant N} \to \{\mathrm{a}, \mathrm{b}\}$ be a experimental observations that satisfies the condition of Theorem 3. We assume without loss of generality that $\sigma(e) = \mathrm{a}$.

Let $\pi : S^n \to S^{n-1}$ be the *parent function* of the tree of instances: $\pi(s_1, \ldots, s_n) = (s_1, \ldots, s_{n-1})$. Let $\leq^n$ stand for the relation "revealed higher" over S^n: For two nodes $\mathbf{s}, \mathbf{t} \in S^n$, $\mathbf{t} \leq^n \mathbf{s}$ whenever $\mathbf{s}$ is revealed higher than $\mathbf{t}$.

We claim first that $\leq^n$ is an interval order for every n. We prove the assertion by induction over $N - n$ (the remaining layers in the tree). For $n = N$ the order over S^n is in fact a ranking, the ranks being the sets $\{\mathbf{s} \in S^N | \sigma(\mathbf{s}) = a\}$ and $\{\mathbf{s} \in S^N | \sigma(\mathbf{s}) = b\}$. Assume now that $\leq^n$ is represented by $I^n_{\mathbf{s}} = [l^n(\mathbf{s}), r^n(\mathbf{s})]$. Without loss of generality, suppose also that $0 < l^n(\mathbf{s}) \leq r^n(\mathbf{s}) < 1$ for every $\mathbf{s} \in S^n$. For $\mathbf{s} \in S^{n-1}$ let $I^{n-1}_{\mathbf{s}} = [l^{n-1}(\mathbf{s}), r^{n-1}(\mathbf{s})]$, where

$$\begin{aligned}
&l^{n-1}(\mathbf{s}) = \lambda(\mathbf{s}) + \min\{l^n(\mathbf{s}') | \mathbf{s}' \in S^n \text{ and } \pi(\mathbf{s}') = \mathbf{s}\}, \text{ and} \\
&r^{n-1}(\mathbf{s}) = \lambda(\mathbf{s}) + \max\{r^n(\mathbf{s}') | \mathbf{s}' \in S^n \text{ and } \pi(\mathbf{s}') = \mathbf{s}\}, \text{ where} \\
&\lambda(\mathbf{s}) = \begin{cases} 1, \text{ if } \sigma(\mathbf{s}) = a \\ 0, \text{ if } \sigma(\mathbf{s}) = b \end{cases} .
\end{aligned}$$

Then $r^{n-1}(\mathbf{t}) \leq l^{n-1}(\mathbf{s})$ if one of the following is satisfied:

- $\sigma(\mathbf{s}) = a$ and $\sigma(\mathbf{t}) = b$.
- $\sigma(\mathbf{s}) = \sigma(\mathbf{t})$ and $\mathbf{s}' <^n \mathbf{t}'$ for every child $\mathbf{s}'$ of $\mathbf{s}$ and every child $\mathbf{t}'$ of $\mathbf{t}$.

It follows from the recursive definition of $\leq^n$ that I^{n-1} is a representation of $\leq^{n-1}$.

We now construct an assignment $p_{\mathbf{s}} \in (0, 1)$ of probabilities for every $\mathbf{s} \in S^{\leqslant N}$ such that

$$p_e > 1/2, \tag{28}$$

$$p_{\mathbf{s}} \in \mathrm{ri}\,(conv\{p(\mathbf{s}') | \mathbf{s}' \text{ is a child of } \mathbf{s}\}) \text{ for every } \mathbf{s} \in S^{<N}, \text{ and} \tag{29}$$

$$p_{\mathbf{s}} > p_{\mathbf{t}} \text{ whenever } \sigma(\mathbf{s}) = a, \sigma(\mathbf{t}) = b \text{ and } d(\mathbf{s}) = d(\mathbf{t}). \tag{30}$$

To construct p, we go over the nodes from the root to the leafs. Choose p_e arbitrary such that $p_e > 1/2$. Assume we defined $p_{\mathbf{s}}$ for $\mathbf{s} \in S^{n-1}$ such that $\mathbf{s} \mapsto p_{\mathbf{s}}$ is $\leq^{n-1}$-strictly monotone and one-to-one. The set S^n is equipped with an interval order $\leq^n$. The parent function function $\pi : S^n \to S^{n-1}$ induces a partition over S^n (the atoms of the partition are $\pi^{-1}(\mathbf{s})$ for $\mathbf{s} \in S^{n-1}$). Let $U = \pi^{-1}(\mathbf{s})$ and $V = \pi^{-1}(\mathbf{t})$ be two such atoms. If $V <^n U$ then, by Definition 5 it follows that $\mathbf{t} <^{n-1}< \mathbf{s}$ and therefore $p_{\mathbf{t}} < p_{\mathbf{s}}$. Therefore Condition (22) of Lemma 7 is satisfied and thus p can be defined over S^n such that (29) p is is satisfied. Therefore p is Bayesian. Finally if $\mathbf{s}, \mathbf{t} \in S^n$ and $\sigma(\mathbf{s}) = a$ and $\sigma(\mathbf{t}) = b$ then $\mathbf{t} <^n \mathbf{s}$ by definition of $\leq^n$ and therefore $p_{\mathbf{t}} < p_{\mathbf{s}}$, since p is $\leq^n$ strictly monotone.

We now claim that for every $1 \leq n \leq N$ there exists some $0 < r_n < \infty$ such that

$$\begin{aligned} &\rho(r_n, p_{\mathbf{s}}) > 1/2 \text{ for every } \mathbf{s} \in S^n \text{ such that } \sigma(\mathbf{s}) = \text{a}, \text{ and} \\ &\rho(r_n, p_{\mathbf{t}}) < 1/2 \text{ for every } \mathbf{t} \in S^n \text{ such that } \sigma(\mathbf{t}) = \text{b}, \end{aligned} \tag{31}$$

where ρ is given in 19. Indeed, fix $1 \leq n \leq N$ and let q be a real number such that

$$\begin{aligned} &q < p_{\mathbf{s}} \text{ for every } \mathbf{s} \in S^n \text{ such that } \sigma(\mathbf{s}) = \text{a}, \text{ and} \\ &p_{\mathbf{t}} < q \text{ for every } \mathbf{t} \in S^n \text{ such that } \sigma(\mathbf{t}) = \text{b}. \end{aligned} \tag{32}$$

Since the function ρ is continuous and monotone in the first argument, and since $\lim_{r\to 0} \rho(q, r) = 0$ and $\lim_{r\to\infty} \rho(q, r) = 1$, it follows that there exists some r_n such that $\rho(r_n, q) = 1/2$. Since the function ρ is monotone in the second argument, (32) implies that

$$\begin{aligned} &\rho(r_n, p_{\mathbf{s}}) > \rho(r_n, q) = 1/2 \text{ for every } \mathbf{s} \in S^n \text{ such that } \sigma(\mathbf{s}) = \text{a}, \text{ and} \\ &\rho(r_n, p_{\mathbf{t}}) < \rho(r_n, q) = 1/2 \text{ for every } \mathbf{t} \in S^n \text{ such that } \sigma(\mathbf{t}) = \text{b}, \end{aligned}$$

as desired. We now define $r_0 > 0$ such that (31) is also satisfied for $n = 0$, and moreover,

$$1 \in \text{ri}\left(\text{conv}\{r_n | n = 0, \ldots, N\}\right).$$

To achieve this, choose $r_0 > 1$ arbitrarily if $r_n < 1$ for some $n \in \{1, \ldots, N\}$; choose $r_0 < 1$ such that $\rho(r_0, p_e) > 1/2$ if $r_n \geq 1$ for every $n \in \{1, \ldots, N\}$ and $r_n > 1$ for some $n \in \{1, \ldots, N\}$; and choose $r_0 = 1$ if $r_n = 1$ for every $n \in \{1, \dots, N\}$. Since $p_e > 1/2$ such a choice can be made and, furthermore, (31) is satisfied. Let $\lambda_n > 0$ be such that $\sum_{n=0}^{N} \lambda_n = 1$ and $\sum_{n=0}^{N} \lambda_n r_n = 1$.

We now construct (α, τ, ζ) such that τ and ζ are independent given α,

$$\mathbb{P}\left(\alpha = \mathrm{a} | \zeta_i = s_i \text{ for } 1 \leq i \leq n\right) = p_{\mathbf{s}}, \tag{33}$$

for every $\mathbf{s} = (s_1, \dots, s_n) \in S^{\leq N}$, and

$$\frac{\mathbb{P}(\tau = n | \alpha = \mathrm{b})}{\mathbb{P}(\tau = n | \alpha = \mathrm{a})} = r_n \tag{34}$$

for every $n \in \{1, \dots, N\}$. By Lemma 1 there exists random variables (α, ζ) over some probability space with values in U and S^N respectively such that (33) is satisfied. Possibly augmenting the underlying probability space, we introduce the random variable τ with values in $\mathbf{N}$ such that τ is independent of ζ given α,

$$\mathbb{P}\left(\tau = n | \alpha = \mathrm{a}\right) = \lambda_n, \text{ and}$$
$$\mathbb{P}\left(\tau = n | \alpha = \mathrm{b}\right) = \lambda_n r_n.$$

Then (34) is satisfied. By Lemma 5, (33), and (34), we get that

$$\mathbb{P}\left(\alpha = \mathrm{a} | \tau = n, \zeta_i = s_i \text{ for } 1 \leq i \leq n\right) = \rho(r_n, p_{\mathbf{s}}).$$

Finally, from the latter equation and (32) it follows that

$$\mathbb{P}\left(\alpha = \mathrm{a} | \tau = n, \zeta = \mathbf{s}\right) > 1/2 \text{ for every } \mathbf{s} \in S^n \text{ such that } \sigma(\mathbf{s}) = \mathrm{a}, \text{ and}$$
$$\mathbb{P}\left(\alpha = \mathrm{b} | \tau = n, \zeta = \mathbf{t}\right) < 1/2 \text{ for every } \mathbf{t} \in S^n \text{ such that } \sigma(\mathbf{t}) = \mathrm{b},$$

as desired. ■

A.3. Unordered Dimensions.

We will use the following version of the alternative theorem (see Border(2003), Theorem 10).

Proposition 1. *[Theorem of the Alternative] Let M be a matrix. Then exactly one of the* *alternatives holds. Either*

$$xM \leq 0$$

for some row vector $x > 0$ *or*

$$My \gg 0$$

for some column vector $y \geq 0$[18]

Proof of Theorem 4. Fix experimental observations σ. Let M be the $(|\mathcal{O}| \cdot (|A| - 1)) \times (|A| \cdot |S|^N)$ matrix whose rows are indexed $(\mathbf{s}, c)$ for instance $\mathbf{s} = (D, \delta)$ and $c \neq \sigma(\mathbf{s})$ and whose columns are indexed $(a, s_1, \ldots, s_N)$ for $a \in A$ and $s_1, \ldots, s_N \in S$ and such the matrix entry $M[\mathbf{s}, c][a, s_1, \ldots, s_N]$ at row $(\mathbf{s}, c)$ and column $(a, s_1, \ldots, s_N)$ is given by

$$M[\mathbf{s}, c][a, s_1, \ldots, s_N] = \begin{cases} 1, & \text{if } \mathbf{s} \text{ agrees with } s_1, \ldots, s_N \text{ and } a = \sigma(\mathbf{s}) \\ -1, & \text{if } \mathbf{s} \text{ agrees with } s_1, \ldots, s_N \text{ and } a = c \\ 0, & \text{otherwise.} \end{cases}$$

The assertion follows from Proposition 1 and from the following two simple lemmas.

Lemma 8. *[Explainable* σ *– Matrix Form] There exists* $y \geq 0$ *such that* $My \gg 0$ *if and only if* σ *can be explained by a restricted conjectured experiment.*

Proof. Assume that $(\alpha, \tau, \zeta_1, \ldots, \zeta_N)$ is a restricted conjectured experiment that explains σ. Then for every instance $\mathbf{s}$ and every $c \neq \sigma(\mathbf{s})$ it follows from the definition of explanation and the fact that τ is independent of (α, ζ) that

$$\mathbb{P}(\alpha = \sigma(\mathbf{s}) | \mathbf{s} \text{ agrees with } \zeta_1, \ldots, \zeta_N) > \mathbb{P}(\alpha = c | \mathbf{s} \text{ agrees with } \zeta_1, \ldots, \zeta_N).$$

Since by (9) $\mathbb{P}(\zeta_1 = s_1, \ldots, \zeta_N = s_N) > 0$, the last equation is equivalent to

$$\mathbb{P}(\alpha = \sigma(\mathbf{s}),\ \mathbf{s} \text{ agrees with } \zeta_1, \ldots, \zeta_N) > \mathbb{P}(\alpha = c,\ \mathbf{s} \text{ agrees with } \zeta_1, \ldots, \zeta_N).$$

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[18] For a vector x, $x \geq 0$ means that all coordinates of x are nonnegative, $x > 0$ means that $x \geq 0$ and $x \neq 0$ and $x \gg 0$ means that all coordinates of x are strictly positive.

Thus, for every instance $\mathbf{s}$ and every $c \neq \sigma(\mathbf{s})$, we get that

$$\sum_{s_1,\dots,s_N \text{ and } \mathbf{s} \text{ agrees with } s_1,\dots,s_N} \mathbb{P}(\alpha = \sigma(\mathbf{s}), \zeta_1 = s_1, \dots, \zeta_N = s_N) - \mathbb{P}(\alpha = c, \zeta_1 = s_1, \dots, \zeta_N = s_N) > 0.$$

Therefore $My \gg 0$ when y be the column vector such that

$$y[a, s_1, \dots, s_N] = \mathbb{P}(\alpha = a, \zeta_1 = s_1, \dots, \zeta_N = s_N). \tag{35}$$

Conversely, assume that $My \gg 0$ for some $y \geq 0$. Since the set of solutions to $My \gg 0$ is open we can assume without loss of generality that $y \gg 0$. Moreover, we can assume without loss of generality that y is normalized so that the sum of its entries is 1. Let α, ζ be random variables whose joint distribution is given by (35) and let τ be a random variable with values in $\mathcal{P}(N)$ and full support which is independent of (α, ζ). Then the above argument can be reversed to show that the restricted conjectured experiment (α, τ, ζ) explains σ. ■

Lemma 9. *[Dutch Books – Matrix Form] There exists $x > 0$ such that $xM \leq 0$ if and only if σ admits a Dutch book.*

Proof. Let $x > 0$ such that $xM \leq 0$. Then every coordinate $(\mathbf{s}, c)$ of x where $\mathbf{s} \in \mathcal{O}$ and $c \neq \sigma(\mathbf{s})$ such that such $z = x[\mathbf{s}, c] > 0$ gives rise to a positive bet $(z, \mathbf{s}, \sigma(\mathbf{s}))$. Since $x > 0$ the set of coordinates is non-empty. Since $xM \leq 0$, the corresponding set of bets is a Dutch book. The argument is reversible. ■

The proof of the Theorem now follows directly. ■