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Game Theory

Note on Repeated Games with Non-Monotonic Value

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We show that even when the information structure is independent of the state of nature, the of the n-stage zero-sum game with incomplete information is not necessarily monotonic with to the length of the game. More precisely, we give an example of such an n-stage game in which $V_2 < V_3$.

ords: Repeated games with incomplete information, value, imperfect monitoring.

ntroduction

ose K is a finite set of states of nature, each corresponding to a payoff matrix zero-sum game. Two players, PI and PII, take part in the n-stage game, ted G_n, played as follows: nature chooses a state k from the set K according robability distribution P. PI is informed about the state chosen. PII has only owledge of P. Then, the players repeat n times the zero-sum game defined e payoff matrix A_k.

he information is standard, that is, after each stage all past actions of both s are announced, then the value V_n of the above described game is tonic (decreasing) with respect to n. This is due to a recursive structure to the sequence V_n, $n \geq 1$. The intuitive reasoning for this is that the ation PII receives from PI increases with the number of stages played. This ation is utilized by PII to improve his average payoff (see Sorin [1980]).

ppose now that the game is played with imperfect monitoring. There exist formation functions, L_1 and L_2 such that after each stage in which player d action a_i, player j gets the information signal $L_j(a_1, a_2)$. Note that games perfect monitoring are a special case of games with imperfect monitoring g the information functions to be one-to-one on the set of ordered pairs of s.

general games with imperfect monitoring, a recursive structure fails to ince PI might be unable to estimate precisely the information gathered

ery grateful to Ehud Lehrer who introduced this question to me.

by PII during the game. Therefore, the proof of monotonicity of V_n is no lo[nger] valid.

It has been shown by Lehrer [1987] that if the information functions a[re] dependent on the state of nature, then the sequence V_n may be non-monoto[nic]. The example Lehrer gives hinges heavilly on the fact that the informatio[n] state-dependent. This note is devoted to the case of state-independent infor[ma]tion structure. Therefore, the signals received during the game imply only [the] actions taken by both players and cannot reveal directly the state of nature. [We] show that V_n may be non-monotonic also when restricting the signalling to [be] state-independent.

2 The Example

Let $S=(s_i)_{i=1}^{6}$ and $T=(t_j)_{j=1}^{5}$ be the possible actions of players PI and P[II] respectively, in the game where the state space is $K=\{1,2,3\}$ and the distribu[tion] over K is $P=\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\}$.

Let the payoff matrices be:

$$A_1=\begin{bmatrix} 40 & 4 & 0 & 4 & 4\\ -100 & -100 & -100 & -100 & -100\\ -100 & -100 & -100 & -100 & -100\\ 40 & 0 & 4 & 4 & 4\\ -100 & -100 & -100 & -100 & -100\\ -100 & -100 & -100 & -100 & -100 \end{bmatrix}$$

$$A_2=\begin{bmatrix} -100 & -100 & -100 & -100 & -100\\ 0 & 40 & 4 & 4 & 4\\ -100 & -100 & -100 & -100 & -100\\ -100 & -100 & -100 & -100 & -100\\ 4 & 40 & 0 & 4 & 4\\ -100 & -100 & -100 & -100 & -100 \end{bmatrix}$$

$$A_3=\begin{bmatrix} -100 & -100 & -100 & -100 & -100\\ -100 & -100 & -100 & -100 & -100\\ 4 & 0 & 40 & 4 & 4\\ -100 & -100 & -100 & -100 & -100\\ -100 & -100 & -100 & -100 & -100\\ 0 & 4 & 40 & 4 & 4 \end{bmatrix}$$

[Sup]pose that the information matrices are:

$$B_1=\begin{bmatrix} 1 & 2 & 3 & N & N\\ 1 & 2 & 3 & N & N\\ 1 & 2 & 3 & N & N\\ 1 & 2 & 3 & N & N\\ 1 & 2 & 3 & N & N\\ 1 & 2 & 3 & N & N \end{bmatrix} \qquad B_2=\begin{bmatrix} N & N & N & 12 & 31\\ N & N & N & 12 & 32\\ N & N & N & 23 & 31\\ N & N & N & 13 & 21\\ N & N & N & 23 & 21\\ N & N & N & 13 & 32 \end{bmatrix}$$

[In o]ther words, when PI plays action s_j and PII plays action t_m player i receives [the] signal $L_i(s_j,t_m)=B_i(j,m)$ for all $i\in\{1,2\}$ $j\in\{1,\dots,6\}$ $m\in\{1,\dots,5\}$.

Note that in each information matrix, there is an informative part (containing [the] signals 12, 13, 23, 21, 23, 32, 1, 2, 3) and a non-informative part (in which the [sig]nal is null: N).

It is assumed that at each stage the players know, in addition to the signals [anno]unced, the actions they themselves chose in all previous stages.

The intuitive description of the proof is as follows. For each state $k\in K$, very [high] payoffs appear in four rows of the payoff matrix A_k. This forces PI to use, [thro]ughout games in which the number of stages is small, one out of two actions [(de]pending on the state k). PII, knowing this, can deduce after the first stage of [an]y short enough game a new partition of the space of states. Namely, he can ["cro]ss out" one of the states and consider only two out of the three states as being [poss]ibly played. Moreover, the information structure allowes PII to do so [with]out enabling PI to evaluate exactly which state has been "crossed out" by [PII]. An action of PII which realizes the information he has gathered contributes [no] additional information to PII. However, such an action discloses completely [hi]s information to PI, who can play optimally against PII from then, on, and [he]nce increase the payoffs.

Specifically, in the 2-stage game PII, realizing the information he has gathered [dur]ing the first stage of the game, can decrease the payoff in the second stage and [he]nce $V_1>V_2$.

In the 3-stage game, if PII uses the information he has gathered on the second [sta]ge, his private knowledge is revealed to PI and the payoff on the third stage will [inc]rease. Otherwise, during stage 2, PII will have to ignore the information he has [alre]ady gathered. This leads to the inequality $V_2<V_3$.

Note that such a construction can not be made when the number of possible [sta]tes of nature is two. "Crossing out" of a matrix in that case is equivalent to PII [kno]wing exactly which state has been chosen (having a degenerate partition of [th]e space of states).

[3] Formal Proof

[De]note by G_n the n-fold repeated game.

In G_1, if PII plays his pure strategy t_4, he guarantees a payoff ≤ 4. If PI plays his pure strategy s_k when informed that state k has been chosen, he ensures himself a payoff ≥ 4. Hence, $V_1 = 4$.

In G_2, PII can ensure a payoff ≤ 3 by using the following strategy:

Stage 1: play t_4 or t_5 with an equal probability of $\frac{1}{2}$.
Stage 2: play t_1 if 23 or 32 have been signalled.
play t_2 if 13 or 31 have been signalled.
play t_3 if 12 or 21 have been signalled.

The logical interpretation of this strategy is that the differences between the payoffs appearing in the game matrices make deceiving for PI (playing as if the state of nature were different from the state k actually chosen, i.e. not playing s_k or s_{k+3}) unworthy. Thus, if the signal $10*i + j$, $1 \leq i, j \leq 3$, has been announced to PII after the first stage of the game, the state of nature is very likely to be i or j.

Mathematically, we may assume, from symmetry considerations, that $k = 1$. If at the first stage PI has chosen s_2, s_3, s_5, or s_6, the lump payoff doesn't exceed $\frac{-100+40}{2} = -30 < 3$. Otherwise, again from symmetry considerations, we may assume s_1 was played at the first stage. Thus PII got the signal 12 or 31. PI does not know which of these signals PII got, but he knows PII got each of these signals with probability $\frac{1}{2}$. Therefore, PI can do no better then to expect PII to play t_2 or t_3 with probability $\frac{1}{2}$ each and to ensure an expected payoff of 2 in the second stage. Hence, $V_2 \leq \max\left(-30, \frac{(4+2)}{2}\right) = 3$.

In G_3, PI can ensure himself a payoff $\geq \frac{10}{3}$ by playing the following strategy assuming k is the state of nature that has been chosen:

Stages 1, 2: play each one of s_k, s_{k+3} with probability $\frac{1}{2}$.
Stage 3: If N was signalled in the second stage play s_k, s_{k+3} with probability $\frac{1}{2}$ each. Otherwise, play the following:

$k = 1$: $(\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0)$ or s_1 or s_4 for 1 or 2 or 3 respectively;
$k = 2$: $(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0)$ or s_2 or s_5 for 2 or 3 or 1 respectively;
$k = 3$: $(0, 0, \frac{1}{2}, 0, 0, \frac{1}{2})$ or s_3 or s_6 for 3 or 1 or 2 respectively;

(the notation $(q_1, \ldots, q_6)$, $\sum_{i=1}^{6} q_i = 1$, is used to denote the one-stage mixed action of PI where the action s_i is played with probability q_i, $1 \leq i \leq 6$)

It suffices to show that PI, using the above strategy, gets payoff of at least $\frac{10}{3}$ against any pure strategy of PII. Let $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ be a pure strategy of PII, σ_i a function from PII's histories of length $i-1$ stages to the space of actions available for him.

We denote $\varphi(23) = \varphi(32) = t_1$, $\varphi(13) = \varphi(31) = t_2$, and $\varphi(12) = \varphi(21) = t_3$; θ_i symbolizes the signal received by PII after stage i.

If $\sigma_1 \in \{t_1, t_2, t_3\}$ then PI gets payoff of 40 with probability $\frac{1}{3}$ and so the expected stage payoff is $\geq \frac{\frac{1}{3}*40}{3} \geq \frac{10}{3}$.

Thus, we may assume $\sigma_1 \in \{t_4, t_5\}$, guaranteeing a payoff of 4 and a signal $\theta_1 \neq N$ on the first stage.

If $\sigma_2(\theta_1) \in \{t_4, t_5\}$, then the payoff in the second stage is 4 and, since PI ensures himself a payoff of 2 on the third stage, the payoff in G_3 is $\geq \frac{4+4+2}{3} = \frac{10}{3}$.

Suppose now that $\sigma_2(\theta_1) \in \{t_1, t_2, t_3\}$ and $\sigma_2(\theta_1) \neq \varphi(\theta_1)$, then with probability $\frac{1}{2}$, $\sigma_2(\theta_1) = t_k$, where k is the state of nature that has been chosen, and the payoff in the second stage of the game is 40. Thus the expected payoff in G_3 is at least $\frac{4 + \frac{1}{2}*40}{3} \geq \frac{10}{3}$.

If $\sigma_2(\theta_1) = \varphi(\theta_1)$, PII ensures a payoff of 2 in the second stage, but no information is augmented for him ($\theta_2 = N$). Moreover, all the information PII has is revealed. PI is, therefore, able to play optimally on the third stage according to the following two cases:

1. If $\sigma_3(\theta_1, N) \in \{t_1, t_2, t_3\}$ and $\sigma_3(\theta_1, N) \neq \sigma_2(\theta_1)$, then the payoff in the third stage is 40 with probability $\frac{1}{2}$ and hence gives expected total payoff of at least $\frac{4 + 2 + \frac{1}{2}*40}{3} = \frac{26}{3} > \frac{10}{3}$ in G_3.
2. If $\sigma_3(\theta_1, N) \in \{\sigma_2(\theta_1), t_4, t_5\}$, then the above strategy of PI clearly ensures a payoff of 4 on the third stage and so guarantees a payoff of $\frac{(4+2+4)}{3} = \frac{10}{3}$ in G_3.

Finally, we deduce $V_3 \geq \frac{10}{3} > 3 \geq V_2 < 4 = V_1$ and the sequence V_n is non-monotonic.

4 Comments

4.1 It is easily verified that restricting the information signals to be common to both players (as well as independent of the state of nature) does not guarantee monotonicity of the value. For instance, suppose we add to each *payoff* matrix in the game described above four columns on the right identical to the right most column in the original matrix. Furthermore, suppose the information matrix is:

$$B = \begin{bmatrix} 1 & 2 & 3 & a & a & b & b & c & c \\ 1 & 2 & 3 & a & b & b & c & c & a \\ 1 & 2 & 3 & b & b & c & c & a & a \\ 1 & 2 & 3 & b & c & c & a & a & b \\ 1 & 2 & 3 & c & c & a & a & b & b \\ 1 & 2 & 3 & c & a & a & b & b & c \end{bmatrix}$$

Then we get a game with a sequence of values (V_n) identical to the one obtain the original example. In particular, the values are not monotonic.

4.2 Similar considerations may be used to construct a game with observ payoffs (i.e., the signals, for both players at each stage, being the payoffs achie where the corresponding sequence (V_n) is non-monotonic. For example, cons the game with state space $K = \{1, 2, 3\}$, distribution over K: $P = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, payoff matrices:

$$A_1 = \begin{bmatrix} h & 4 & -4 & 1 & 1 & 2 & 2 & 3 & 3 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ h & -4 & 4 & 2 & 3 & 3 & 1 & 1 & 2 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -4 & h & 4 & 2 & 2 & 3 & 3 & 1 & 1 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ 4 & h & -4 & 3 & 1 & 1 & 2 & 2 & 3 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ 4 & -4 & h & 3 & 3 & 1 & 1 & 2 & 2 \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -h & -h & -h & -h & -h & -h & -h & -h & -h \\ -4 & 4 & h & 1 & 2 & 2 & 3 & 3 & 1 \end{bmatrix}$$

where $h = 100$.

Analogous calculations to those introduced in this note lead to the conc that for this game, $V_1 = 2 > V_2 = 1 < V_3 = \frac{4}{3}$.

References

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Game Theory

ectivity Functions and Simple Games

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: The rather new notion of effectivity function is related to the notion of simple game. Every function is associated with a simple game. So theory about simple games may be applicable ivity functions. E.g. if the effectivity function is additive, then the associated simple game is . Via a characterization of weighted simple games it is possible to characterize maximal effectivity functions.

we characterize additive effectivity functions and their associated simple games.

ds: Effectivity functions, simple games, additive, weighted, k-trade robust, asummability, maximal, monotonic, strongly monotonic, power preserving

Introduction

r to a cooperative game a coalitional game form describes for each on how much it can achieve on its own. The main difference between rative game and a coalitional game form is that in the latter the "value" of tion is neither a real number nor a set of vectors, but a family of subsets the coalition is effective for. One interpretation of a coalitional game form s follows. Let B be in that family of subsets of coalition S. Then S can force its veto power) the outcome to be in B. These game forms are especially in all kind of social situations.

example let there be six applicants for a job and a committee of five ers who decides which of the applicants is assigned to that job. As a final n mechanism each committee member has one veto vote by which he can precisely one candidate of getting the job. Then every committee member e the outcome to be in any subset of at least five candidates. Moreover, coalition of k members is effective for every subset of at least $6 - k$ ants. Let none of the applicants be a committee member. Then the ng (weighted) simple game is related to the example above. Let the set of s be the union of the committee and the applicants. A coalition of players is whenever it consists of at least six players and not all coalition members

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