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A Simple Result on Comparative Statics

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Let X and P be sets and let $f\colon X \times P \to \boldsymbol{R}$. Assume that $x^*(p)$ maximizes $f(\cdot, p)$ over X. The question of *comparative statics* is, "How does x^* change as p changes?"

For the case where X and P are real intervals, we have the following basic result:

Proposition 1 *Let X and P be open intervals in $\boldsymbol{R}$, and let $f\colon X \times P \to \boldsymbol{R}$ be twice continuously differentiable. Assume that for all $x \in X$ and all $p \in P$,*

$$\frac{\partial^2 f(x, p)}{\partial p \partial x} > 0.$$

Let x^0 maximize $f(\cdot, p^0)$ over X and x^1 maximize $f(\cdot, p^1)$ over X. Then

$$(p^1 - p^0)(x^1 - x^0) \geqslant 0.$$

That is, if $p^1 > p^0$, then $x^1 \geqslant x^0$.

Proof: By hypothesis

$$f(x^0, p^0) \geqslant f(x^1, p^0) \qquad \text{and} \qquad f(x^1, p^1) \geqslant f(x^0, p^1).$$

Therefore, subtracting the first inequality from the second we have

$$f(x^1, p^1) - f(x^1, p^0) \geqslant f(x^0, p^1) - f(x^0, p^0). \tag{1}$$

Now write

$$g(x) = f(x, p^1) - f(x, p^0)$$

so that (1) becomes

$$g(x^1) - g(x^0) \geqslant 0, \tag{2}$$

and note that g is twice continuously differentiable. Therefore by the Second Fundamental Theorem of Calculus [1, Theorem 5.3, p. 205],

$$0 \leqslant g(x^1) - g(x^0) = \int_{x_0}^{x^1} g'(x)\, dx = \int_{x_0}^{x^1} \frac{\partial f(x, p^1)}{\partial x} - \frac{\partial f(x, p^0)}{\partial x}\, dx.$$

But again,

$$\frac{\partial f(x, p^1)}{\partial x} - \frac{\partial f(x, p^0)}{\partial x} = \int_{p^0}^{p^1} \frac{\partial}{\partial p} \frac{\partial f(x, p)}{\partial x}\, dp.$$

Thus (2) becomes

$$\int_{x^0}^{x^1} \int_{p^0}^{p^1} \frac{\partial^2 f(x,p)}{\partial p \partial x} \, dp \, dx \geqslant 0.$$

Therefore, if $p^1 > p^0$, then the inner integral is strictly positive, so the second integral is nonnegative only if $x^1 \geqslant x^0$. (Recall that if $b < a$, then $\int_a^b = -\int_b^a$.) Similarly, if $p^1 < p^0$, then $x^1 \leqslant x^0$. Either way the conclusion follows. ∎

Some remarks are in order.

- Note that this argument assumes nothing about the continuity of the function $x^*(p)$. Indeed, it need not even be a function—there could be several maximizers.
- The result is not a local result about derivatives—it applies to discrete parameter changes.
- However, if x^* is a differentiable function of p, then $\frac{dx^*}{dp} \geqslant 0$.
- There is no explicit appeal to second order conditions. (The second order condition is that $\frac{\partial^2 f(x^*,p)}{\partial x^2} \leqslant 0$.)
- The standard local argument goes like this: The first order condition is that

  $$\frac{\partial f(x^*,p)}{\partial x} = 0.$$

  Implicitly differentiating with respect to p gives

  $$\frac{\partial^2 f(x^*,p)}{\partial x^2} \frac{dx^*}{dp} + \frac{\partial^2 f(x^*,p)}{\partial p \partial x} = 0,$$

  The Implicit Function Theorem says that this is valid if the second order condition holds strictly,

  $$\frac{\partial^2 f(x^*,p)}{\partial x^2} < 0,$$

  in which case x^* is locally C^1, and

  $$\frac{dx^*}{dp} = -\frac{\dfrac{\partial^2 f(x^*,p)}{\partial p \partial x}}{\dfrac{\partial^2 f(x^*,p)}{\partial x^2}} > 0.$$

- The assumption that $\frac{\partial^2 f(x,p)}{\partial p \partial x} > 0$ could be weakened, as long as the Fundamental Theorem of Calculus holds. Also if this inequality is reversed, the inequality in the conclusion is reversed.

- The same logic applied to minimization reverses the inequality in the conclusion.
- There is an easy extension to the separable multivariate case.

Proposition 2 *Let X and P be open convex subsets of $\boldsymbol{R}^{\mathrm{n}}$, and let $f\colon X \times P \to \boldsymbol{R}$ be twice continuously differentiable. Assume that for all $x \in X$ and $p \in P$, and all $i, j = 1, \dots, n$*

$$\frac{\partial^2 f(x,p)}{\partial p_i \partial x_i} > 0 \qquad \textit{and} \qquad \frac{\partial^2 f(x,p)}{\partial p_i \partial x_j} = 0.$$

Let p^1 differ from p^0 only in the k^{th} coordinate. Let x^0 maximize $f(\cdot, p^0)$ over X and x^1 maximize $f(\cdot, p^1)$ over X. Then

$$(p_k^1 - p_k^0)(x_k^1 - x_k^0) \geqslant 0.$$

That is, if $p_k^1 > p_k^0$, then $x_k^1 \geqslant x_k^0$.

Proof: By the Second Fundamental Theorem of Calculus for Line Integrals [2, Theorem 10.3, p. 334],

$$g(x) = f(x,p^1) - f(x,p^0) = \int_0^1 \sum_{i=1}^n \frac{\partial f(x, p^0 + t(p^1 - p^0))}{\partial p_i}(p_i^1 - p_i^0)\, dt.$$

Now write $h(s) = g(x^0 + s(x^1 - x^0))$, so that

$$\begin{aligned} g(x^1) - g(x^0) \quad &= \quad = h(1) - h(0) \quad = \quad = \int_0^1 h'(s)\, ds \\ &= \int_0^1 \int_0^1 \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f(x^0 + s(x^1 - x^0), p^0 + t(p^1 - p^0))}{\partial p_i \partial x_j}(p_i^1 - p_i^0)(x_j^1 - x_j^0)\, dt\, ds \\ &= \int_0^1 \int_0^1 \sum_{i=1}^n \frac{\partial^2 f(x^0 + s(x^1 - x^0), p^0 + t(p^1 - p^0))}{\partial p_i \partial x_i}(p_i^1 - p_i^0)(x_i^1 - x_i^0)\, dt\, ds \\ &= (p_k^1 - p_k^0)(x_k^1 - x_k^0) \int_0^1 \int_0^1 \frac{\partial^2 f(x^0 + s(x^1 - x^0), p^0 + t(p^1 - p^0))}{\partial p_k \partial x_k}\, dt\, ds, \end{aligned}$$

and the conclusion follows as before. ∎

Now separability is a strong assumption, but is satisfied by the most common economic application, in which p is a vector of prices, and

$$f(x,p) = p \cdot x = \sum_{i=1}^n p_i x_i.$$

The argument given here is then reminiscent of Samuelson's [4, pp. 80–81] argument that conditional factor demands are downward sloping, and also Rochet [3].

References

[1] T. M. Apostol. 1967. *Calculus*, 2d. ed., volume 1. Waltham, Massachusetts: Blaisdell.

[2] ——— . 1969. *Calculus*, 2d. ed., volume 2. Waltham, Massachusetts: Blaisdell.

[3] J.-C. Rochet. 1987. A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of Mathematical Economics* 16:191–200.

[4] P. A. Samuelson. 1965. *Foundations of economic analysis*. New York: Athenaeum. Reprint of edition published by Harvard University Press, 1947.