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The Second Welfare Theorem

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Consider an Arrow–Debreu model economy

$$E = \big((X_i, \succcurlyeq_i)_{i=1}^m, (Y_j)_{j=1}^n, \omega\big).$$

Second Welfare Theorem *Assume the economy E satisfies the following conditions.*

1. *For each consumer $i = 1, \dots, m$*
   (a) *X_i is nonempty and convex.*
   (b) *$\succcurlyeq_i$ is continuous, locally nonsatiated, and convex.*
2. *For each producer $j = 1, \dots, n$,*
   (a) *Y_j is nonempty and convex.*

Let $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$ be an efficient allocation. Then there is a nonzero price vector $\bar{p}$ satisfying

1. *For each consumer $i = 1, \dots, m$, $\bar{x}^i$ minimizes $\bar{p} \cdot x$ over the upper contour set $\{x \in X_i : x \succcurlyeq \bar{x}^i\}$.*

   Thus if there is a cheaper point $\tilde{x} \in X_i$ satisfying $\bar{p} \cdot \tilde{x} < \bar{p} \cdot \bar{x}^i$, then $\bar{x}^i$ actually maximizes $\succcurlyeq_i$ over the budget set $\{x \in X_i : \bar{p} \cdot x \leqslant \bar{p} \cdot \bar{x}^i\}$.
2. *For each producer $j = 1, \dots, n$, $\bar{y}^j$ maximizes profit over Y_j at prices $\bar{p}$. That is,*

$$\bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot y \qquad \text{for all } y \in Y_j.$$

That is, $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n, \bar{p})$ is a ***valuation quasiequilibrium****. If the cheaper point condition holds for each i, then it is a* ***valuation equilibrium****.*

Proof: Since $(\bar{x}^1, \ldots, \bar{x}^m, \bar{y}^1, \ldots, \bar{y}^n)$ is efficient, it is impossible to make everyone better off. So define the set "Scitovsky set" S by

$$S = \sum_{i=1}^{m} P_i(\bar{x}^i),$$

and define the aggregate consumption possibility set A by

$$A = \omega + \sum_{j=1}^{n} Y_j.$$

By efficiency $A \cap S = \varnothing$. (For suppose, $x \in A \cap S$. Since $x \in S$, we can write $x = \sum_{i=1}^{m} x^i$, where each $x^i \in P(\bar{x}^i)$, or $x \succ \bar{x}^i$. Since $x \in A$, we can write $x = \omega + \sum_{j=1}^{n} y^j$. But then $(x^1, \ldots, x^m, y^1, \ldots, y^n)$ is an allocation, and $x^i \succ \bar{x}^i$ for each i, contradicting the efficiency of $(\bar{x}^1, \ldots, \bar{x}^m, \bar{y}^1, \ldots, \bar{y}^n)$.)

It follows from Lemmas 5 and 4 below that S is open and convex since each summand is, and is nonempty by local nonsatiation. Similarly A is convex. Thus by the Separating Hyperplane Theorem, there is a nonzero price vector $\bar{p}$ satisfying

$$\bar{p} \cdot x \geqslant \bar{p} \cdot y \quad \text{for each } x \in S,\ y \in A.$$

From Lemma 2 below, each $\bar{x}^i$ belongs to the closure of $P_i(\bar{x}^i)$, so $\sum_{i=1}^{m} \bar{x}^i$ belongs to the closure of S. Now $\sum_{i=1}^{m} \bar{x}^i = \omega + \sum_{j=1}^{n} \bar{y}^j$ so it also belongs to A. It follows that

$$\bar{p} \cdot x \geqslant \bar{p} \cdot \sum_{i=1}^{m} \bar{x}^i = \bar{p} \cdot \left(\omega + \sum_{j=1}^{n} \bar{y}^j \right) \geqslant \bar{p} \cdot y \quad \text{for each } x \in S,\ y \in A.$$

From the Summation Principle, we then have

$$\bar{p} \cdot \bar{x}^i \leqslant \bar{p} \cdot x \quad \text{for all } x \in P(\bar{x}^i) \qquad \text{and} \qquad \bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot y \quad \text{for all } y \in Y_j.$$

Since $U(\bar{x}^i)$ is the closure of $P(\bar{x}^i)$ we also have

$$\bar{p} \cdot \bar{x}^i \leqslant \bar{p} \cdot x \quad \text{for all } x \in U(\bar{x}^i).$$

This proves that we have a valuation quasiequilibrium. The role of the cheaper point condition is well known. ∎

Preliminary results on preferences

We start with some preliminary lemmas on preference relations. For our purposes, a preference relation $\succcurlyeq$ is quasiorder, or preorder, on a set X. That is, $\succcurlyeq$ is a total, transitive, reflexive binary relation on X. The binary relations $\succ$ and $\sim$ are the **asymmetric** and **symmetric parts** of $\succcurlyeq$, defined by

$$x \succ y \quad \text{if } x \succcurlyeq y \text{ and not } y \succcurlyeq x$$

and

$$x \sim y \quad \text{if } x \succcurlyeq y \text{ and } y \succcurlyeq x$$

Recall that a function $u \colon X \to \boldsymbol{R}$ is a **utility for** $\succcurlyeq$ if

$$x \succcurlyeq y \quad \Longleftrightarrow \quad u(x) \geqslant u(y).$$

Nonsatiation

A preference relation $\succcurlyeq$ on a set X has a **satiation point** x if x is a greatest element, that is, if $x \succcurlyeq y$ for all $y \in X$. A preference relation is **nonsatiated** if it has no satiation point. That is for every x there is some $y \in X$ with $y \succ x$.

If (X, d) is a metric space, the preference relation $\succcurlyeq$ is **locally nonsatiated** if for every $x \in X$ and every $\varepsilon > 0$, there exists a point $y \in X$ with $d(y, x) < \varepsilon$ and $y \succ x$. Note that this is a joint condition on X and $\succcurlyeq$. In particular, if X is nonempty, it must be that for each point $x \in X$ and every $\varepsilon > 0$ there is a point $y \neq x$ belonging to X with $d(y, x) < \varepsilon$. That is, X may have no isolated points.

Continuity

Given a preference relation $\succcurlyeq$ on a set X, define the **strict** and **weak upper contour sets**

$$P(x) = \{y \in X : y \succ x\} \qquad \text{and} \qquad U(x) = \{y \in X : y \succcurlyeq x\}.$$

We also define the **strict** and **weak lower contour sets**

$$P^{-1}(x) = \{y \in X : x \succ y\} \qquad \text{and} \qquad U^{-1}(x) = \{y \in X : x \succcurlyeq y\}.$$

When (X, d) is a metric space, we say that $\succcurlyeq$ is **continuous** if its graph is closed. There are other equivalent characterizations.

Lemma 1 *For a total, transitive, reflexive preference relation* $\succcurlyeq$ *on a metric space* X*, the following are equivalent.*

1. *The graph of* $\succcurlyeq$ *is closed. That is, if* $y_n \to y$*,* $x_n \to x$*, and* $y_n \succcurlyeq x_n$ *for each* n*, then* $y \succcurlyeq x$*.*

2. *The graph of* $\succ$ *is open. That is, if* $y \succ x$*, there is an* $\varepsilon > 0$ *such that if* $d(y', y) < \varepsilon$ *and* $d(x', x) < \varepsilon$*, then* $y' \succ x'$*.*

3. *For each* x*, the weak contour sets* $U(x) = \{y \in X : y \succcurlyeq x\}$ *and* $U^{-1}(x) = \{y \in X : x \succcurlyeq y\}$ *are closed.*

4. *For each* x*, the strict contour sets* $P(x) = \{y \in X : y \succ x\}$ *and* $P^{-1}(x) = \{y \in X : x \succ y\}$ *are open.*

Proof: Since $\succcurlyeq$ is total, it is clear that (1) $\iff$ (2) and (3) $\iff$ (4). Moreover it is also immediate that (1) $\implies$ (3) and (2) $\implies$ (4). So it suffices to prove that (4) implies (1).

So assume by way of contradiction that $y_n \to y$, $x_n \to x$, and $y_n \succcurlyeq x_n$ for each n, but $x \succ y$. Since $P(y)$ is open by condition (4) and $x \in P(y)$ by hypothesis, there is some $\varepsilon > 0$ such that $d(z, x) < \varepsilon$ implies $z \in P(y)$, or $z \succ y$. Similarly, since $P^{-1}(x)$ is open and $y \in P^{-1}(x)$ there is some $\varepsilon' > 0$ such that $d(w, y) < \varepsilon'$ implies $x \succ w$. Since $x_n \to x$ and $y_n \to y$, for large enough n, we have $d(x_n, x) < \varepsilon$ and $d(y_n, y) < \varepsilon'$, so

$$x \succ y_n \succcurlyeq x_n \succ y$$

for these large n. Pick one such n, call it n_0, and observe that

$$x \succ x_{n_0} \succ y.$$

Now condition (4) implies $P(x_{n_0})$ is open and since $x \in P(x_{n_0})$, there is some $\eta > 0$ such that $d(z, x) < \eta$ implies $z \succ x_{n_0}$. Similarly, since $P^{-1}(x_{n_0})$ and $y \in P^{-1}(x_{n_0})$, there is $\eta' > 0$ such that $d(w, y) < \eta'$ implies $x_{n_0} \succ w$. Now for large enough n we have $d(x_n, x) < \eta$ and $d(y_n, y) < \eta'$, so

$$x_n \succ x_{n_0} \succ y_n,$$

which contradicts $y_n \succcurlyeq x_n$ for all n. ∎

We also say that $\succcurlyeq$ is **upper semicontinuous** if for each x, the set $U(x) = \{y \in X : y \succcurlyeq x\}$ is closed, or equivalently, $P^{-1}(x) = \{y \in X : x \succ y\}$ is open in X. Similarly, $\succcurlyeq$ is **lower semicontinuous** if for each x, the set $U^{-1}(x) = \{y \in X : x \succcurlyeq y\}$ is closed, or equivalently, $P(x) = \{y \in X : y \succ x\}$ is open in X. Observe that a preference relation is continuous if and only if it is both upper and lower semicontinuous.

Lemma 2 *If $\succcurlyeq$ is continuous and locally nonsatiated, then $U(x)$ is the closure of $P(x)$.*

Proof: $\overline{P}(x) \subset U(x)$: Let y belong to $\overline{P}(x)$. That is, there is a sequences y_n in $P(x)$ with $y_n \to y$. Then for each n, we have $y_n \succ x$, so a fortiori $y_n \succcurlyeq x$. Since $y_n \to y$, we have $(y_n, x) \to (y, x)$, so by continuity, $y \succcurlyeq x$, that is, $y \in U(x)$.

$U(x) \subset \overline{P}(x)$: Let y belong to $U(x)$. By local nonsatiation, for each n there is a y_n satisfying $d(y_n, y) < \frac{1}{n}$ and $y_n \succ y$. Since $y_n \succ y$ and $y \succcurlyeq x$, we have $y_n \succ y$, so $y_n \in P(x)$. But $y_n \to y$, so $y \in \overline{P}(x)$. ∎

Convexity

When X is a subset of a linear space, we say that $\succcurlyeq$ is

- **weakly convex** if

$$y \succcurlyeq x \qquad \Longrightarrow \qquad \lambda y + (1-\lambda)x \succcurlyeq x \quad \text{for all } 0 < \lambda < 1.$$

- **convex** if

$$y \succ x \qquad \Longrightarrow \qquad \lambda y + (1-\lambda)x \succ x \quad \text{for all } 0 < \lambda < 1.$$

- **strictly convex** if

$$y \succcurlyeq x \qquad \Longrightarrow \qquad \lambda y + (1-\lambda)x \succ x \quad \text{for all } 0 < \lambda < 1.$$

To simplify the discussion of these properties let say that ***z* is between *x* and *y*** if (i) $x \neq y$, and (ii) $z = \lambda x + (1-\lambda)y$ for some $0 < \lambda < 1$.

The property of weak convexity is not actually weaker than convexity.

Example 3 Let $X = [-1, 1]$ and define $\succcurlyeq$ by means of the utility function

$$u(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $\succcurlyeq$ is convex, but not weakly convex. Why? □

The preference relation in the example above is not continuous, which brings up the next lemma.

Lemma 4 *If $\succcurlyeq$ is convex and upper semicontinuous, then it is weakly convex.*

Proof: Assume that $y \succcurlyeq x$. In case $y \succ x$, then by convexity $\lambda y + (1-\lambda)x \succ x$ for $0 < \lambda < 1$, so a fortiori $\lambda y + (1-\lambda)x \succcurlyeq x$. So now consider the case $y \sim x$ and assume by way of contradiction that for some $0 < \bar{\lambda} < 1$ we have $x \succ \bar{\lambda}y + (1-\bar{\lambda})x = z$. By upper semicontinuity, we may choose $\tilde{\lambda}$ close to $\bar{\lambda}$, but with $\tilde{\lambda} > \bar{\lambda}$ so that $x \succ \tilde{\lambda}y + (1-\bar{\lambda})x = w$. See Figure 1. But this means that z is between w and x, and since $x \succ w$, convexity implies $z \succ w$. On the other hand, w is between y and z, and $y \sim x \succ z$, so convexity implies $w \succ z$, a contradiction. ∎

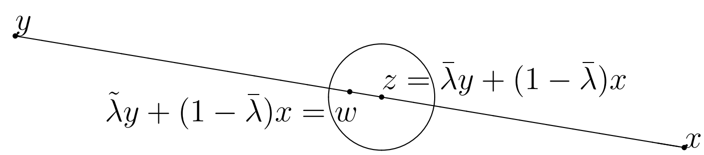

Figure 1: $y \succ z \implies w \succ z$ and $x \succ w \implies z \succ w$, oops.

Lemma 5 *If X is convex and $\succcurlyeq$ is weakly convex, then for each x, both $U(x)$ and $P(x)$ are convex sets.*

If X is convex and $\succcurlyeq$ is convex and continuous, then for each x, both $U(x)$ and $P(x)$ are convex sets.

Proof: The first statement is easy to prove. The second statement follows from the first and Lemma 4. ∎

The next result gives conditions that rules out "thick" indifference classes.

Lemma 6 *If X is convex, and $\succcurlyeq$ is convex, continuous, and nonsatiated, then $P(x)$ is the interior of $U(x)$.*

Proof: Since $P(x) \subset U(x)$ and $P(x)$ is open by lower semicontinuity, we have $P(x) \subset \operatorname{int} U(x)$. For the reverse inclusion, let y belong to the interior of $U(x)$, so there is some $\varepsilon > 0$ such that the ε-ball centered at y lies wholly in $U(x)$. Assume by way of contradiction that $y \notin P(x)$. Then since $y \in U(x)$, it must be that $y \sim x$. Since $\succcurlyeq$ is nonsatiated, there is a point $z \in X$ with $z \succ y$. Choose $\alpha < 0$ but close enough to zero, so that the point $w = (1-\alpha)y + \alpha z$ is within ε of y and also so that $z \succ w$, which can be done by upper semicontinuity of $\succcurlyeq$. See Figure 2. Then $z \succ w \succcurlyeq x \sim y$. But since y lies between z and w, by convexity we must have $y \succ w$, a contradiction. ∎

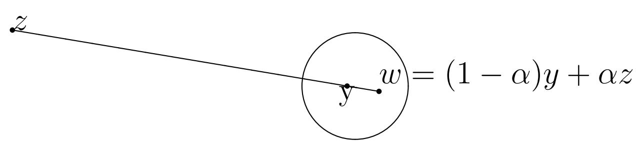

Figure 2: $w \succcurlyeq x \sim y$ and $z \succ y \succ w$, oops.

Example 7 Lemma 6 may fail without convexity. Let $X = \boldsymbol{R}$ and let $\succcurlyeq$ be defined by the utility $u(x) = x^2$. Then $\succcurlyeq$ is locally nonsatiated and continuous, but $P(0) = \boldsymbol{R} \setminus \{0\} \neq \boldsymbol{R} = \operatorname{int} U(0)$. □