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What to remember about metric spaces

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These notes are supposed to be a gentle introduction to the topological concepts used in economic theory. If the term "metric space" disturbs you, just mentally replace it by "Euclidean space." You won't lose too much by doing that. I provide almost no proofs here, but it is a good exercise to try to prove these statements. Proofs of these facts can be found in any good introduction to analysis. I personally like Walter Rudin's blue book [4].[1] I am also fond of *The Hitchhiker's Guide* [1].

1 Metrics

A **metric** on a nonempty set X is a function $d\colon X \times X \to \boldsymbol{R}$ satisfying the following four properties that are designed to capture intuitive properties of distance in the real world.

1. Positivity: The distance between two points is a nonnegative real number, and the distance from a point to itself is zero. Formally, for all $x, y \in X$
$$d(x,y) \geqslant 0 \quad \text{and} \quad d(x,x) = 0.$$
2. Discrimination: The distance between two distinct points is strictly positive. Formally,
$$d(x,y) = 0 \implies x = y.$$
3. Symmetry: The distance from point x to point y is the same as the distance from point y to point x. Formally, for all $x, y \in X$,
$$d(x,y) = d(y,x).$$
4. Triangle Inequality: The shortest distance from point x to point y is less than or equal to the distance from point x to point y stopping by point z along the way. (See Figure 1.) Or given a triangle xyz, the length of any side is less than or equal to the sum of the lengths of the remaining sides. Formally, for all $x, y, z \in X$,
$$d(x,y) \leqslant d(x,z) + d(z,y).$$

If d is a metric on a set X, then the pair (X, d) is called a **metric space**.

1 Example (Examples of metric spaces)

- The natural metric on $\boldsymbol{R}$ is
$$d(x,y) = |x - y|.$$

[1] At least the American edition is blue, but I am told that it comes in other colors.

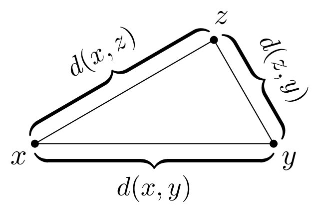

Figure 1. The triangle inequality.

- There are several natural metrics on $\boldsymbol{R}^{\mathrm{m}}$. The **Euclidean metric** is defined by
$$d(x,y) = \Big(\sum_{i=1}^{m} |x_i - y_i|^2\Big)^{\frac{1}{2}}.$$
The $\boldsymbol{\ell_1}$ **metric** is defined by
$$d(x,y) = \sum_{i=1}^{m} |x_i - y_i|.$$
The **sup metric** or **uniform metric** is defined by
$$d(x,y) = \max_{i=1,\dots,m} |x_i - y_i|.$$

- These definitions can be extended to spaces of infinite sequences, by replacing the finite sum with an infinite series. The infinite dimensional spaces of sequences where every point has a finite distance from zero under these metrics (that is, when the infinite series is absolutely convergent) are called ℓ_2, ℓ_1, and ℓ_∞, respectively. (Although for the metric on ℓ_∞, the maximum must replaced by a supremum: $d(x,y) = \sup_n |x_n - y_n|$.) In general, for $p \geqslant 1$,
$$\ell_p = \Big\{(x_1, x_2, \dots) \in \boldsymbol{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty\Big\}.$$
And the ℓ_p metric is
$$d_p(x,y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{1/p}.$$
It can be shown that $\lim_{p\to\infty} d_p(x,y) = \sup_n |x_n - y_n|$, hence the term ℓ_∞.

- The same ideas apply to spaces of functions. Let $C[a,b]$ be the vector space of all continuous real-valued functions on the interval $[a,b]$. Then we can define metrics on $C[a,b]$ by
$$d_p(f,g) = \Big(\int_a^b \big|f(x) - g(x)\big|^p \, dx\Big)^{1/p} \qquad 0 < p < \infty$$
and
$$d_\infty(f,g) = \max\Big\{\big|f(x) - g(x)\big| : a \leqslant x \leqslant b\Big\}.$$

- In general, a norm $\|\cdot\|$ on a vector space defines a metric by

$$d(x, y) = \|x - y\|.$$

- A peculiar metric is the **discrete metric** on a set, defined by $d(x, y) = 1$ whenever $x \neq y$.
- The **Baire space** is $\mathbb{N}^{\mathbb{N}}$, the space of sequences of natural numbers. A useful metric on this space is the **tree metric**,

$$d(x, y) = \frac{1}{\min\{n : x_n \neq y_n\}}.$$

□

2 Open balls and neighborhoods

Let (X, d) be a metric space. The **open ε-ball** centered at a point $\in X$ is

$$B_\varepsilon(x) = \{y \in X : d(y, x) < \varepsilon\},$$

A point x is an **interior point** of a subset A of X if there is some $\varepsilon > 0$ such that $B_\varepsilon(x)$ is included in A. In this case we also say that A is a **neighborhood** of x.

The set of interior points is called the **interior** of X, denoted $\operatorname{int} X$ or sometimes X°.

3 Open sets

A set G is **open** if every point in G is an interior point, that is, $G = \operatorname{int} G$. The letter G is often used to denote an open set, perhaps in reference to the German word *geöffnet*. The most important class of open sets are the open balls:

- Every open ε-ball $B_\varepsilon(x)$ is an open set.

As a result of this,

- The interior of a set is open. Indeed it is the largest (in the sense of inclusion) open subset.

These simple facts about open sets are important to remember.

- The union of an arbitrary family of open sets is open.
- The intersection of a finite family of open sets is open.
- The empty set and X are each open.

2 Definition *Let A be set in a metric space (X, d). The **ε-neighborhood** $N_\varepsilon(A)$ of A is defined by*

$$N_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x).$$

Note that as union of open balls $B_\varepsilon(x)$, the set $N_\varepsilon(A)$ is open. If X is also a vector space with translation-invariant metric d, then we also have $N_\varepsilon(A) = A + B_\varepsilon(0)$.

4 Topology

The collection of open sets in a metric space is called the **topology** of the metric space. Two metrics generating the same topology are **equivalent**. The Euclidean, ℓ_1, and sup metrics on $\boldsymbol{R}^{\mathrm{m}}$ are equivalent metrics for the topology of $\boldsymbol{R}^{\mathrm{m}}$. A property of a metric space that can be expressed in terms of open sets without mentioning a specific metric is called a **topological property**. It is possible to take the notion of open set as a primitive, and define “topological spaces” that may not arise from any metric.[2] These general topological spaces have their crucial uses, but I won’t go into that here.

5 Relative topology

Let A be a nonempty subset the metric space (X, d). The metric d restricted to A is a metric on A. The topology it defines is the **relative topology** on A with respect to X, or the topology on A relative to X. The ε-ball centered at x in A is just

$$B_\varepsilon(x) = \{y \in A : d(x, y) < \varepsilon\} = A \bigcap \{y \in X : d(x, y) < \varepsilon\}.$$

Now let $E \subset A$. A point x is a **relative interior point of $\boldsymbol{E}$ with respect to $\boldsymbol{A}$** if there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \{y \in A : d(y, x) < \varepsilon\}$ is included in E. The set of relative interior points is called the **relative interior** of E. A set E is **relatively open in $\boldsymbol{A}$** if every point in G is a relative interior point. Note that A is always relatively open in A itself, but not necessarily open in X.

For example, let X be the real line with its usual metric and let $A = [0, 1]$, which is not an open subset of X as neither 0 nor 1 is an interior point. But both 0 and 1 are interior points of $[0, 1]$ relative to itself. Thus the interval $(1/2, 1]$ is relatively open in $[0, 1]$.

The following fact shows how to define the relative topology of A in terms of the topology of X without resorting to the metric. That is, it shows that the relative topology of A (with respect to X) is a topological property.

- If $A \subset X$, then G is relatively open in A if and only if there is an open subset $\hat{G}$ of X such that $G = \hat{G} \bigcap A$.

6 Closed sets

A set is **closed** if its complement is open. Thus:

- The intersection of any family of closed sets is closed.
- The union of a finite family of closed sets is closed.
- The empty set and X are both closed.

The letter F is often used to stand for a closed set, after the French *fermé*.

[2] A **topological space** is a nonempty set X together with a topology, where a topology is any family τ of sets satisfying the following properties: (i) $\varnothing, X \in \tau$, (ii) τ is closed under finite intersections, and (iii) τ is closed under arbitrary unions. Members of the topology τ, which are thus subsets of X, are by definition **open**.

7 Closure of a set

The smallest closed set including a set A is called the **closure of A**, denoted $\overline{A}$ or $\operatorname{cl} A$. It is the intersection of all closed sets that include A. Clearly $A \subset \overline{A}$, and $A = \overline{A}$ if and only if A is closed.

- If F is closed, then $\overline{F} = F$, consequently $\overline{\overline{A}} = \overline{A}$ for any set A.
- If $A \subset B$, then $\overline{A} \subset \overline{B}$.
- A point x does not belong to $\overline{A}$ if and only if there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \bigcap A = \varnothing$. (This is because the complement of $B_\varepsilon(x)$ is a closed set.)
- A point x belongs to $\overline{A}$ if and only if for every $\varepsilon > 0$, we have $B_\varepsilon(x) \bigcap A \neq \varnothing$.
- $\overline{B_\varepsilon(x)} = \overline{\{y : d(y,x) < \varepsilon\}} \subset \{y : d(y,x) \leqslant \varepsilon\}$, and in a general metric space the inclusion may be proper. (To see why the inclusion may be proper, think about the open ball of radius 1 in a discrete metric space X: $B_1(x) = \{x\}$ and $\{y \in X : d(y,x) \leqslant 1\} = X$.)
- However, in $\boldsymbol{R}^{\mathrm{n}}$, we do have $\overline{B_\varepsilon(x)} = \{y : d(y,x) \leqslant \varepsilon\}$.

8 Boundary of a set

The **boundary** of a set A, denoted ∂A or $\operatorname{bdy} A$, is defined by

$$\partial A = \overline{A} \bigcap \overline{(X \setminus A)}.$$

In other words,

- $x \in \partial A$ if and only if for every $\varepsilon > 0$,

$$B_\varepsilon(x) \bigcap A \neq \varnothing \quad \text{and} \quad B_\varepsilon(x) \bigcap A^c \neq \varnothing.$$

For example, in $\boldsymbol{R}^2$, let D be the unit disc,

$$D = \{(x,y) \in \boldsymbol{R}^2 : x^2 + y^2 \leqslant 1\}.$$

Then its boundary is the unit circle,

$$\partial D = \{(x,y) \in \boldsymbol{R}^2 : x^2 + y^2 = 1\}.$$

Also, the boundary of X is the empty set. If $E \subset A \subset X$, then it is not surprising that the boundary of E relative to A, which is $\overline{E} \bigcap \overline{(A \setminus E)}$, need not be the same as the boundary of E relative to X, which is $\overline{E} \bigcap \overline{(X \setminus E)}$.

9 Dense sets

A subset D of a set A is **dense in A** if the closure of D includes A. That is, any closed set that includes D also includes A.

- A set D is dense in A if and only if for every $x \in A$ and every $\varepsilon > 0$,

  $$B_\varepsilon(x) \bigcap D \neq \varnothing.$$

  Another way to say this is that every point in A can be "approximated" arbitrarily well by a point in D.

- The boundary of a dense subset of X is X

A metric space (X, d) is **separable** if there is a dense subset of X that is countable. The following fact is not trivial.

- Every subset of a separable metric space is separable (with its relative topology).

10 Compactness

A set $\mathcal{G}$ of sets is a **cover** of a set A (or **covers** A) if A is included in the union of $\mathcal{G}$,

$$A \subset \bigcup \mathcal{G} = \bigcup_{E \in \mathcal{G}} E.$$

If $\mathcal{G}$ is a cover of A, the set $\mathcal{E}$ of sets is a **subcover** of A if every set that belongs to $\mathcal{E}$ also belongs to $\mathcal{G}$ and $\mathcal{E}$ covers A.[3] If every set belonging to $\mathcal{G}$ is open, then $\mathcal{G}$ is called an **open cover** of A.

For example, the set $\mathcal{G} = \{(n-2, n+2) : n \in \mathbb{N}\}$ of intervals is an open cover of $\boldsymbol{R}$, and $\mathcal{E} = \{(n-2, n+2) : n \in \mathbb{N},\ n \text{ even}\}$ is an open subcover of $\boldsymbol{R}$.

An important topological property is compactness. A set K is **compact** if every collection $\mathcal{G}$ of open sets satisfying $K \subset \bigcup_{G \in \mathcal{G}} G$ includes a finite subcollection $G_1, \ldots, G_k$ satisfying $K \subset \bigcup_{i=i}^{k} G_i$. This is usually phrased as, "*every open cover of K has a finite subcover.*"

There is an equivalent characterization of compact sets that is perhaps more useful. A family $\mathcal{A}$ of sets has the **finite intersection property** if every finite subset $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ has a nonempty intersection, $\bigcap_{i=1}^{n} A_i \neq \varnothing$. For instance, the family $\{(a, \infty) \subset \boldsymbol{R} : a \in \boldsymbol{R}\}$ of intervals has the finite intersection property.

3 Theorem *A set K is compact if and only if every family of closed subsets of K having the finite intersection property has a nonempty intersection.*

Hint for proof: If $\mathcal{F}$ is a family of closed subsets of K, then the family $\{K \setminus F : F \in \mathcal{F}\}$ of relatively open complements covers K if and only if $\bigcap \mathcal{F} = \varnothing$. ∎

[3] This terminology is a bit unsatisfactory, since $\mathcal{E}$ is a subset of $\mathcal{G}$ and a cover of A. It would be more precise to say that $\mathcal{E}$ is a subcover of $\mathcal{G}$ of A (or should it be $\mathcal{E}$ is a subcover of A of $\mathcal{G}$?), but that sounds awkward.

The following facts about compact sets are easy consequences of the definitions.

- The empty set is compact.
- Any finite subset of a metric space is compact.
- A compact subset of a metric space is closed.
- A closed subset of a compact set is also compact.
- Finite unions of compact sets are compact.
- If $K \subset A \subset X$, then K is compact relative to A if and only if K is compact relative to X. (Note that by taking $A = K$, we see that compactness in X, unlike openness or closedness in X, is determined by the relative topology of K.)

The next results are only a little less straightforward.

4 Lemma *Let G be an open subset of a metric space (X, d) and let K be a nonempty compact subset of G. Then there is an $\varepsilon > 0$ such that*

$$K \subset N_\varepsilon(K) \subset G.$$

Hint: For each $x \in K$ there is $\varepsilon_x > 0$ with $B_{\varepsilon_x}(x) \subset G$. Let $B_{\varepsilon_{x_1}}(x_1), \dots, B_{\varepsilon_{x_n}}(x_n)$ be a finite subcover, and set $\varepsilon = \min_i \varepsilon_{x_i}$. ∎

5 Lemma *Let K be a compact subset of metric space, and let $\mathcal{G}$ be an open cover of K. Then there exists some $\delta > 0$, called a* ***Lebesgue number*** *of the cover, such that for each $x \in K$ we have $B_\delta(x) \subset G$ for at least one $G \in \mathcal{G}$.*

Hint: For each $x \in K$ there is a $G_x \in \mathcal{G}$ and δ_x with $x \in B_{2\delta_x}(x) \subset G_x$. Then $\{B_{\delta_x}(x) : x \in K\}$ is an open cover of K. Take a finite subcover $B_{\delta_{x_1}}(x_1), \dots, B_{\delta_{x_n}}(x_n)$, and set $\delta = \min_i \delta_{x_i}$. ∎

11 Boundedness and total boundedness

For a nonempty subset A of a metric space (X, d) its **diameter** is $\sup\{d(x, y) : x, y \in A\}$. A nonempty set is **bounded** if its diameter is finite. A nonempty subset of a metric space is **totally bounded** if for every $\varepsilon > 0$, it can be covered by finitely many ε-balls. Boundedness and total boundedness are not topological properties—they depend on the metric. In fact, if (X, d) is a metric space, define the new metric d' by $d'(x, y) = \max\{d(x, y), 1\}$. Then (X, d') is a bounded metric space of diameter 1, but has the same topology (open sets) as (X, d).

6 Heine–Borel–Lebesgue Theorem *A nonempty subset of $\boldsymbol{R}^{\mathrm{m}}$ is compact if and only if it is both closed and bounded in the Euclidean metric.*

More generally, any set in a Euclidean space that is bounded in the Euclidean metric is totally bounded. This result is special. In general, a set may be closed and bounded without being totally bounded or compact. However, in a metric space, a set is compact if and only if it is totally bounded and complete (see section 19 for completeness).

12 Product spaces

The Cartesian product of two metric spaces (X, d_X) and (Y, d_Y) is a metric space under several natural equivalent metrics, such as:

$$d_1\big((x,y),(x',y')\big) = d_X(x,x') + d_Y(y,y'), \qquad d_\infty\big((x,y),(x',y')\big) = \max\{d_X(x,x'), d_Y(y,y')\}.$$

(Compare these to the ℓ_p metrics on $\boldsymbol{R}^{\mathrm{m}}$ in section 1.) The **product topology** on $X \times Y$ is the topology defined by one of the metrics above.

The most important things to know about the product topology are:

- If (x, y) belongs to the interior of a subset G of $X \times Y$, then there exist open neighborhoods U of x (in X) and V of y (in Y) such that
$$(x, y) \in U \times V \subset G.$$
- If C and K are compact subsets of X and Y, then $C \times K$ is a compact subset of $X \times Y$.

These notions generalize in a straightforward way to products of finitely many metric spaces. (Think of $\boldsymbol{R}^{\mathrm{m}}$ as the product of m copies of $\boldsymbol{R}$.)

We can even put a metric on a countable product of metric spaces. Let (X_n, d_n), $n = 1, 2, \ldots$, be a sequence of *bounded* metric spaces of diameter at most 1. (See section 11 to see why the boundedness assumption is not restrictive.) Let $X = \prod_{n=1}^{\infty} X_n$, the infinite product, and define the metric d on X by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} d_n(x_n, y_n)/2^n,$$

where

$$\boldsymbol{x} = (x_1, x_2, \ldots) \quad \text{and} \quad \boldsymbol{y} = (y_1, y_2, \ldots).$$

The metric space (X, d) is also bounded with diameter at most 1, and its topology is called the **product topology**.

It is possible to define product topologies for arbitrary products of metric spaces, but unless the set of factors is countable, the result will not be a metric space. Nonetheless, the notion of compactness is topological, so the next result makes sense.

7 Tychonoff Product Theorem *The Cartesian product of an arbitrary family of compact sets is compact.*

13 Sequences and subsequences

Formally, a **sequence** in a set A is a function from the natural numbers $\mathbb{N}$ into A. Traditionally, sequences are denoted using lower case Latin letter near the end of the alphabet, such as x or y. Also traditionally, instead of writing $x(n)$ for the value of the function, we write x_n. Occasionally, we may write something like $(x_1, x_2, \ldots)$ or $\boldsymbol{x} = (x_1, x_2, \ldots)$ to denote a sequence, particularly if we wish to treat it as a point in a vector space. We may also refer to a sequence with the notation $x_1, x_2, \ldots$, or (rather unfortunately) $\{x_1, x_2, \ldots\}$.

A strictly increasing function φ from $\mathbb{N}$ into $\mathbb{N}$ defines a **subsequence** of a sequence as follows. Let x be a sequence in A, that is, $x\colon \mathbb{N} \to A$. Define the sequence $y\colon \mathbb{N} \to A$ by $y(k) = x\big(\varphi(k)\big)$. Then y is a subsequence of x. But it is traditional to write the value of y at k as x_{n_k}.

A **finite sequence** is a function on $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. We may occasionally use the pleonasm "infinite sequence" to distinguish a sequence from a finite sequence.

The ***n*-tail** of the sequence x is the subsequence defined by the increasing function $k \mapsto k + n - 1$, and is traditionally denoted something like $(x_n, x_{n+1}, \dots,)$.

14 Convergence

Let (X, d) be a metric space. A sequence $x_1, x_2, \dots$ in X **converges** to a point x in X, written

$$x_n \xrightarrow[n\to\infty]{} x$$

or simply $x_n \to x$, if $d(x_n, x) \to 0$ as a sequence of real numbers. In other words, if

$$(\forall \varepsilon > 0)\,(\exists N)\,(\forall n \geqslant N)\;[d(x_n, x) < \varepsilon].$$

Or in yet other words, if the sequence **eventually** lies in any neighborhood of x. We often say "$\mathcal{P}$ for n **large enough**" as a shorthand for the expression "$(\exists N)\,(\forall n \geqslant N)\;[\mathcal{P}]$."

- Limits are unique. That is, if $x_n \to y$ and $x_n \to z$, then $y = z$.
- The closure of a set A in a metric space consists of all points that are limits of sequences in A.

15 Sequential Compactness

A topological space X is **sequentially compact** if every sequence in X has a subsequence that converges to a point in X. One reason for the terminology is the following result.

8 Theorem *A set K in a metric space is compact if and only if is is sequentially compact.*

Here is a sketch of the proof of half the theorem. Assume that K is compact and nonempty, and let $x_1, x_2, \dots$ be a sequence in K. (If K is empty, the theorem is vacuously true.) Define $F_n = \overline{\{x_n, x_{n+1}, \dots\}}$, the closure of the se of values of the n-tail of x. Then $\{F_n : n \in \mathbb{N}\}$ is a family of closed subsets of K having the finite intersection property. (Why?) Since K is compact $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$ (Theorem 3). Now let y belong to this intersection. That is, $y \in \overline{\{x_n, x_{n+1}, \dots\}} \subset K$ for every n. Construct the strictly increasing function $k \mapsto n_k$ from $\mathbb{N}$ into $\mathbb{N}$ inductively as follows. Let $n_1 = 1$. Given $n_1 < \cdots < n_k$, pick n_{k+1} to satisfy $n_{k+1} \geqslant n_k + 1$ and $d(x_{n_{k+1}}, y) < 1/(k+1)$. Since $y \in F_{n_k+1}$, there is always such an $x_{n_{k+1}}$ in $\{x_{n_k+1}, x_{n_k+2}, \dots\}$. Then $x_{n_k} \xrightarrow[k\to\infty]{} y$ is the desired subsequence.

The converse is harder.

16 Continuity

Let (X,d) and (Y,ρ) be metric spaces. A function $f\colon X \to Y$ is **continuous** if inverse images of open sets are open. That is,

$$U \text{ is an open subset of } Y \quad \implies \quad f^{-1}(U) \text{ is an open subset of } X.$$

Equivalently, f is continuous if and only if

$$x_n \to x \quad \implies \quad f(x_n) \to f(x).$$

Or equivalently, there is the "ε-δ" definition,

$$(\forall x \in X)\,(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall z \in X)\ \big[d(x,z) < \delta \implies \rho\big(f(x), f(z)\big) < \varepsilon\big].$$

We may also have occasion to talk about continuity at a point. We say that f is **continuous at the point *x*** if any of the following equivalent conditions holds.

$$U \text{ is a neighborhood of } f(x) \quad \implies \quad f^{-1}(U) \text{ is a neighborhood of } x.$$

$$x_n \to x \quad \implies \quad f(x_n) \to f(x).$$

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall z \in X)\ \big[d(x,z) < \delta \implies \rho\big(f(x), f(z)\big) < \varepsilon\big].$$

A function is continuous if and only if it is continuous at every point in its domain.

17 Semicontinuity

A real function $f\colon X \to \boldsymbol{R}$ is **upper semicontinuous** if for each $\alpha \in \boldsymbol{R}$, the upper contour set $[f \geqslant \alpha]$ is closed. It is **lower semicontinuous** if every lower contour set $[f \leqslant \alpha]$ is closed. Equivalently, f is lower semicontinuous if and only if its **epigraph**,

$$\operatorname{epi} f = \{(x,\alpha) \in X \times \boldsymbol{R} : f(x) \leqslant \alpha\}$$

is a closed subset of $X \times \boldsymbol{R}$, and f is upper semicontinuous if and only if its **hypograph**,

$$\operatorname{hypo} f = \{(x,\alpha) \in X \times \boldsymbol{R} : f(x) \geqslant \alpha\}$$

is a closed subset of $X \times \boldsymbol{R}$.

Note that f is upper semicontinuous if and only if $-f$ is lower semicontinuous. A real-valued function is continuous if and only if it is both upper and lower semicontinuous.

We can even talk about semicontinuity at a point. The real-valued function f is **upper semicontinuous at the point *x*** if

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\ [d(y,x) < \delta \implies f(y) < f(x) + \varepsilon].$$

Similarly, f is **lower semicontinuous at the point *x*** if

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\ [d(y,x) < \delta \implies f(y) > f(x) - \varepsilon].$$

Equivalently, f is upper semicontinuous at x if

$$f(x) \geqslant \limsup_{y \to x} f(y) = \inf_{\varepsilon > 0} \sup_{0 < d(y,x) < \varepsilon} f(y).$$

Similarly, f is lower semicontinuous at x if

$$f(x) \leqslant \liminf_{y \to x} f(y) = \sup \varepsilon > 0 \inf_{0 < d(y,x) < \varepsilon} f(y).$$

The function f is lower semicontinuous if and only if it is lower semicontinuous at each point. Likewise f is upper semicontinuous if and only if it is upper semicontinuous at each point.

18 Weierstrass's Theorem

9 Theorem *The continuous image of a compact set is compact.*

More pedantically, the above theorem asserts that if X and Y are topological spaces, and if $f \colon X \to Y$ is continuous, and if K is a compact subset of X, then $f(K)$ is a compact subset of Y.

10 Weierstrass's Theorem *If K is a nonempty compact set and $f \colon K \to \boldsymbol{R}$ is continuous, then f has both a maximizer and a minimizer in K.*

11 Theorem *If K is a nonempty compact set and $f \colon K \to \boldsymbol{R}$ is upper semicontinuous, then f has a maximizer in K.*

If K is a nonempty compact set and $f \colon K \to \boldsymbol{R}$ is lower semicontinuous, then f has a minimizer in K.

Outline of proof: Assume K is a nonempty compact set and $f \colon K \to \boldsymbol{R}$ is upper semicontinuous. Then $\mathcal{F} = \{[f \geqslant \alpha] : \alpha \in \operatorname{range} f\}$ is a family of closed sets having the finite intersection property. The set of maximizers is the nonempty set $\bigcap \mathcal{F}$. ∎

19 Completeness

A sequence $(x_1, x_2, \ldots)$ in a metric space is a **Cauchy sequence** if

$$\lim_{n \to \infty} \operatorname{diam}\{x_n, x_{n+1}, \ldots\} = 0.$$

Any convergent sequence is a Cauchy sequence. A metric space is **complete** if every Cauchy sequences is convergent (to some point in the space). Every Euclidean space is complete. Every closed subset of a complete metric space is complete. Completeness is *not* a topological property.

12 Cantor Intersection Lemma *Let $F_1 \supset F_2 \supset \cdots$ be a nested decreasing sequence of nonempty closed subsets of a complete metric space. If $\lim_{n \to \infty} \operatorname{diam} F_n = 0$, then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.*

This conclusion of this lemma may fail if the hypothesis that $\lim_{n\to\infty} \operatorname{diam} F_n = 0$ is not satisfied. Try to think of a simple example. Hint: Use the complete metric space of natural numbers with the usual metric, $d(n,m) = |n-m|$.

20 Distance functions

The distance from a point x to a nonempty set A in a metric space is defined by

$$d(x,A) = \inf\{d(x,y) : y \in A\}.$$

- Given a nonempty set A, the distance function is continuous. In fact, this stronger condition is true:

$$\bigl|d(x,A) - d(y,A)\bigr| \leqslant d(x,y).$$

  This is easy to see: If you want to go from x to A, you can always go to y first, and then go to A, and vice versa, so

$$d(x,A) \leqslant d(x,y) + d(y,A) \quad \text{and} \quad d(y,A) \leqslant d(x,y) + d(x,A),$$

  so

$$d(x,A) - d(y,A) \leqslant d(x,y) \quad \text{and} \quad d(y,A) - d(x,A) \leqslant d(x,y).$$

  That is,

$$\bigl|d(x,A) - d(y,A)\bigr| \leqslant d(x,y).$$

- $\overline{A} = \{x \in X : d(x,A) = 0\}$.
- $N_\varepsilon(A) = \{x \in X : d(x,A) < \varepsilon\}$

21 Convergence of sequences of functions

Let $f_1, f_2, \ldots$ be a sequence of functions from a set X into a metric space (Y,d). The sequence **converges pointwise** to a function f if for each point $x \in X$ the sequence $f_n(x) \to f(x)$ in Y. In other words,

$$(\forall x \in X)\,(\forall \varepsilon > 0)\,(\exists N)\,(\forall n \geqslant N)\ \bigl[d\bigl(f_n(x), f(x)\bigr) < \varepsilon\bigr].$$

In particular N may be chose to depend on x as well as ε. The sequence **converges uniformly** to f if N can be chosen independently of x. In other words,

$$(\forall \varepsilon > 0)\,(\exists N)\,(\forall x \in X)\,(\forall n \geqslant N)\ \bigl[d\bigl(f_n(x), f(x)\bigr) < \varepsilon\bigr].$$

Note the subtle difference in the order of the quantifiers.

Define the metric ρ for functions by

$$\rho(f,g) = \sup_x d\bigl(f(x), g(x)\bigr).$$

(Technically this is not a metric since the supremum may be infinite. We can fix that by setting $\rho(f,g) = \sup_x \min\bigl\{d\bigl(f(x), g(x)\bigr), 1\bigr\}$.) Then $f_n \to f$ uniformly if and only if $\rho(f_n, f) \to 0$. Pointwise convergence cannot described by a metric unless X is countable.

Clearly uniform convergence implies pointwise convergence. The converse is false.

13 Example (Pointwise vs. uniform convergence) Let $f_n\colon [0,1] \to [0,1]$ be defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [1/2n, 1/n] \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to 0$ pointwise, but not uniformly. □

22 ⋆ Lipschitz continuity

A function f from the metric space (X, d) into the metric space (Y, δ) is **Lipschitz continuous** with **Lipschitz modulus** α if for every x, x' in X,

$$\delta\big(f(x), f(x')\big) \leqslant \alpha d(x, x').$$

We sometimes simply say that f is a **Lipschitz function** or **Lipschitzian**. We have already seen that distance functions are Lipschitz functions with modulus 1.

14 Rademacher's Theorem *Let G be an open subset of $\boldsymbol{R}^{\mathrm{m}}$ and let $f\colon G \to \boldsymbol{R}$ be Lispschitz continuous. Then f is (Fréchet-)differentiable almost everywhere in G.*

This result is not so elementary and may be found, for instance, in Clarke, et. al. [2].

The second fundamental theorem of calculus is usually stated for continuously differentiable functions, but it is also true for Lipschitz continuous functions.

15 Theorem *Let $f\colon [a,b] \to \boldsymbol{R}$ be Lipschitz continuous. Then*

$$f(b) = f(a) + \int_a^b f'(x)\,dx.$$

A word about the theorem above: Note that f' may not exist everywhere, but the set of points at which it does not exist has measure zero, and so cannot affect the value of the Lebesgue integral of f'. This theorem is a special case of a more general result on *absolutely continuous* functions, see for instance H. L. Royden [3].

If the Lipschitz modulus of a Lipschitz function is strictly less than 1, then it is a **contraction**. For a function f mapping a set into itself, a point x is a **fixed point** of f if $f(x) = x$. The following theorem plays a key rôle in the theory and practice of dynamic programming.

16 Contraction Fixed Point Theorem *Let (X, d) be a complete metric space and let $f\colon X \to X$ be a contraction. Then f has a unique fixed point $\bar{x}$.*

Moreover, for any x_0, the sequence defined recursively by $x_{n+1} = f(x_n)$ satisfies $x_n \to \bar{x}$.

23 ⋆ Bases for topologies

A collection $\mathcal{B}$ of open sets is a **base** for the topology on the metric space X if every open subset X is a union of sets from $\mathcal{B}$. For instance, the collection of open balls

$$\{B_\varepsilon(x) : x \in X,\ \varepsilon > 0\}$$

is a base for the topology on X. So is the following collection of open balls:

$$\{B_{1/n}(x) : x \in X,\ n \in \mathbb{N}\}.$$

In $\boldsymbol{R}^{\mathrm{m}}$, a base for the topology is given by

$$\{B_{1/n}(x) : x \in \mathbb{Q}^{\mathrm{m}},\ n \in \mathbb{N}\}.$$

This base is countable, that is, it can be put into one-to-one correspondence with the natural numbers $\mathbb{N}$.

A metric space that has a countable base is called a **second countable** metric space. Recall that a metric space is separable if it has a countable dense subset.

17 Theorem *A metric space is second countable if and only if it is separable.*

24 ⋆ Separation properties

Every metric space has the **Hausdorff property**: For every pair x, y of distinct points, there are disjoint neighborhoods U of x and V of y. (For instance, take open balls of radius $d(x, y)/2$ centered at each point.)

A stronger property is true, namely every metric space is **regular**. That is, if F is closed and $x \notin F$, there are disjoint open sets G and U with $F \subset G$ and $x \in U$. (Let $\alpha = d(x, F)/2$. Since F is closed, and $x \notin F$, we must have $\alpha > 0$. Set $G = \bigcup\{B_\alpha(y) : y \in F\}$ and $U = B_\alpha(x)$.)

The strongest separation property possessed by metric spaces is this:

18 Theorem *Let (X, d) be a metric space and let E and F be disjoint closed subsets of X. Then there is a continuous function $f\colon X \to [0, 1]$ satisfying*

$$f^{-1}[E] = \{0\} \quad \textit{and} \quad f^{-1}[F] = \{1\}.$$

25 ⋆ Semimetric spaces

A **semimetric** on a set X is a function $d\colon X \times X \to \boldsymbol{R}$ satisfying all the properties of a metric except discrimination. That is, it is possible for distinct points x and y to satisfy $d(x, y) = 0$. Semimetrics arise naturally in spaces of functions. Let f be an integrable (Riemann integrable or Lebesgue integrable, it doesn't matter) function on the interval $[a, b]$, and let g differ from f at only finitely many points. Then

$$d(f, g) = \int_a^b \bigl|f(x) - g(x)\bigr|\, dx = 0,$$

so the metric described in section 1 on the space of continuous functions is only a semimetric on the space of integrable functions. (Note that if f and g are continuous and disagree at single point, then they disagree on an open interval about that point.)

Many of the results presented here are true for semimetrics, others need to be modified. But a common method for dealing with semimetric spaces is to convert them to metric spaces as follows. Given a semimetric space (X, d) define the binary relation $\sim$ by

$$x \sim y \quad \text{if} \quad d(x, y) = 0.$$

Then you can show that $\sim$ is reflexive, symmetric, and transitive, in other words an **equivalence relation**. The **equivalence class** of x, usually denoted $[x]$, is defined by

$$[x] = \{y \in X : x \sim y\} = \{y \in X : d(x, y) = 0\}.$$

The set of equivalence classes, denoted $X/\sim$, is called the **quotient of X modulo $\sim$**, and is a partition of X. We can define a true metric on $X/\sim$, abusively denoted d, by

$$d\big([x], [y]\big) = d(x, y).$$

When dealing with functions, we often do not make a distinction between $x \in X$ and $[x] \in X/\sim$, casually referring to either simply as x. We sometimes say that we **identify** x and y if $x \sim y$, and we may also casually refer to $X/\sim$ simply as X.

Suggested references

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