

## Lucas's "Tree" Model

KC Border  
Revised May 2005  
v. 2012.02.12::15.18

This is a gentle introduction to consumption-based asset pricing, based on Sargent's [2] treatment of Lucas [1].

As a notational convention, variables that are random from the point of view of time  $t$  are denoted by boldface symbols.

### 1 Consumers

Consumers live forever and care about the expected value

$$\mathbf{E} \sum_{n=0}^{\infty} \beta^n u(\mathbf{c}_{t+n}),$$

where  $c_t$  is consumption at time  $t$ . The per period utility,  $u$  is assumed to be strictly increasing, concave, and twice continuously differentiable. (Later on, we shall take  $u(c) = \ln c$ .)

Each period  $t$ , the consumer must allocate wealth  $w_t$  between consumption  $c_t$  and asset accumulation  $a_t$ . Thus the budget at time  $t$  is

$$c_t + a_t \leq w_t.$$

Assets grow at the random rate  $\mathbf{R}_{t+1} - 1$  between period  $t$  and  $t + 1$ ,

$$\mathbf{w}_{t+1} = \mathbf{R}_{t+1} a_t.$$

The Bellman optimality equation for the consumer with wealth  $w$  is

$$V(w) = \max_{0 \leq a \leq w} u(w - a) + \beta \mathbf{E} V(\mathbf{R}a).$$

The first order condition for an interior maximum at  $a^*$  is

$$-u'(w - a^*) + \beta \mathbf{E} (V'(\mathbf{R}a^*)\mathbf{R}) = 0. \tag{1}$$

By the Envelope Theorem (see, e.g., [3, Theorem 4.11]),

$$V'(w) = \frac{\partial}{\partial w} \left\{ u(w - a^*) + \beta \mathbf{E} V(\mathbf{R}a^*) \right\} = u'(w - a^*). \quad (2)$$

So

$$u'(c_t^*) = \beta \mathbf{E} (u'(c_{t+1}^*) \mathbf{R}_{t+1}),$$

where  $c^* = w - a^*$ , or

$$\beta \mathbf{E} \frac{u'(c_{t+1}^*)}{u'(c_t^*)} \mathbf{R}_{t+1} = 1. \quad (3)$$

### Comparison with the two period non-random case

In the two period non-random case, the consumer maximizes

$$u(c_1) + \beta u(c_2)$$

subject to

$$\begin{aligned} c_1 + a &= w \\ c_2 &= Ra \end{aligned}$$

or equivalently

$$c_1 + \frac{1}{R} c_2 = w.$$

The Lagrangean is

$$u(c_1) + \beta u(c_2) + \lambda \left( w - c_1 - \frac{1}{R} c_2 \right)$$

and the first order conditions are

$$\begin{aligned} u'(c_1^*) - \lambda &= 0 \\ \beta u'(c_2^*) - \lambda \frac{1}{R} &= 0 \end{aligned}$$

so rearranging, and dividing the latter by former yields

$$\beta \frac{u'(c_2^*)}{u'(c_1^*)} R = 1,$$

which is the analog of (3).

## 2 The orchard economy

There are  $n$  identical consumers and  $n$  trees. Trees live forever and bear fruit every season. Fruit is perishable and cannot be stored. Trees are the only asset, and fruit is the only consumption good.

The quantity of fruit per tree at time  $t$  is  $d_t$ . It is the same for all trees. The fruit dividend follows a Markov process, bounded between  $0 < \underline{d} < \bar{d}$ . Trees are traded each period after the fruit has been harvested. The market price of a tree (in units of fruit) at time  $t$  is  $p_t$ .

Let  $s_t$  denote the number of trees owned at the start of period  $t$ . Then an individual's wealth  $w_t$  (in units of fruit) is simply the the fruit he has harvested plus the value of the tree, times the number of trees:

$$w_t = (p_t + d_t)s_t.$$

Thus

$$\mathbf{R}_{t+1} = \frac{\mathbf{p}_{t+1} + \mathbf{d}_{t+1}}{p_t}$$

and (3) becomes

$$\beta \mathbf{E} \frac{u'(\mathbf{c}_{t+1}^*)}{u'(\mathbf{c}_t^*)} \frac{\mathbf{p}_{t+1} + \mathbf{d}_{t+1}}{p_t} = 1, \quad (4)$$

## 3 The price of trees

From (4) we get

$$p_t = \beta \mathbf{E}_t \left\{ \frac{u'(\mathbf{c}_{t+1}^*)}{u'(\mathbf{c}_t^*)} (\mathbf{p}_{t+1} + \mathbf{d}_{t+1}) \right\}$$

Now recurse forward,

$$\mathbf{p}_{t+1} = \beta \mathbf{E}_{t+1} \left\{ \frac{u'(\mathbf{c}_{t+2}^*)}{u'(\mathbf{c}_{t+1}^*)} (\mathbf{p}_{t+2} + \mathbf{d}_{t+2}) \right\},$$

so

$$p_t = \mathbf{E}_t \left\{ \beta \frac{u'(\mathbf{c}_{t+1}^*)}{u'(\mathbf{c}_t^*)} \mathbf{d}_{t+1} + \beta^2 \frac{u'(\mathbf{c}_{t+2}^*)}{u'(\mathbf{c}_t^*)} \mathbf{d}_{t+2} + \beta^2 \frac{u'(\mathbf{c}_{t+2}^*)}{u'(\mathbf{c}_t^*)} \mathbf{p}_{t+2} \right\}$$

etc., so

$$p_t = \mathbf{E}_t \left\{ \sum_{n=1}^{\infty} \beta^n \frac{u'(\mathbf{c}_{t+n}^*)}{u'(\mathbf{c}_t^*)} \mathbf{d}_{t+n} \right\} \quad (5)$$

## 4 Market Clearing

In order for the the prices to clear the market for trees, then at each  $t$  we must have

$$\begin{aligned} s_t^* &= 1 \\ a_t^* &= p_t \\ c_t^* &= d_t. \end{aligned}$$

Thus, according to (5), the equilibrium price must satisfy

$$p_t = \mathbf{E}_t \left\{ \sum_{n=1}^{\infty} \beta^n \frac{u'(d_{t+n})}{u'(d_t)} d_{t+n} \right\}.$$

### Special case

If

$$u(c) = \ln c,$$

(the Cobb–Douglas case), then  $u'(c) = 1/c$  and we have

$$p_t = \mathbf{E}_t \left\{ \sum_{n=1}^{\infty} \beta^n \frac{d_t}{d_{t+n}} d_{t+n} \right\} = \frac{\beta}{1 - \beta} d_t. \quad (6)$$

## 5 Taxation

In this section we maintain the assumption that  $u(c) = \ln c$ , and add a government sector. The government's consumption per capita  $g_t$  is given by

$$g_t = \varepsilon_t d_t,$$

where  $\varepsilon_t$  is a Markov process taking values in  $(0, 1)$ .

Consider the following two tax regimes.

- Lump sum taxation of individuals.
- Tax on trees.

## 5.1 Lump sum taxation

In this regime, the consumer is taxed an amount  $\tau_t$ , so the budget is

$$c_t + a_t = w_t - \tau_t.$$

Let  $p_t^L$  denote the equilibrium price of trees at time  $t$  under this scheme. Then

$$w_t = (p_t^L + d_t) s_t,$$

so

$$\mathbf{R}_{t+1} = \frac{\mathbf{p}_{t+1}^L + \mathbf{d}_{t+1}}{p_t^L}.$$

Market clearing requires that  $s_t = 1$  and  $c_t^* = (1 - \varepsilon_t)d_t$ , so (4) becomes

$$\beta \mathbf{E} \frac{u'((1 - \varepsilon_{t+n})\mathbf{d}_{t+n})}{u'((1 - \varepsilon_t)d_t)} \frac{\mathbf{p}_{t+1}^L + (1 - \varepsilon_{t+1})\mathbf{d}_{t+1}}{p_t^L} = 1,$$

so (5) becomes

$$p_t^L = \mathbf{E}_t \left\{ \sum_{n=1}^{\infty} \beta^n \frac{u'((1 - \varepsilon_{t+n})\mathbf{d}_{t+n})}{u'((1 - \varepsilon_t)d_t)} \mathbf{d}_{t+n} \right\}.$$

For logarithmic utility we have

$$\begin{aligned} p_t^L &= \mathbf{E}_t \left\{ \sum_{n=1}^{\infty} \beta^n \frac{(1 - \varepsilon_t)d_t}{(1 - \varepsilon_{t+n})\mathbf{d}_{t+n}} \mathbf{d}_{t+n} \right\} \\ &= (1 - \varepsilon_t)d_t \left\{ \sum_{n=1}^{\infty} \beta^n \mathbf{E}_t \frac{1}{1 - \varepsilon_{t+n}} \right\}. \end{aligned} \tag{7}$$

## 5.2 Tax on trees

In this regime, there is a tax  $\tau_t$  on each tree owned at the beginning of the period. Let  $p_t^T$  denote the equilibrium price of trees at time  $t$  under this scheme. Then

$$w_t = ((p_t^T - \tau_t) + d_t) s_t,$$

so the budget remains

$$c_t + a_t = w_t,$$

but

$$\mathbf{R}_{t+1} = \frac{\mathbf{p}_{t+1}^T + \mathbf{d}_{t+1} - \tau_{t+1}}{p_t^T}.$$

Thus (4) becomes

$$\beta \mathbf{E} \frac{u'(\mathbf{c}_{t+1})}{u'(c_t)} \frac{\mathbf{p}_{t+1}^T + \mathbf{d}_{t+1} - \tau_{t+1}}{p_t^T} = 1,$$

So using  $\tau_t = \varepsilon_t d_t$ ,

$$p_t^T = \beta \mathbf{E} \frac{u'(\mathbf{c}_{t+1})}{u'(c_t)} (\mathbf{p}_{t+1}^T + (\mathbf{d}_{t+1} - \tau_{t+1})) = 1.$$

Recurring forward leads to this analog of (5)

$$p_t^T = \mathbf{E}_t \left\{ \sum_{n=1}^{\infty} \beta^n (1 - \varepsilon_{t+n}) \frac{u'(\mathbf{c}_{t+n})}{u'(c_t)} \mathbf{d}_{t+n} \right\}$$

or using  $c_t = (1 - \varepsilon_t)d_t$

$$p_t^T = \mathbf{E}_t \left\{ \sum_{n=1}^{\infty} \beta^n (1 - \varepsilon_{t+n}) \frac{u'((1 - \varepsilon_{t+n})\mathbf{d}_{t+n})}{u'((1 - \varepsilon_t)d_t)} \mathbf{d}_{t+n} \right\}.$$

For log utility this becomes

$$\begin{aligned} p_t^T &= \mathbf{E}_t \left\{ \sum_{n=1}^{\infty} \beta^n (1 - \varepsilon_{t+n}) \frac{(1 - \varepsilon_t)d_t}{(1 - \varepsilon_{t+n})\mathbf{d}_{t+n}} \mathbf{d}_{t+n} \right\} \\ &= \frac{\beta}{1 - \beta} (1 - \varepsilon_t)d_t. \end{aligned} \tag{8}$$

### 5.3 Comparison

Since  $0 < \varepsilon_t < 1$ , we have  $1/(1 - \varepsilon_t) > 1$  for each  $t$ , so  $\mathbf{E}_t 1/(1 - \varepsilon_{t+n}) > 1$  for each  $n$ , so

$$\sum_{n=1}^{\infty} \beta^n \mathbf{E}_t \frac{1}{1 - \varepsilon_{t+n}} > \sum_{n=1}^{\infty} \beta^n = \frac{\beta}{1 - \beta}.$$

Thus from (7) and (8), we have

$$p_t^L > p_t^T.$$

In other words, under a lump sum tax on consumers, the price of trees is higher than under a tax directly on trees—even though the taxes have no real effects in this model.

## References

- [1] Lucas, Jr., R. E. 1978. Asset prices in an exchange economy. *Econometrica* 46(6):1429–1445.
- [2] Sargent, T. J. 1987. *Dynamic macroeconomic theory*. Cambridge, Massachusetts: Harvard University Press.
- [3] Stokey, N., R. E. Lucas, Jr., and E. C. Prescott. 1989. *Recursive methods in economic dynamics*. Cambridge, Mass.: Harvard University Press.