

Preliminary Notes on Lattices*

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1 Partially ordered sets

A **partial order** is a binary relation \succeq on a set X that satisfies the following properties:

1. \succeq is **reflexive**, that is, for all $x \in X$,

$$x \succeq x.$$

2. \succeq is **transitive**, that is, for all $x, y, z \in X$

$$x \succeq y \ \& \ y \succeq z \quad \implies \quad x \succeq z.$$

3. \succeq is **antisymmetric**, that is, for all $x, y \in X$,

$$x \succeq y \ \& \ y \succeq x \quad \implies \quad x = y.$$

We write $x \succ y$ to mean $x \succeq y$, but $x \neq y$. We may also write $y \preceq x$ to mean $x \succeq y$, or $y \prec x$ to mean $x \succ y$. A pair (X, \succeq) , where X is a nonempty set and \succeq is a partial order is called a **partially ordered set**, sometimes called a **poset**. Two elements x and y of a partially ordered set are **ordered** if either $x \succeq y$ or $y \succeq x$. A partial order \succeq is a **linear order** if it is **complete**, that is, every pair is ordered. A linearly ordered subset of a partially ordered set is called a **chain**. Note that a partial order (and hence a linear order) does not allow for “indifference,” that is, we cannot have $x \succeq y$, $y \succeq x$, and $x \neq y$.

*This is primarily intended as a very brief but self-contained introduction to the main results of Topkis [8].

1 Example Here are some familiar examples of partially ordered sets. They are also the ones you are most likely to encounter in economics.

1. The usual ordering \geq of the real numbers \mathbf{R} is a partial order, in fact a linear order.
2. Set inclusion \subset is a partial order on the power set $\mathcal{P}(X)$ of a set X .
3. The pointwise ordering \geq of real-valued functions on a set X is a partial order, where $f \geq g$ if $f(x) \geq g(x)$ for all $x \in X$. This includes as a special case the coordinatewise ordering \geq on \mathbf{R}^n , where $x \geq y$ if $x_i \geq y_i$ for each $i = 1, \dots, n$.
4. First order stochastic dominance is a partial order of probability distributions (or random variables).
5. Von Neumann–Morgenstern utility functions are partially ordered by de Finetti–Arrow–Pratt risk aversion. □

It is often useful for examples to describe a finite partially ordered set in terms of its minimal directed graph. In this kind of diagram, points x and y are connected by an arrow from x to y if $x \succ y$ and there is no z with $x \succ z \succ y$. (In this case we might say that x is an immediate successor of y . If there were such a z , transitivity would imply $x \succ y$, so the arrow from x to y would be redundant.) See Example 6 and Figure 1 for an example of such a diagram. This kind of diagram, minus the arrowheads, is sometimes known as a **Hasse diagram**. Without the arrowheads, the direction of the relation ($x \succ y$ or $y \succ x$) is to be inferred from their relative vertical positions.

An element x is an **upper bound** for a set A in a partially ordered set X if $x \succeq y$ for each $y \in A$. It is a **lower bound** if $y \succeq x$ for each $y \in A$. An element x is the **greatest element** of A if it belongs to A and is an upper bound for A . An element x is the **least element** of A if it belongs to A and is an lower bound for A . The element x is a **maximal element** of A if it belongs to A and there is no y belonging to A with $y \succ x$. The element x is a **minimal element** of A if it belongs to A and there is no y belonging to A with $y \prec x$.

The element x is the **least upper bound** or **supremum** of A if it is the least element of the set of upper bounds of A . The element x is the **greatest lower bound** or **infimum** of A if it is the greatest element of the set of lower bounds of A .

2 Exercise *The greatest element of A , if it exists, is unique. The greatest element is the unique maximal element of A .*

An **interval** in X is a set A with the property that if $x, y \in A$ and $x \succ z \succ y$, then $z \in A$. The **closed interval** $[x, y]$ is $\{z \in X : x \preceq z \preceq y\}$. Note that this set is empty unless $x \preceq y$. The **open interval** (x, y) is $\{z \in X : x \prec z \prec y\}$. We may also write $(-\infty, x]$ for $\{y \in X : y \preceq x\}$, $(-\infty, x)$ for $\{y \in X : y \prec x\}$, $[x, \infty)$ for $\{y \in X : y \succeq x\}$, and (x, ∞) for $\{y \in X : y \succ x\}$. (N.B. This use of the symbol ∞ should not be confused with the extended real number ∞ .)

Given a family $\{(X_i, \succeq_i) : i \in I\}$ of partially ordered sets, the **product order** is the partial order \succeq on the Cartesian product $\prod_{i \in I} X_i$ defined by

$$(x_i)_{i \in I} \succeq (y_i)_{i \in I} \quad \text{if } \forall i \in I \quad x_i \succeq_i y_i.$$

Note that the pointwise order on functions is really the product order.

2 Lattices

A **lattice** is a partially ordered set (X, \succeq) where every pair x, y in X has a supremum and an infimum (in X). The supremum of a pair x, y is also called the **join** of x and y , denoted $x \vee y$. The infimum is also called the **meet**, denoted $x \wedge y$. Note that by Exercise 2, the meet and join are unique whenever they exist.

The meet and join are called **lattice operations**. It is convenient to restate the definitions as follows. For any x, y, z , to show that $z = x \wedge y$, that is, z is the greatest lower bound of the set $\{x, y\}$, we need to show three things: $x \succeq z$, $y \succeq z$, and

$$(x \succeq u \ \& \ y \succeq u) \quad \implies \quad z \succeq u. \quad (*)$$

Likewise, to show that $z = x \vee y$, that is, z is the least upper bound of the set $\{x, y\}$, we need to show: $z \succeq x$, $z \succeq y$, and

$$(u \succeq x \ \& \ u \succeq y) \quad \implies \quad u \succeq z. \quad (**)$$

The following facts are now obvious (given the reflexivity of \succeq) and will be used over and over without any special mention. Let X be a lattice. For every $x, y, z \in X$,

1. $x \vee y \succeq x \succeq x \wedge y$.¹
2. If $x \succeq y$, then $x = x \vee y$ and $y = x \wedge y$.
3. $x = x \wedge x = x \vee x$.
4. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.
5. $z \succeq x \vee y$ if and only if ($z \succeq x$ & $z \succeq y$). Likewise $z \preceq x \wedge y$ if and only if ($z \preceq x$ & $z \preceq y$).

The next facts are only a little less obvious.

3 Exercise (Associativity) Let X be a lattice. For every $x, y, z \in X$,

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad \text{and} \quad x \vee (y \vee z) = (x \vee y) \vee z.$$

4 Exercise Every nonempty finite lattice has a greatest and a least element. Every linearly ordered set is a lattice.

5 Example Here are some familiar examples of lattices and partially ordered sets that are or are not lattices.

1. The numbers \mathbf{R} with the usual ordering \geq is a lattice. (This is obvious, but it also follows from Exercise 4, as it is a linearly ordered set.)
2. The power set $\mathcal{P}(X)$ is a lattice under set inclusion \supseteq . Indeed $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.
3. The set \mathbf{R}^n with the coordinatewise order \geq is a lattice, where $x \wedge y = (x_1 \wedge y_1, \dots, x_n \wedge y_n)$ and $x \vee y = (x_1 \vee y_1, \dots, x_n \vee y_n)$.
But note that under the order \succeq , defined by $x \succeq y$ if $x = y$ or $x_i > y_i$ for $i = 1, \dots, n$, the set \mathbf{R}^n is not a lattice.
4. The set of linear subspaces of a linear space X is a lattice under set inclusion, but here $M \vee N = \text{span } M \cup N$.
5. The set of continuous real-valued functions on a topological space is a lattice under the pointwise order, and $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$ for each x . I leave it to you to prove that $f \vee g$ and $f \wedge x$ are continuous.

¹A word on precedence: $x \vee y \succeq x$ means $(x \vee y) \succeq x$, which should be apparent from the fact that $x \vee (y \succeq x)$ is meaningless.

6. The set of differentiable functions on a real interval is *not* a lattice under the pointwise order. To see this, let $f(x) = x$ and $g(x) = -x$, and ask yourself what $f \vee g$ would have to be. □

If you are like me, you might have guessed that there is a distributive law for lattices, and you would be wrong.

6 Example (No distributive law) Let $X = \{u, v, x, y, z\}$ and define \succeq by $u \succeq x \succeq v$, $u \succeq y \succeq v$, and $u \succeq z \succeq v$. See Figure 1. Then (X, \succeq) is a lattice. But

$$x \wedge (y \vee z) = x \wedge u = x \quad \text{and} \quad (x \wedge y) \vee (x \wedge z) = v \vee v = v,$$

and

$$x \vee (y \wedge z) = x \vee v = x \quad \text{and} \quad (x \vee y) \wedge (x \vee z) = u \wedge u = u.$$

Thus no distributive law holds. □

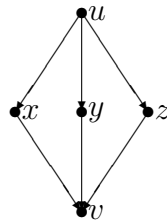


Figure 1: A non-distributive lattice.

A **sublattice** of (X, \succeq) is a subset A of X that contains the meet and join (in X) of each pair of elements of A . This is not the same as A being a lattice in its own right. It is possible that A (ordered by \succeq) is a lattice, but the meet and join of x and y may be different in A than in X , so be careful.

7 Example (Not a sublattice) Let $X = \{x, y, z, u, v\}$ and define \succeq by $x \succ y \succ u \succ z$ and $x \succ y \succ v \succ z$ (and all the other relations implied by reflexivity and transitivity, but no more). See Figure 2. In particular, u and v are unordered with $y = u \vee v$ and $z = u \wedge v$. The set $A = \{x, z, u, v\}$ is a lattice, but is not a sublattice of X , for in A , $x = u \vee_A v$ (as $y \notin A$). □

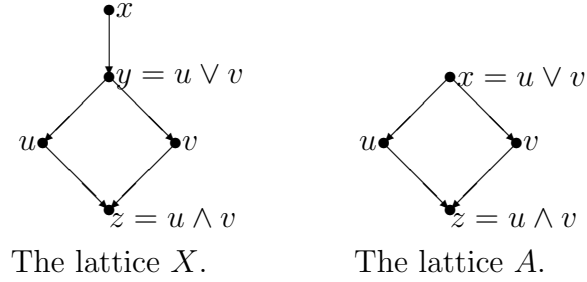


Figure 2: The lattice A is not a sublattice of X .

8 Exercise (Topkis [8, Example 2.2.5, p. 14]) A closed interval in a lattice is a sublattice.

9 Exercise If (X_i, \succeq_i) is a lattice for each i in some index set I , then $\prod_{i \in I} X_i$ is a lattice in the product order, and $(x_i)_{i \in I} \vee (y_i)_{i \in I} = (x_i \vee y_i)_{i \in I}$ and $(x_i)_{i \in I} \wedge (y_i)_{i \in I} = (x_i \wedge y_i)_{i \in I}$.

3 Induced set order

Given a lattice (X, \succeq) , we define the **induced set order** \sqsupseteq on the family $\mathcal{P}_0(X)$ of nonempty subsets of X by

$$A \sqsupseteq B \quad \text{if } x \in A, y \in B \text{ imply } x \vee y \in A, x \wedge y \in B.$$

Note that when A and B are singletons, the induced set order agrees with the order on X . That is,

$$\{x\} \sqsupseteq \{y\} \quad \text{if and only if } x \succeq y.$$

(This is because $x \succeq y$ if and only if $x = x \vee y$ and $y = x \wedge y$.)

10 Exercise (Topkis [8, Theorem 2.4.1, p. 33]) The induced set order is antisymmetric and transitive on $\mathcal{P}_0(X)$. Moreover $A \sqsupseteq A$ if and only if A is a sublattice.

11 Lemma Let X be a lattice. Let $I = [a, b]$ and $I' = [a', b']$ be closed intervals (where of course $a \preceq b$ and $a' \preceq b'$). Then $I \sqsupseteq I'$ if and only if $a \succeq a'$ and $b \succeq b'$.

Proof: (\implies) Assume $I \sqsupseteq I'$. Then in particular $a \wedge a' \in I' = [a', b']$. Thus $a' \preceq a \wedge a'$, and by definition $a \wedge a' \preceq a$. By transitivity, $a' \preceq a$. Similarly, since $I \sqsupseteq I'$, we have $b \vee b' \in I = [a, b]$, so $b \succeq b \vee b'$. But $b \vee b' \succeq b'$, so by transitivity, $b \succeq b'$.

(\impliedby) Assume $a \succeq a'$ and $b \succeq b'$, and let $x \in I = [a, b]$ and $y \in I' = [a', b']$.

Then $b \succeq x$ and $b \succeq b' \succeq y$. But $b \succeq x$ and $b \succeq y$ imply $b \succeq x \vee y$. In addition, $x \vee y \succeq x \succeq a$, so $x \vee y \in [a, b] = I$.

Also, $a' \preceq a \preceq x$ and $a' \preceq y$, so $a' \preceq x \wedge y$. And $b' \succeq y \succeq x \wedge y$, so $x \wedge y \in [a', b'] = I'$. This completes the proof that $I \sqsupseteq I'$. ■

12 Example Let us say that a set A in a partially ordered set is an **increasing set** if $x \in A$ and $y \succeq x$ imply $y \in A$. Let also say that a set B is a **decreasing set** if $x \in B$ and $x \succeq y$ imply $y \in B$.

In a lattice, if A is an increasing set and B is a decreasing set, then $A \sqsubseteq B$. □

4 Monotone functions

A function f from a partially ordered set (X, \succeq_X) to a partially ordered set (Y, \succeq_Y) is **(weakly) increasing** if $x \succeq_X x'$ implies $f(x) \succeq_Y f(x')$. (Henceforth the subscript on the order may be omitted if it is clear from the context.) The function f is **(weakly) decreasing** if $x \succeq x'$ implies $f(x) \preceq f(x')$. It is **strictly increasing** if $x \succ x'$ implies $f(x) \succ f(x')$ and **strictly decreasing** if $x \succ x'$ implies $f(x) \prec f(x')$. A function is monotone if it is either increasing or decreasing. Topkis points out that the term “nondecreasing” usually means “weakly increasing” rather than “not decreasing,” as a weakly increasing function is monotone, whereas any function that is not monotone is *a fortiori* not decreasing.

5 Supermodular functions

Let f be a real-valued function on a lattice (X, \succeq) . We say that f is **supermodular** if for every $x, y \in X$,

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y).$$

Note that if $x \succeq y$ (or vice versa), then the above is satisfied as an equality. So say that f is **strictly supermodular** if the above inequality is strict whenever x and y are unordered. Finally, f is **submodular** if the inequality is always reversed.

13 Example (Supermodular functions) Here are some familiar examples of supermodular functions.

1. Let X be the real numbers with their usual order. Then any real-valued function is both supermodular and submodular. This is because the reals are linearly ordered, so either $x = x \wedge y$ and $y = x \vee y$ or $y = x \wedge y$ and $x = x \vee y$. Either way the supermodularity inequality has $f(x) + f(y)$ on both sides.
2. On a less trivial note, the function $f(x, y) = xy$ is strictly supermodular on \mathbf{R}^2 (with the coordinatewise order). To see this, let (x, y') and (x', y) be unordered. Without loss of generality, let $x' = x + \varepsilon$, $y' = y + \delta$ where $\varepsilon, \delta > 0$. Then $(x, y') \vee (x', y) = (x', y')$ and $(x, y') \wedge (x', y) = (x, y)$. Now $x'y + y'x = (x + \varepsilon)y + (y + \delta)x = 2xy + \varepsilon y + \delta x < 2xy + \varepsilon y + \delta x + \varepsilon \delta = xy + (x + \varepsilon)(y + \delta) = xy + x'y'$, which proves strict supermodularity.

By the way, this example and the previous one show that a supermodular function need not be monotone.

3. Consider an algebra \mathcal{A} of subsets of a set X . It is a lattice under set inclusion, where the join of two sets is their union, and the meet is their intersection. Let P be a probability measure on \mathcal{A} . Then we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

so rearranging terms we see that P is both supermodular and submodular.

Topkis [8, Example 2.6.2, p. 46] provides additional examples. □

6 Maximization and comparative statics

We now present some basic results on optimization of supermodular functions.

14 Theorem (Topkis [8, Theorem 2.7.1, p. 66]) *Let X be a lattice. If f is a real-valued supermodular function on X , then the set of maximizers of f is a sublattice. If f is a real-valued submodular function on X , then the set of minimizers of f is a sublattice.*

Proof: Since the empty set is a sublattice, assume x and y are maximizers. In particular, $f(x) \geq f(x \wedge y)$ and $f(y) \geq f(x \vee y)$. By supermodularity, $f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$. Thus $f(x) = f(y) = f(x \wedge y) = f(x \vee y)$, so that $x \wedge y$ and $x \vee y$ are also maximizers.

The proof for minimization is gotten by reversing the above inequalities. ■

The next result is the main result on monotonicity of solutions to parameterized optimization problems.

15 Theorem (Topkis [8, Lemma 2.8.1, p. 75]) *Let X be a lattice and P be a partially ordered set of parameters. Let $\varphi: P \rightarrow X$ be a constraint correspondence from P into X . Let $f: X \times P \rightarrow \mathbf{R}$, let*

$$\mu(p) = \{x \in \varphi(p) : x \text{ maximizes } f(\cdot, p) \text{ over } \varphi(p)\}$$

be the set of constrained maximizers of f over $\varphi(p)$ (the “argmax” correspondence), and let $F(p) = f(x, p)$ for $x \in \mu(p)$ be the value function.

Assume that φ is lattice-valued and increasing with respect to the induced set order. That is,

$$p \succeq q \implies \varphi(p) \supseteq \varphi(q).$$

Assume also that

$$f(y, q) + f(x, p) \leq f(x \wedge y, q) + f(x \vee y, p)$$

whenever $p \succeq q$, $x \in \varphi(p)$, and $y \in \varphi(q)$. (This is almost saying that f is supermodular on the graph of φ , except that we don't require either P or the graph of φ to be a lattice.)

Then μ is increasing. That is, if $\mu(p)$ and $\mu(q)$ are nonempty, then

$$p \succeq q \implies \mu(p) \supseteq \mu(q).$$

Proof: Let $p \succeq q$ and assume $\mu(p)$ and $\mu(q)$ are nonempty. Pick $x \in \mu(p)$ and $y \in \mu(q)$. Since φ is increasing, $x \wedge y \in \varphi(q)$ and $x \vee y \in \varphi(p)$. Therefore by the definition of μ and F , we have

$$f(x \wedge y, q) \leq f(y, q) = F(q) \quad \text{and} \quad f(x \vee y, p) \leq f(x, p) = F(p).$$

Combining this with our hypothesis on f yields

$$f(y, q) + f(x, p) = f(x \wedge y, q) + f(x \vee y, p) = F(q) + F(p).$$

A little thought should convince you that in fact

$$f(y, q) = f(x \wedge y, q) = F(q) \quad \text{and} \quad f(x \vee y, p) = f(x, p) = F(p).$$

That is, $x \vee y \in \mu(p)$ and $x \wedge y \in \mu(q)$. This shows that $\mu(p) \sqsupseteq \mu(q)$. ■

7 Supermodularity and increasing differences

Let X and P be partially ordered sets, $S \subset X \times P$, and let $f: S \rightarrow \mathbf{R}$ be a real-valued function on a subset of their product. We say that f has **increasing differences on S** if

$$f(x, p) - f(y, p) \geq f(x, q) - f(y, q) \quad \text{whenever} \quad p \succ q \text{ and } x \succ y,$$

and of course $(x, p) \in S$ and $(y, q) \in S$. Note that this condition is equivalent to the following:

$$f(x, p) - f(x, q) \geq f(y, p) - f(y, q) \quad \text{whenever} \quad p \succ q \text{ and } x \succ y.$$

Similarly, f has **decreasing differences** if $f(x, p) - f(y, p) \leq f(x, q) - f(y, q)$ whenever $p \succ q$ and $x \succ y$. And of course, f has **strictly increasing differences** if $f(x, p) - f(y, p) > f(x, q) - f(y, q)$ or **strictly decreasing differences** if $f(x, p) - f(y, p) < f(x, q) - f(y, q)$ whenever $p \succ q$ and $x \succ y$.

Supermodularity on the product implies increasing differences, but the concept of supermodularity only makes sense when X and P are lattices, whereas increasing differences only requires partial orders.

16 Exercise (Topkis [8, Theorem 2.6.1, p. 44]) *Let X and P be lattices and let $f: X \times P \rightarrow \mathbf{R}$ be supermodular. Then f has increasing differences on $X \times P$.*

The converse of this result is not true. For instance, let g be a non-supermodular function on X and define $f(x, p) = g(x)$ for all $p \in P$. Then f has increasing differences (albeit very weakly, as $f(x, p) - f(x, q) = 0 \geq 0 = f(y, p) - f(y, q)$), but is not supermodular since g is not. The problem is that increasing differences cannot guarantee supermodularity in each variable. However, if f is supermodular in each variable and has increasing differences, then f is supermodular on the product.

17 Theorem (Topkis [8, Theorem 2.6.2, p. 45]) *Let X and P be lattices and let $f: X \times P \rightarrow \mathbf{R}$ be supermodular in each variable, and have increasing differences on $X \times P$. Then f is supermodular on $X \times P$.*

Proof: Let (x, p) and (y, q) belong to $X \times P$. Since f is supermodular in its second argument, fixing the first argument at $x \vee y$, we have

$$f(x \vee y, p \wedge q) + f(x \vee y, p \vee q) \geq f(x \vee y, p) + f(x \vee y, q),$$

or, rewriting,

$$\underbrace{f(x \vee y, p \vee q) - f(x \vee y, q)}_A \geq \underbrace{f(x \vee y, p) - f(x \vee y, p \wedge q)}_{A'}.$$

Since f is supermodular in its first argument, fixing the second at q , we have

$$f(x \wedge y, q) + f(x \vee y, q) \geq f(x, q) + f(y, q),$$

or, rewriting,

$$\underbrace{f(x \vee y, q) - f(y, q)}_B \geq \underbrace{f(x, q) - f(x \wedge y, q)}_{B'}.$$

Now observe

$$\begin{aligned} & f(x \vee y, p \vee q) \\ &= \underbrace{f(x \vee y, p \vee q) - f(x \vee y, q)}_A + \underbrace{f(x \vee y, q) - f(y, q)}_B + f(y, q) \\ &\geq \underbrace{f(x \vee y, p) - f(x \vee y, p \wedge q)}_{A'} + \underbrace{f(x, q) - f(x \wedge y, q)}_{B'} + f(y, q) \\ &\geq \underbrace{f(x, p) - f(x, p \wedge q)}_{A''} + \underbrace{f(x, p \wedge q) - f(x \wedge y, p \wedge q)}_{B''} + f(y, q) \\ &= f(x, p) + f(y, q) - f(x \wedge y, p \wedge q), \end{aligned}$$

where $A' \geq A''$ and $B' \geq B''$ follow from increasing differences. (The equalities are obtained by netting out terms that add to zero.) Rearranging gives the supermodularity inequality,

$$f(x \vee y, p \vee q) + f(x \wedge y, p \wedge q) \geq f(x, p) + f(y, q).$$

■

An important application of this theorem is when X and P are linearly ordered spaces, such as the real line. In that case, every function is supermodular (in one real variable), so supermodularity and increasing differences are equivalent on \mathbf{R}^2 . By the way Topkis [8, Theorem 2.6.2, p. 45] proves this result for any finite product of lattices, not just two. The proof is the same, but notationally more challenging.

8 Supermodularity and derivatives

Increasing differences, and hence supermodularity, has obvious implications for mixed partial derivatives.

18 Theorem *Let X and P be open intervals of \mathbf{R}^n and \mathbf{R}^m respectively, and let $f: X \times P \rightarrow \mathbf{R}$ be twice differentiable. If f has increasing differences, then*

$$\frac{\partial^2 f(x, p)}{\partial x_i \partial p_j} \geq 0.$$

Proof: Since f has increasing differences, if $v > 0$ and $w > 0$, then

$$H(v, w) = f(x + ve^i, p + we^j) - f(x, p + we^j) - (f(x + ve^i, p) - f(x, p)) \geq 0,$$

where e^i and e^j are unit coordinate vectors in \mathbf{R}^n and \mathbf{R}^m . But

$$\frac{\partial^2 f(x, p)}{\partial x_i \partial p_j} = \lim_{v \rightarrow 0} \lim_{w \rightarrow 0} \frac{H(v, w)}{vw} \geq 0.$$

See Loomis and Sternberg [3, p. 188] (or my optimization notes) if you doubt me. ■

A converse is also true. I suspect that I can weaken the continuous differentiability assumption, and also extend the result to more than one dimension in each variable.

19 Theorem *Let X and P be open intervals of \mathbf{R} , and let $f: X \times P \rightarrow \mathbf{R}$ be twice continuously differentiable. If*

$$\frac{\partial^2 f(x, p)}{\partial x \partial p} \geq 0$$

for all x and p , then f has increasing differences.

Proof: By the Second Fundamental Theorem of Calculus [1, Theorem 5.3, p. 205] (which assumes continuous differentiability),

$$\begin{aligned} & (f(x_1, p_1) - f(x_0, p_1)) - (f(x_1, p_0) - f(x_0, p_0)) \\ &= \int_{x_0}^{x_1} \frac{\partial f(x, p_1)}{\partial x} dx - \int_{x_0}^{x_1} \frac{\partial f(x, p_0)}{\partial x} dx \\ &= \int_{x_0}^{x_1} \left(\frac{\partial f(x, p_1)}{\partial x} - \frac{\partial f(x, p_0)}{\partial x} \right) dx \\ &= \int_{x_0}^{x_1} \int_{p_0}^{p_1} \frac{\partial^2 f(x, p)}{\partial x \partial p} dp dx. \end{aligned}$$

Now this is nonnegative whenever $p_1 \geq p_0$ and $x_1 \geq x_0$, so f has increasing differences. ■

9 Quasisupermodularity

Let X be a lattice and let Y be partially ordered. A function $f: X \rightarrow Y$ is **quasisupermodular** if

$$f(x \wedge z) \preceq f(x) \implies f(z) \preceq f(x \vee z)$$

$$f(x \wedge z) \prec f(x) \implies f(z) \prec f(x \vee z).$$

Note that supermodularity is defined for real-valued functions, while quasisupermodularity is defined for poset-valued functions.² Sensibly enough, a real-valued supermodular function is also quasisupermodular (assuming, of course, the usual order on the reals).

²Milgrom and Shannon [5] introduced the term quasisupermodularity in the context of real-valued functions, but Topkis [8] extends it to poset-valued functions.

10 Single crossing property

Let X be a lattice and let P and Y be partially ordered, and let $S \subset X \times P$. A function $f: S \rightarrow Y$ has the **single crossing property in (x, p) on S** if for every $x, z \in X$ and $p, q \in P$ such that $\{x, z\} \times \{p, q\} \subset S$, we have

$$x \succ z \ \& \ p \succ q \ \& \ f(x, q) \succeq f(z, q) \quad \implies \quad f(x, p) \succ f(z, p).$$

11 Log-supermodularity

The real-valued function g on a lattice is **log-supermodular** if

$$g(x)g(y) \leq g(x \wedge y) g(x \vee y).$$

The motivation for this term is this: Assuming g is strictly positive, taking the logarithm of the above inequality implies

$$\ln(g(x)g(y)) \leq \ln(g(x \wedge y) g(x \vee y)),$$

so

$$\ln(g(x)) + \ln(g(y)) \leq \ln(g(x \wedge y)) + \ln(g(x \vee y)).$$

That is, $\ln \circ g$ is supermodular. The virtue of the definition given is that g need not be strictly positive.

12 Preservation

The property of supermodularity is preserved by a number of natural transformations. *****

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Answers to exercises

Exercise 2

The greatest element of A , if it exists, is unique.

Proof: Let x and y be greatest elements of A . By definition of greatest, we must have $x \succeq y$ as x is the greatest element of A and y belongs to A . Likewise $y \succeq x$. By antisymmetry $x = y$. Thus any greatest element is unique. ■

The greatest element is the unique maximal element of A .

Proof: Suppose x is the greatest element of A and y is a maximal element of A . Then $x \succeq y$ as x is greatest, so by definition of maximality, $y = x$. Thus a greatest element is the unique maximal element. ■

Exercise 3

Let X be a lattice. For every $x, y, z \in X$,

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad \text{and} \quad x \vee (y \vee z) = (x \vee y) \vee z.$$

Proof: Let $u = x \wedge (y \wedge z)$. That is, $u \preceq x$, $u \preceq y \wedge z$, and

$$(w \preceq x \ \& \ w \preceq y \wedge z) \implies w \preceq u.$$

Now $u \preceq y \wedge z$ if and only if $u \preceq y$ and $u \preceq z$, and $w \preceq y \wedge z$ if and only if $w \preceq y$ and $w \preceq z$. Thus u is characterized by the following statements:

$$u \preceq x, \quad u \preceq y, \quad u \preceq z,$$

and

$$(w \preceq x \ \& \ w \preceq y \ \& \ w \preceq z) \implies w \preceq u.$$

Regrouping these conjunctions yields the characterization of $u = (x \wedge y) \wedge z$.

The proof of $x \vee (y \vee z) = (x \vee y) \vee z$ is similar. ■

Exercise 4

Every nonempty finite lattice has a greatest and a least element.

Proof: Let $A = \{x_1, \dots, x_m\}$ be a nonempty finite lattice. Put $y_1 = x_1$, and inductively define

$$y_{n+1} = y_n \vee x_{n+1}$$

for $n = 2, \dots, m - 1$. I claim the following

Claim: For $n = 1, \dots, m$, we have $y_n \in A$ and $y_n \succeq x_k$ for each $k = 1, \dots, n$.

Clearly the claim is true for $n = 1$, so assume the induction hypothesis that the claim is true $n < m$. It suffices to show the claim is true for $n + 1$: Since by hypothesis $y_n \in A$ and A is a lattice, then $y_{n+1} = y_n \vee x_{n+1}$ belongs to A . Further, by the definition of join, $y_{n+1} \succeq x_{n+1}$ and $y_{n+1} \succeq y_n$. Moreover, for any $k \leq n$, the induction hypothesis guarantees that $y_n \succeq x_k$, so by transitivity $y_{n+1} \succeq x_k$. This completes the inductive proof of the claim.

Now notice that the claim shows that y_m is the greatest element of A . A similar argument (replacing join by meet) shows that A has a least element. ■

Every linearly ordered set is a lattice.

Proof: Let A be a linearly ordered set, and let x and y belong to A . By definition of linear order either $x \succeq y$ in which case $x = x \vee y$ and $y = x \wedge y$, or $y \succeq x$ in which case $y = x \vee y$ and $x = x \wedge y$. Either way, A contains both the meet and join of x and y , and so is a lattice. ■

Exercise 8

A closed interval in a lattice is a sublattice.

Proof: Let $[a, b]$ be a closed interval in the lattice X . Since the empty set is a sublattice of X , assume that $[a, b]$ is nonempty. Let x and y belong to $[a, b]$. That is, $b \succeq x \succeq a$ and $b \succeq y \succeq a$. Since b is an upper bound for $\{x, y\}$, the definition of join implies $b \succeq x \vee y$. Moreover $x \vee y \succeq x \succeq a$, so by transitivity, $x \vee y \succeq a$. Thus $x \vee y \in [a, b]$. A similar argument show that $x \wedge y \in [a, b]$. That is, $[a, b]$ is a sublattice of X . ■

Exercise 9

If (X_i, \succeq_i) is a lattice for each i in some index set I , then $\prod_{i \in I} X_i$ is a lattice in the product order, and $(x_i)_{i \in I} \vee (y_i)_{i \in I} = (x_i \vee y_i)_{i \in I}$ and $(x_i)_{i \in I} \wedge (y_i)_{i \in I} = (x_i \wedge y_i)_{i \in I}$.

Proof: To see that $(x_i)_{i \in I} \vee (y_i)_{i \in I} = (x_i \vee y_i)_{i \in I}$ note that $(x_i \vee y_i)_{i \in I}$ is indeed a pointwise upper bound for $\{(x_i)_{i \in I}, (y_i)_{i \in I}\}$ and that is pointwise the least upper bound. Similarly $(x_i)_{i \in I} \wedge (y_i)_{i \in I} = (x_i \wedge y_i)_{i \in I}$, which proves that $\prod_{i \in I} X_i$ is a lattice in the product (pointwise) order. ■

Exercise 10

The induced set order is antisymmetric and transitive on $\mathcal{P}_0(X)$. Moreover $A \sqsupseteq A$ if and only if A is a sublattice.

Proof: Antisymmetry: Assume $A \sqsupseteq B$ and $B \sqsupseteq A$, and let $x \in A$ and $y \in B$. Then $x \wedge y \in A$ (as $B \sqsupseteq A$). But $y \succeq x \wedge y$, so $y = (x \wedge y) \vee y \in A$ (as $A \sqsupseteq B$). Since y is an arbitrary element of B , we have $B \subset A$. But symmetrically, $A \subset B$, so $A = B$.

Transitivity: Assume $A \sqsupseteq B$ and $B \sqsupseteq C$. We wish to show that $A \sqsupseteq C$. That is, we need to show that if $x \in A$ and $y \in C$, then $x \vee y \in A$ and $x \wedge y \in C$.

So let $x \in A$ and $y \in C$. Pick any $z \in B$. Since $B \sqsupseteq C$, we have $z \vee y \in B$. Let $u = z \vee y$. Then $u \succeq y$, so $u \wedge y = y$. Moreover since $u \in B$ and $x \in A$, we have $x \wedge u \in B$ (as $A \sqsupseteq B$). Therefore since $B \sqsupseteq C$, we have

$$\underbrace{(x \wedge u)}_{\in B} \wedge y \in C.$$

But by associativity (Exercise 3),

$$(x \wedge u) \wedge y = x \wedge (u \wedge y) = x \wedge y,$$

so $x \wedge y \in C$.

To see that $x \vee y \in A$, let $w = x \wedge z$. Then $w \in B$ (as $A \sqsupseteq B$) and $x \vee w = x$. Then $w \vee y \in B$ (as $B \sqsupseteq C$). Therefore

$$x \vee \underbrace{(w \vee y)}_{\in B} \in A,$$

as $A \sqsupseteq B$. But

$$x \vee (w \vee y) = (x \vee w) \vee y = x \vee y,$$

so $x \vee y \in A$, and we are done.

Reflexivity: By definition $A \sqsupseteq A$ if and only if for each $x, y \in A$ both $x \wedge y$ and $x \vee y$ belong to A , which is to say that A is a sublattice. ■

Exercise 16

Let X and P be lattices and let $f: X \times P \rightarrow \mathbf{R}$ be supermodular. Then f has increasing differences on $X \times P$.

Proof: Let $p \succ q$ and $x \succ y$. By supermodularity,

$$f(x, q) + f(y, p) \leq f((x, q) \wedge (y, p)) + f((x, q) \vee (y, p)).$$

By Exercise 9, $(x, q) \vee (y, p) = (x, p)$ and $(x, q) \wedge (y, p) = (y, q)$, so

$$f(x, q) + f(y, p) \leq f(y, q) + f(x, p),$$

or rearranging,

$$f(x, p) - f(y, p) \geq f(x, q) - f(y, q),$$

which says that f has increasing differences. ■