CALIFORNIA INSTITUTE OF TECHNOLOGY
Division of the Humanities and Social Sciences

Notes on Simple Job Search Models

KC Border
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These notes elaborate somewhat on the treatment in Sargent [1] of the decision problem faced by an unemployed worker seeking a job. In each period of unemployment, the worker draws unemployment compensation c. The compensation could be negative, representing a search cost. Also, in each period the unemployed worker receives a wage offer w at random. Wage offers are identically and independently distributed according to the known cumulative distribution function F. For simplicity, I assume that the density of F exists and denote it by f. I only use this assumptions so that I can be cavalier when writing expressions like $\int_a^b g\,dF$. Is it the integral over $[a, b]$ or $(a, b]$ or (a, b) or $[a, b)$? If F has a density, then it doesn't matter. I also need to be more careful when integrating by parts without a density. Finally, a density implies that $F(0) = 0$.

The worker is risk neutral and maximizes the expected value of present discounted value of income over an infinite horizon. The discount factor is $\beta < 1$.

1 No firing, no recall

In this section the worker cannot recall an offer. Once rejected, it is gone. Furthermore, it is assumed that once an offer is accepted, the worker receives the offered wage forever after. We assume that the wage captures all the relevant aspects of the job, so that nothing is learned after taking the job. Thus the worker has no incentive ever to quit.

A natural state space for this problem is just the set of wage offers and there are two actions available to the worker: accept or reject. Upon receiving an offer w, the worker may accept the offer and earn w forever, or he may reject the offer, collect his compensation c and next period search again. If he takes the offer, the present discounted value of his income stream is just $w/(1 - \beta)$. If he rejects the offer, the present discounted value of his income is c plus β times the expected value of being on the market again. Thus the value of having an offer w and making the optimal choice of rejecting or accepting the offer is the unique solution to the following functional equation.

$$V(w) = \max\left\{\frac{w}{1-\beta},\ c + \beta\int_0^\infty V(s)\,dF(s)\right\} \tag{1}$$

It is immediate from (1) that the optimal decision rule takes the form of a reservation wage, w^*. That is, the policy is to reject any offer less than w^* and accept any greater offer. This is

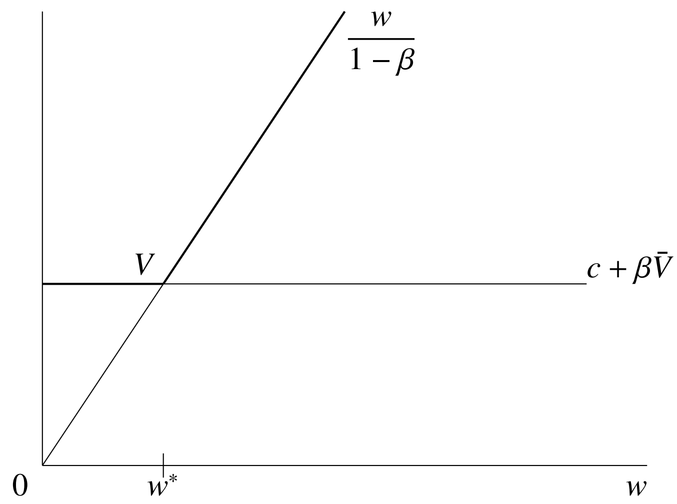

Figure 1. Value Function and Reservation Wage

because the optimal value function is the maximum of two terms, a constant term corresponding to rejecting the offer and searching again, and an increasing linear term corresponding to accepting the offer. The optimal reservation wage w^* is the point where the increasing term overtakes the constant term and so solves

$$\boxed{\tfrac{w^*}{1-\beta} = c + \beta\bar{V}} \tag{2}$$

where

$$\bar{V} = \int_0^\infty V(s)\, dF(s).$$

Now

$$V(w) = \begin{cases} \dfrac{w}{1-\beta} & w \geqslant w^* \\ c + \beta\bar{V} & w \leqslant w^*. \end{cases} \tag{3}$$

Thus

$$\boxed{\bar{V} = \int_0^{w^*} c + \beta\bar{V}\, dF(s) + \int_{w^*}^\infty \tfrac{s}{1-\beta}\, dF(s).} \tag{4}$$

This gives us two equations in the two unknowns w^* and $\bar{V}$. Let's proceed to eliminate $\bar{V}$.

Subtracting $c + \beta\bar{V} = \int_0^\infty c + \beta\bar{V}\, dF(s) = \int_0^{w^*} c + \beta\bar{V}\, dF(s) + \int_{w^*}^\infty c + \beta\bar{V}\, dF(s)$ from both sides of equation (4) gives

$$(1-\beta)\bar{V} - c = \int_{w^*}^\infty \frac{s}{1-\beta} - (c + \beta\bar{V})\, dF(s) \tag{5}$$

Solving (2) for $\bar{V}$ gives

$$\bar{V} = \frac{w^*}{\beta(1-\beta)} - \frac{c}{\beta}.$$

Substituting this into the left hand side of (5) and using (2) more directly on the right hand side gives

$$\frac{w^*}{\beta} - \frac{(1-\beta)c}{\beta} - c = \int_{w^*}^{\infty} \frac{s - w^*}{1-\beta}\, dF(s).$$

Multiply both sides by β to get

$$w^* - c = \frac{\beta}{1-\beta} \int_{w^*}^{\infty} (s - w^*)\, dF(s). \tag{6}$$

Note that V does not appear in this equation.

Define the function h by

$$\boxed{h(w) = \int_{w}^{\infty} (s - w)\, dF(s).}$$

Then $\frac{h(w)}{1-F(w)}$ is the expected increase in the wage offer conditional on the offer being at least w. This means that $h(w)$ is the conditional expected increment times the probability of a better offer. Observe that

$$h'(w) = -[1 - F(w)] \leqslant 0,$$

$$h''(w) = F'(w) = f(w) \geqslant 0,$$

that is, h is decreasing and convex. Further, $h(0) = Ew = \int_0^\infty w\, dF(w)$, and $h(w) \to 0$ as $w \to \infty$.

Then we can rewrite (6) as

$$\boxed{w^* - c = \frac{\beta}{1-\beta} h(w^*).} \tag{7}$$

The lhs of (7) is the present cost of searching once more when an offer of w^* is in hand, and the rhs is the discounted present value of the expected increase in income from searching again, times the probability of an increase. That is, the optimal reservation wage equates the marginal cost and the marginal benefit of continuing to search. This should be obvious to any good economist.

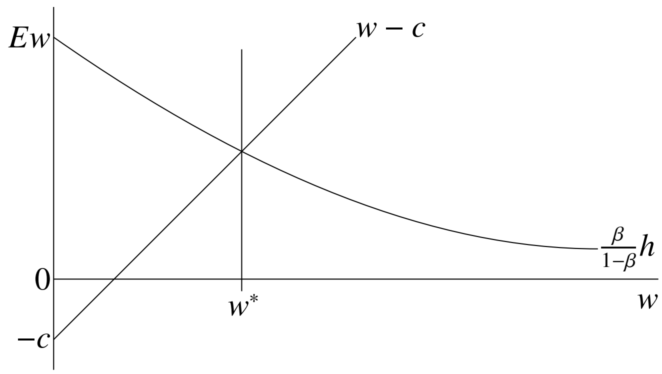

Figure 2. Determination of w^*

1.1 Comparative statics

Equation (7) says that w^* is the abscissa of the intersection of two curves, a linear curve with slope 1, $w - c$, and a decreasing convex curve, $\dfrac{\beta}{1-\beta}h$. Since $\dfrac{\beta}{1-\beta}$ is an increasing function of β, an increase in β shifts the h curve upward, so w^* shifts to the right. Also, we can rewrite

$$h(w) = \int_{w^*}^{\infty} (s - w)\,dF(s) = \int_0^{\infty} (s - w)^+\,dF(s).$$

Since $(s-w)^+$ is nondecreasing and convex, a first order stochastically dominating increase in F and an increase in the riskiness of F both shift the h curve upward. The following comparative statics for w^* follow immediately.

- An increase in c increases w^*.
- An increase in β increases w^*.
- An increase in F in the sense of first order stochastic dominance increases w^*.
- An increase in the riskiness of F increases w^*.

2 No firing; recall

In this section we drop the assumption that offers may not be recalled. The appropriate state variable is now the best offer to date. The value of having best offer to date b and making the correct decision satisfies the following functional equation.

$$V(b) = \max\left\{\frac{b}{1-\beta},\ c + \beta V(b)F(b) + \beta\int_b^{\infty} V(s)\,dF(s)\right\}. \tag{8}$$

The first term on the right hand side is the value of accepting the wage w. The second term, corresponding to rejecting the offer b and searching again, has two parts: the current period payoff c plus the discounted expected value of searching again next period. The wage offer next period is either less than or equal to b, which happens with probability $F(b)$, in which case b remains the best offer to date, or it is larger than b and becomes the new best offer to date. The expected sum of these pieces is the second term on the right hand side of (8).

It is no longer obvious that a reservation wage strategy is optimal, but it seems plausible. For suppose the best offer to date is b and it pays to search again rather accept b. Then this will always be true. Thus if an offer is ever accepted, then it must be accepted the first time it is offered. Thus a reservation wage is optimal. But this means that it must be the same reservation wage as if there were no recall. Thus you should use exactly the same rule with or without recall! This implies that the solution to (1) is also a solution to (8), which we now show.

Claim: Let U denote the solution to (1), then U also solves (8), that is,

$$U(b) = \max\left\{\frac{b}{1-\beta},\ c + \beta U(b)F(b) + \beta\int_b^\infty U(s)\,dF(s)\right\}.$$

Proof of claim: To see this, recall that equations (2) and (3) imply

$$U(b) = \begin{cases} \dfrac{b}{1-\beta} & b \geqslant w^* \\[2ex] \dfrac{w^*}{1-\beta} & b \leqslant w^*. \end{cases}$$

Now for $b \leqslant w^*$ we have

$$\begin{aligned}
&c + \beta U(b)F(b) + \beta\int_b^\infty U(s)\,dF(s) \\
&\quad = c + \beta\int_0^b U(b)\,dF(s) + \beta\int_b^{w^*} U(s)\,dF(s) + \beta\int_b^\infty U(s)\,dF(s) \\
&\quad = c + \beta\int_0^b \frac{w^*}{1-\beta}\,dF(s) + \beta\int_b^{w^*} \frac{w^*}{1-\beta}\,dF(s) + \beta\int_{w^*}^\infty \frac{s}{1-\beta}\,dF(s) \\
&\quad = c + \beta\int_0^{w^*} \frac{w^*}{1-\beta}\,dF(s) + \beta\int_{w^*}^\infty \frac{s}{1-\beta}\,dF(s) \\
&\quad = c + \beta\bar{U} \\
&\quad = \frac{w^*}{1-\beta} \\
&\quad \geqslant \frac{b}{1-\beta}.
\end{aligned}$$

This shows that (8) holds for $b \leqslant w^*$. For the case $b \geqslant w^*$, we have

$$
\begin{aligned}
& c + \beta U(b)F(b) + \beta \int_b^\infty U(s)\,dF(s) \\
&= c + \beta \int_0^{w^*} U(b)\,dF(s) + \beta \int_{w^*}^b U(b)\,dF(s) + \beta \int_b^\infty U(s)\,dF(s) \\
&= c + \beta \int_0^{w^*} \frac{b}{1-\beta}\,dF(s) + \beta \int_{w^*}^b \frac{b}{1-\beta}\,dF(s) + \beta \int_b^\infty U(s)\,dF(s) \\
&= c + \beta \int_0^{w^*} \frac{w^* + b - w^*}{1-\beta}\,dF(s) + \beta \int_{w^*}^b \frac{s + b - s}{1-\beta}\,dF(s) + \beta \int_b^\infty U(s)\,dF(s) \\
&= c + \beta \bar{U} + \beta \int_0^{w^*} \frac{b - w^*}{1-\beta}\,dF(s) + \beta \int_{w^*}^b \frac{b - s}{1-\beta}\,dF(s) \\
&\leqslant c + \beta \bar{U} + \beta \int_0^{w^*} \frac{b - w^*}{1-\beta}\,dF(s) + \beta \int_{w^*}^b \frac{b - w^*}{1-\beta}\,dF(s) \\
&= c + \beta \bar{U} + \beta \int_0^b \frac{b - w^*}{1-\beta}\,dF(s) \\
&= \frac{w^*}{1-\beta} + \beta \frac{b - w^*}{1-\beta} F(b) \\
&\leqslant \frac{w^*}{1-\beta} + \frac{b - w^*}{1-\beta} \\
&= \frac{b}{1-\beta}.
\end{aligned}
$$

This shows that (8) holds for $b \leqslant w^*$, which completes the proof that U satisfies (8). ∎

3 Search when you can be fired

In this section there is no recall of offers, but every period on the job there is a probability α of being fired. For simplicity, we assume that this probability is independent of time on the job or the wage at hiring.

The value of having an offer w and acting optimally is the solution to the following functional equation.

$$V(w) = \max\left\{c + \beta \bar{V},\ w + \beta(\alpha \bar{V} + \gamma V(w))\right\} \tag{9}$$

where

$$\gamma = 1 - \alpha$$

and again

$$\bar{V} = \int_0^\infty V(s)\,dF(s).$$

(*Note:* We have not adequately defined the state space for this problem. Having an offer of w and holding a job at wage w are not really the same state. We should therefore replace the $V(w)$ on the right in (9) by $\hat{V}(w)$, the value to an employed person at wage w of not being fired. Then $\hat{V}(w) = s + \beta(\alpha\bar{V} + \gamma\hat{V}(w))$. But if a person became employed as a result of having taken offer w it must be that the maximum in (9) occurs in the second term, $w + \beta(\alpha\bar{V} + \gamma\hat{V}(w))$, in which case $V(w) = \hat{V}(w)$.)

Now $c + \beta\bar{V}$ is independent of w and if V is nondecreasing then $w + \beta(\alpha\bar{V} + \gamma V(w))$ is nondecreasing, and so a reservation wage policy is optimal. Letting $\bar{w}$ denote the reservation wage, we have

$$V(w) = \begin{cases} c + \beta\bar{V} & w \leqslant \bar{w} \\ \\ w + \beta(\alpha\bar{V} + \gamma V(s)) & w \geqslant \bar{w}. \end{cases}$$

For $w \geqslant \bar{w}$, $V(w)$ appears on both sides of (10),

$$V(w) = w + \beta(\alpha\bar{V} + \gamma V(w)).$$

Collecting terms,

$$(1 - \gamma\beta)V(w) = w + \alpha\beta\bar{V}$$

or

$$V(w) = \frac{w + \alpha\beta\bar{V}}{1 - \gamma\beta}.$$

Thus the value function satisfies

$$V(w) = \begin{cases} c + \beta\bar{V} & w \leqslant \bar{w} \\ \\ \dfrac{w + \alpha\beta\bar{V}}{1 - \gamma\beta} & w \geqslant \bar{w} \end{cases}$$

and

$$c + \beta\bar{V} = \frac{\bar{w} + \alpha\beta\bar{V}}{1 - \gamma\beta} \tag{10}$$

This leads to two equations for $\bar{V}$,

$$\bar{V} = \int_0^{\bar{w}} c + \beta\bar{V}\,dF(s) + \int_{\bar{w}}^{\infty} \frac{s + \alpha\beta\bar{V}}{1 - \gamma\beta}\,dF(s)$$

and

$$\bar{V} = \int_0^{\bar{w}} \bar{V}\,dF(s) + \int_{\bar{w}}^{\infty} \bar{V}\,dF(s).$$

Multiply the second by β, add c and subtract from the first to get

$$(1 - \beta)\bar{V} - c = \int_{\bar{w}}^{\infty} \frac{s + \alpha\beta\bar{V}}{1 - \gamma\beta} - (c + \beta\bar{V})\,dF(s).$$

Now use (10) to get

$$\bar{V} = \frac{\bar{w} - (1-\gamma\beta)c}{\gamma\beta(1-\beta)}.$$

Using this on the left and (10) directly on the right gives

$$(1-\beta)\frac{\bar{w} - (1-\gamma\beta)c}{\gamma\beta(1-\beta)} - c = \int_{\bar{w}}^{\infty} \frac{s + \alpha\beta\bar{V} - \bar{w} - \alpha\beta\bar{V}}{1-\gamma\beta}\, dF(s).$$

Multiplying by $\gamma\beta$ and simplifying yields

$$\boxed{\bar{w} - c = \frac{\gamma\beta}{1-\gamma\beta} h(\bar{w}).}$$

This is the same as (7) with discount rate $\gamma\beta$ in place of β! Since $\gamma < 1$, the possibility of being fired acts like a decrease in the discount factor, which we have seen decreases the reservation wage. This is somewhat intuitive as a chance of being fired acts to decrease the value of any wage offer and so leads to first order stochastic decrease in the wage distribution, which depresses the reservation wage. The effects of changes in β and c are the same as before.

4 A non-recursive approach

We could have analyzed the problem of optimal job search without using the Bellman equation at all. The analysis would proceed like this.

First it is easy to show that using reservation wage strategy is part of an optimal scheme. The question is how to choose the reservation wage optimally. Let us write down the lifetime expected income from choosing the reservation wage w.

Define $g(w) = \int_w^\infty s\, dF(s)$. Note that $g'(w) = -w f(w)$, where $f(w) = F'(w)$ is the density of wage offers.

Then the expected discounted income is:

$$\begin{aligned}
\frac{g(w)}{1-\beta} + cF(w) + \beta F(w)\left(\frac{g(w)}{1-\beta} + cF(w)\right) + \beta^2 F^2(w)\left(\cdots\right) + \cdots \\
= \left(\frac{g(w)}{1-\beta} + cF(w)\right) \cdot \sum_{t=0}^{\infty} \beta^t F^t(w) \\
= \left(\frac{g(w)}{1-\beta} + cF(w)\right) \cdot \frac{1}{1-\beta F(w)}.
\end{aligned}$$

The derivative of this with respect to w is

$$\begin{aligned}
&\left(\frac{-wf(w)}{1-\beta} + cf(w)\right)\frac{1}{1-\beta F(w)} \\
&+ \left(\frac{g(w)}{1-\beta} + cF(w)\right) \cdot \frac{1}{\left(1-\beta F(w)\right)^2}\beta f(w).
\end{aligned}$$

Setting this equal to zero and multiplying by $\frac{\left(1-\beta F(w)\right)^2}{f(w)}$ yields the first order condition:

$$-\frac{1-\beta F(w)}{1-\beta}w + (1-\beta F(w))c + \frac{\beta}{1-\beta}g(w) + \beta F(w)c = 0.$$

Now use the identity

$$\frac{1-\beta F}{1-\beta} = \frac{(1-\beta) + 1 - \beta F - (1-\beta)}{1-\beta} = 1 + \frac{\beta}{1-\beta}(1-F),$$

and rearrange the first order condition to get

$$\begin{aligned} w - c &= \frac{\beta}{1-\beta}\Big(g(w) - (1 - F(w)w)\Big) \\ &= \frac{\beta}{1-\beta}h(w), \end{aligned}$$

where $h(w) = \int_w^\infty (s-w)dF(s)$. This (fortunately) agrees with the recursive method.

References

[1] Sargent, T. J. 1987. *Dynamic macroeconomic theory*. Cambridge, Massachusetts: Harvard University Press.

5 Exercises

5.1 Mortality

Consider the no firing, no recall case, where the probability of dying is α in each period. If the utility of being in the dead state is taken to be 0, write out the Bellman equation, and use it to show that the worker behaves as if his discount rate is $(1-\alpha)\beta$.

5.2 Is dying worse than being fired?

Compare the reservation wage from problem 5.1 (dying with probability α) with the reservation wage from section 3 (being fired with probability α). Now compare the value functions. In which scenario is the worker better off? Why?

5.3 Possibility of no offer

Suppose that each period, an offer is drawn from F with probability $1-\alpha$ and with probability α an offer of 0 (interpreted as no offer) is received. Compare this case to the case where the worker receives an offer drawn form F each period. What happens to the reservation wage? (Hint: Use a stochastic dominance argument.)

5.4 Change of state variable

Characterize the solution of

$$V(x) = \max\left\{x,\ \beta\int_0^\infty V(s)\,dF(s)\right\}.$$

In what sense is this a change of variable?

5.5 Risk aversion ⋆

Compare the search behavior of a risk averse worker with the search behavior of a risk neutral worker. In particular, show that a more risk averse (in the Arrow–Pratt sense) searcher sets a lower reservation wage.