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Partial Notes On A Simple Matching Model

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These notes treat Sargent's [4] version of Prescott and Townsend's [3] simplified version of Jovanovic's [1] matching model. It goes beyond Sargent's treatment by dispensing with the normality hypotheses on signals and the quality of matches. Instead, all that is assumed is that the wage is stochastically increasing with the signal. This setting provides a nice framework for showing off how to work with stochastic dominance relations.

1 The Story

At time $t = 0$, an unemployed worker is paired with a firm. For each pair nature selects a random vector (s, w), where w is the worker's marginal product for that particular firm. The pair does not immediately observe w, though. Instead, they observe s, a signal of the worker's productivity. The firm then offers a probationary salary equal to her conditional expected marginal product for one period to the worker. If the worker accepts the probationary offer, then she earns that offer for the current period and at time $t = 1$ the true marginal product w is offered to the worker. If she accepts the offer, she remains employed at that wage forever. If she rejects the offer, she must sit out the job market for one period and draw a new match at $t = 2$.

Note that this is a very partial equilibrium analysis, in that firms make no real decisions—they simply pay the worker her expected wage. A complete model would have to explain why firms do this, particularly since, given the behavior of workers in this model, it is not in their interest to do so.

As usual, workers maximize the expected present discounted value of income over their lifetime. For simplicity of notation, we assume that the payoff from not taking an offer is zero. The discount factor is β.

2 Assumptions and Notation

The signal-wage pair (s, w) is distributed on $S \times W \subset R^2_+$ according to the joint density h. For concreteness, we may assume $S = W = [0, \infty)$. Let G be the marginal cumulative distribution of s, i.e.,

$$G(s^*) = \int_0^{s^*} \int_W h(s, w)\, dw\, ds$$

with density g:

$$g(s) = \int_W h(s, w)\, dw.$$

Let F be the marginal cumulative distribution of w and $F(\cdot|s)$ be the conditional cumulative distribution of w given s:

$$F(w^*) = \int_S \int_0^{w^*} h(s,w)\,dw\,ds$$

$$F(w^*|s) = \frac{\int_0^{w^*} h(s,w)\,dw}{g(s)}.$$

Let $m(s)$ denote the conditional mean of w given s, i.e.,

$$m(s) = \frac{\int_W wh(s,w)\,dw}{g(s)} = \int_W w\,dF(w|s). \tag{1}$$

Assumption 1 *$F(\cdot|s)$ is strictly increasing in the sense of first order stochastic dominance in s, that is, if $s_1 > s_2$, then $F(\cdot|s_1) \succ_1 F(\cdot|s_2)$.*

Lemma 2 *The expected wage conditional on s, $m(s)$, is an increasing function of s.*

Proof: The identity function w is strictly increasing in w. By assumption, $F(\cdot|s)$ is stochastically increasing (in the sense of first order stochastic dominance) in s. By (1) and the definition of first order stochastic dominance, m is increasing in s. ∎

3 The Worker's Dynamic Program

A reasonable state space for the worker is:

- (u,s) – unemployed and observing signal s.
- (p,w) – on probation and observing w.
- (e,w) – employed at wage w.

The law of motion has already been described above. Note that an unemployed worker's state is sufficiently described by her signal s, since she knows that the firm will offer her $m(s)$ as a probationary wage.

4 The Optimality Equation

Let V be the value function. To simplify notation set $V_p(w) = V(p,w)$ and $V_u(s) = V(u,s)$.

The optimality equation can be written as

$$V(e,w) = \frac{w}{1-\beta} \tag{2}$$

$$V_u(s) = \max\{\beta\bar{V}_u,\ m(s) + \beta\bar{W}_p(s)\} \tag{3}$$

$$V_p(w) = \max\{\beta\bar{V}_u,\ \frac{w}{1-\beta}\}, \tag{4}$$

where

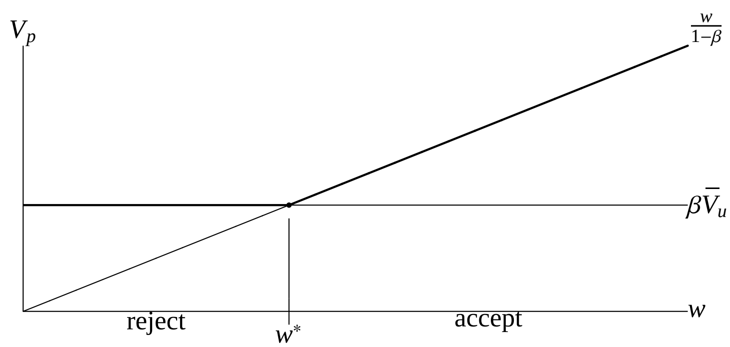
$$\bar{V}_u = \int_S V_u(s)\,dG(s),$$

and

$$\bar{W}_p(s) = \int_W V_p(w)\,dF(w|s).$$

5 Reservation Policies

Look first at (4): It is immediate that probationary workers ought to use a reservation



wage, w^*, which satisfies

$$\frac{w^*}{1-\beta} = \beta\bar{V}_u. \tag{5}$$

Then

$$V_p(w) = \begin{cases} \beta\bar{V}_u & w \leqslant w^* \\ \dfrac{w}{1-\beta} & w \geqslant w^*. \end{cases} \tag{6}$$

This is plainly a nondecreasing function of w, so that $\bar{W}_p(s)$ is nondecreasing in s (since $F(\cdot|s)$ is stochastically increasing in s).

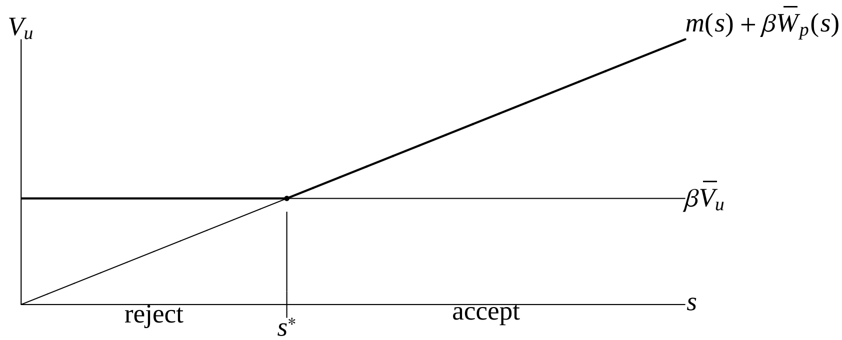
Turning to (3), we see then that $m(s) + \beta\bar{W}_p(s)$ is increasing, so that a reservation strategy is again optimal at this stage:

Thus

$$\bar{V}_u(s) = \begin{cases} \beta\bar{V}_u & s \leqslant s^* \\ m(s) + \beta\bar{W}_p(s) & s \geqslant s^* \end{cases} \tag{7}$$

where the reservation signal s^* satisfies

$$\beta\bar{V}_u = m(s^*) + \beta\bar{W}_p(s^*). \tag{8}$$



6 Wages Increase on Average after Probation

The reservation wage property of the worker's optimal policy has the consequence that average wages go up for workers remaining after probation.

The average wage of a worker on probation is just the average value of $m(s)$ conditional on $s \geqslant s^*$,

$$\frac{\int_{s^*}^{\infty} m(s)\, dG(s)}{1 - G(s^*)}. \tag{9}$$

The average wage of workers who remain after probation is the average of w conditional on $s \geqslant s^*$ and $w \geqslant w^*$:

$$\frac{\int_{s^*}^{\infty} \int_{w^*}^{\infty} w\, dF(w|s)\, dG(s)}{\int_{s^*}^{\infty} \int_{w^*}^{\infty} 1\, dF(w|s)\, dG(s)}. \tag{10}$$

It is virtually immediate that (10) is larger than (9), and indeed this follows from the Neyman-Pearson lemma. (See, e.g. [2].) A proof however is probably instructive.

Rewrite (9) as

$$\frac{\int_{s^*}^{\infty} \int_{w^*}^{\infty} w\, dF(w|s)\, dG(s) + \int_{s^*}^{\infty} \int_{0}^{w^*} w\, dF(w|s)\, dG(s)}{\int_{s^*}^{\infty} \int_{w^*}^{\infty} 1\, dF(w|s)\, dG(s) + \int_{s^*}^{\infty} \int_{0}^{w^*} 1\, dF(w|s)\, dG(s)}. \tag{11}$$

The first term in this numerator is the numerator of (10) and similarly for the denominators. Thus (10) is of the form a/b, while (9) is of the form $(a+c)/(b+d)$. Furthermore, the ratio of the integrands in a/b is greater than the ratio of the integrands in c/d, so the additional in terms intuitively bring down the average. Formally, $a/b > (a+c)/(b+d)$

if and only if $ad > bc$. (All these terms are positive.) So what we need to show is

$$\int_{s^*}^{\infty}\int_{w^*}^{\infty} w\,dF(w|s)\,dG(s)\cdot\int_{s^*}^{\infty}\int_{0}^{w^*} 1\,dF(w|s)\,dG(s) > \int_{s^*}\int_{0}^{w^*} w\,dF(w|s)\,dG(s)\cdot\int_{s^*}^{\infty}\int_{w^*}^{\infty} 1\,dF(w|s)\,dG(s).$$

But this is easy:

$$\begin{aligned}
&\int_{s^*}^{\infty}\int_{w^*}^{\infty} w\,dF(w|s)\,dG(s)\cdot\int_{s^*}^{\infty}\int_{0}^{w^*} 1\,dF(w|s)\,dG(s)\\
&> \int_{s^*}^{\infty}\int_{w^*}^{\infty} w^*\,dF(w|s)\,dG(s)\cdot\int_{s^*}^{\infty}\int_{0}^{w^*} 1\,dF(w|s)\,dG(s)\\
&= w^*\int_{s^*}^{\infty}\int_{w^*}^{\infty} 1\,dF(w|s)\,dG(s)\cdot\int_{s^*}^{\infty}\int_{0}^{w^*} 1\,dF(w|s)\,dG(s)\\
&= w^*\int_{s^*}^{\infty}\int_{0}^{w^*} 1\,dF(w|s)\,dG(s)\cdot\int_{s^*}^{\infty}\int_{w^*}^{\infty} 1\,dF(w|s)\,dG(s)\\
&= \int_{s^*}^{\infty}\int_{0}^{w^*} w^*\,dF(w|s)\,dG(s)\cdot\int_{s^*}^{\infty}\int_{w^*}^{\infty} 1\,dF(w|s)\,dG(s)\\
&> \int_{s^*}^{\infty}\int_{0}^{w^*} w\,dF(w|s)\,dG(s)\cdot\int_{s^*}^{\infty}\int_{w^*}^{\infty} 1\,dF(w|s)\,dG(s).
\end{aligned}$$

∎

7 Relationship between Reservation Levels

How are s^* and w^*, or $m(s^*)$ and w^* related? From (6) it follows that

$$\begin{aligned}
\bar{W}_p(s) &= \int_W V_p(w)\,dF(w|s)\\
&= \int_0^{w^*}\beta\bar{V}_u\,dF(w|s) + \int_{w^*}^{\infty}\frac{w}{1-\beta}\,dF(w|s)\\
&= \beta\bar{V}_u F(w^*|s) + \frac{1}{1-\beta}\int_{w^*}^{\infty} w\,dF(w|s).
\end{aligned}$$

We can use this to rewrite (7) as

$$\bar{V}_u(s) = \begin{cases} \beta\bar{V}_u & s \leqslant s^* \\ m(s) + \beta\left[\beta\bar{V}_u F(w^*|s) + \frac{1}{1-\beta}\int_{w^*}^{\infty} w\,dF(w|s)\right] & s \geqslant s^* \end{cases} \tag{12}$$

Using (5), i.e., $\beta\bar{V}_u = \dfrac{w^*}{1-\beta}$, we get

$$V_u(s) = \begin{cases} \dfrac{w^*}{1-\beta} & s \leqslant s^* \\ m(s) + \frac{\beta}{1-\beta}w^*F(w^*|s) + \frac{\beta}{1-\beta}\int_{w^*}^{\infty} w\,dF(w|s) & s \geqslant s^*. \end{cases}$$

Evaluating this at $s = s^*$, we conclude

$$\frac{w^*}{1-\beta} = m(s^*) + \frac{\beta}{1-\beta} w^* F(w^*|s^*) + \frac{\beta}{1-\beta} \int_{w^*}^{\infty} w \, dF(w|s^*).$$

Using our tricks that $F = 1 - (1 - F)$ and $1 - F(x) = \int_x^\infty 1 \, dF$, we can rearrange terms to get

$$\begin{aligned}
\frac{w^*}{1-\beta} &= m(s^*) + \frac{\beta}{1-\beta} w^* F(w^*|s^*) + \frac{\beta}{1-\beta} \int_{w^*}^{\infty} w \, dF(w|s^*) \\
&= m(s^*) + \frac{\beta}{1-\beta} w^* \Big(1 - (1 - F(w^*|s^*))\Big) + \frac{\beta}{1-\beta} \int_{w^*}^{\infty} w \, dF(w|s^*) \\
&= m(s^*) + \frac{\beta}{1-\beta} w^* - \frac{\beta}{1-\beta} w^* (1 - F(w^*|s^*))\Big) + \frac{\beta}{1-\beta} \int_{w^*}^{\infty} w \, dF(w|s^*) \\
&= m(s^*) + \frac{\beta}{1-\beta} w^* - \frac{\beta}{1-\beta} \int_{w^*}^{\infty} w^* \, dF(w|s^*) + \frac{\beta}{1-\beta} \int_{w^*}^{\infty} w \, dF(w|s^*) \\
&= m(s^*) + \frac{\beta}{1-\beta} w^* + \frac{\beta}{1-\beta} \int_{w^*}^{\infty} (w - w^*) \, dF(w|s^*).
\end{aligned}$$

Or in other words,

$$w^* - m(s^*) = \frac{\beta}{1-\beta} \int_{w^*}^{\infty} (w - w^*) \, dF(w|s^*). \tag{13}$$

Since the right hand side of this positive, we conclude

$$w^* > m(s^*).$$

That is, the wage needed to induce a worker to remain with the firm is larger than the wage needed to attract her on a probationary basis. This is true even though quitting means sitting out a period.

8 The Probability of Quitting

The probability of quitting is just the probability that the realized wage w is less than the reservation wage w^* conditional on s, the signal. Thus the probability of quitting depends on s, or equivalently on $m(s)$, the probationary salary, since m is an increasing function of s. This probability is

$$\int_0^{w^*} 1 \, dF(w|s)$$

or equivalently,

$$\int_W I_{[0,w^*]}(w) \, dF(w|s),$$

where $I_{[0,w^*]}$ is the indicator function of $[0, w^*]$. As such it is a nonincreasing function of w. Since an increase in s stochastically increases $F(\cdot|s)$ (in the sense of first order stochastic dominance), an increase in s leads to a decrease in the probability of quitting. Since probationary wage offers are an increasing function of s, quit rates are negatively correlated with probationary (starting) salaries in this model.

References

[1] Jovanovic, B. 1979. Job matching and the theory of turnover. *Journal of Political Economy* 87(5):972–990.

[2] Lehmann, E. L. 1959. *Testing statistical hypotheses*. Wiley Series in Probability and Mathematical Statistics. New York: John Wiley and Sons.

[3] Prescott, E. C. and R. M. Townsend. 1980. Equilibrium under uncertainty: Multi-agent statistical decision theory. In *Bayesian Analysis in Econometrics and Statistics*, pages 169–194. North Holland, Amsterdam.

[4] Sargent, T. J. 1987. *Dynamic macroeconomic theory*. Cambridge, Massachusetts: Harvard University Press.