

## Investment and Risk Aversion

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Fall 2003

There are two assets, a safe asset that returns  $(1 + r_0)$  for each dollar invested and risky asset that returns  $(1 + \mathbf{r})$  for each dollar invested, where  $\mathbf{r}$  is a nondegenerate random variable.

If his wealth is  $\hat{w}$ , an expected utility maximizing investor will choose the amount  $x$  to invest in the risky asset to maximize

$$E u((\hat{w} - x)(1 + r_0) + x(1 + \mathbf{r})).$$

The difference  $\mathbf{r} - r_0$  is the excess of  $\mathbf{r}$  over the safe return, so for convenience, let us call it  $\mathbf{q}$ , i.e.,  $\mathbf{q} = \mathbf{r} - r_0$ , and set  $w = (1 + r_0)\hat{w}$ . Thus  $x$  is chosen to maximize

$$E u(w + x\mathbf{q}),$$

which is a prettier problem.

There are some questions that are frequently glossed over in the literature. One is whether we want to restrict  $x$  to lie in the interval  $[0, w]$ . If so, we have to worry about boundary conditions. We also have to worry whether  $w + x\mathbf{q}$  lies in the domain of the utility function with probability one. For instance, a utility function that is commonly studied is the logarithmic utility  $u(w) = \ln w$  (where  $u(0) = -\infty$  is allowed). If we make the limited liability assumption that  $1 + \mathbf{r} \geq 0$  a.s., and also restrict  $x$  to  $[0, w]$ , then we have no problems in that regard. On the other hand, we may actually want to allow borrowing ( $x > w$ ) and/or short selling ( $x < 0$ ). In that case, we probably need to have the utility defined on the whole real line, which rules out the logarithmic utility, among others.

In what follows, I shall assume that utilities are defined on an interval  $D$  of the real line, are continuous strictly increasing functions on  $D$  that are twice continuously differentiable, with strictly positive derivatives everywhere on the interior of  $D$ , and that a solution exists and is interior to the domain.

The first order necessary condition for an interior maximum is

$$E u'(w + x^*\mathbf{q})\mathbf{q} = 0. \tag{*}$$

Observe that  $(\star)$  has a solution only if  $\mathbf{q} < 0$  with positive probability, which makes perfect economic sense. (Otherwise there would be an arbitrage opportunity: borrow at  $r_0$  and invest at  $\mathbf{r}$ , earning a riskless profit.)

The second order necessary condition is

$$\mathbf{E} u''(w + x^* \mathbf{q}) \mathbf{q}^2 \leq 0.$$

If  $u$  is concave, then  $u'' \leq 0$ , so the second order condition is automatically satisfied. I may also assume that the strong second order condition

$$\mathbf{E} u''(w + x^* \mathbf{q}) \mathbf{q}^2 < 0$$

holds at a particular solution. This is usually necessary to make the solution a differentiable function of the parameters.

## 1 A trivial lemma

**Lemma 1** *Let  $f$  be a nondecreasing real function on an interval  $I$ , let  $x$  belong to  $I$ , and let  $\alpha > 0$ . Then for any  $v$  for which  $x + \alpha v \in I$ , we have*

$$f(x + \alpha v)v \geq f(x)v.$$

*This equality is reversed if  $\alpha < 0$  or if  $f$  is nonincreasing. The inequality is strict provided  $v \neq 0$  and  $f$  is not constant on the interval from  $x$  to  $x + \alpha v$ .*

*Proof:* We prove the claim for  $\alpha > 0$ , the others are obvious from its proof. There are two interesting cases:  $v > 0$  and  $v < 0$ . When  $v > 0$ , then the monotonicity of  $f$  implies  $f(x + \alpha v) \geq f(x)$ , so  $f(x + \alpha v)v \geq f(x)v$ . And when  $v < 0$ , then  $f(x + \alpha v) \leq f(x)$ , but multiplying by the negative quantity  $v$  reverses the inequality, so again  $f(x + \alpha v)v \geq f(x)v$ . ■

## 2 Decreasing risk aversion

A natural comparative statics question is: What happens to  $x^*$  as a function of  $w$ ?

**Proposition 2** *Assume  $u$  is  $C^2$  and  $u' > 0$ , and define the Arrow-Pratt coefficient of risk aversion  $r(w) = \frac{-u''(w)}{u'(w)}$ . Fix  $w_0$ , and assume that  $x_0^*$*

satisfies the strong second order condition. Then there is a neighborhood of  $w_0$  on which  $x^*$  is a  $C^1$  function of  $w$ .

Moreover, if  $r$  is decreasing at  $w_0$ , then  $x^*$  is increasing at  $w_0$  if  $x_0^*$  is positive and decreasing if  $x_0^*$  is negative. If, on the other hand,  $r$  is increasing at  $w_0$ , then  $x^*(w)$  is decreasing when  $x_0^*$  is positive and increasing when  $x_0^*$  is negative.

*Proof:* Now  $x_0^*$  satisfies the first order condition

$$\mathbf{E} u'(w_0 + x_0^* \mathbf{q}) \mathbf{q} = 0.$$

By the strong second order condition, the Implicit Function Theorem implies that  $x^*$  is a  $C^1$  function of  $w$  on an appropriate neighborhood of  $w_0$ . Thus differentiating the first order condition with respect to  $w$  gives

$$\mathbf{E} u''(w_0 + x_0^* \mathbf{q}) \mathbf{q} \left( 1 + \mathbf{q} \frac{dx^*(w_0)}{dw} \right) = 0$$

or

$$\frac{dx^*(w_0)}{dw} = - \frac{\mathbf{E} u''(w_0 + x_0^* \mathbf{q}) \mathbf{q}}{\mathbf{E} u''(w_0 + x_0^* \mathbf{q}) \mathbf{q}^2}.$$

The strong second order condition implies that the denominator is negative so the sign of  $\frac{dx^*(w_0)}{dw}$  is the sign of  $\mathbf{E} u''(w_0 + x_0^* \mathbf{q}) \mathbf{q}$ .

Now suppose  $r(w)$  is decreasing at  $w_0$ . Consider first the case  $x_0^* > 0$ . By Lemma 1,

$$r(w_0 + x_0^* \mathbf{q}) \mathbf{q} \leq r(w_0) \mathbf{q}.$$

Therefore, recalling the definition of  $r$  and multiplying by the negative quantity  $-u'(w_0 + x_0^* \mathbf{q})$ , we have

$$u''(w_0 + x_0^* \mathbf{q}) \mathbf{q} \geq -r(w_0) u'(w_0 + x_0^* \mathbf{q}) \mathbf{q}.$$

Taking the expectation of both sides gives

$$\mathbf{E} u''(w_0 + x_0^* \mathbf{q}) \mathbf{q} \geq -r(w_0) \mathbf{E} u'(w_0 + x_0^* \mathbf{q}) \mathbf{q} = 0$$

where the equality follows from the first order condition ( $\star$ ). Thus

$$\text{sign} \frac{dx^*(w_0)}{dw} = \text{sign} \mathbf{E} u''(w_0 + x_0^* \mathbf{q}) \mathbf{q} \geq 0$$

when  $r$  is decreasing at  $w_0$ . Similarly,  $\frac{dx^*(w_0)}{dw} \leq 0$  when  $r$  is increasing at  $w_0$ .

These conclusions are reversed if  $x_0^* < 0$ . ■

### 3 What if $u$ is more risk averse than $v$ ?

**Proposition 3** *Assume  $u$  is more risk averse than  $v$ . If  $v$  is risk averse or the two preferences are “sufficiently close” (in a sense to be made precise in the proof), then*

$$0 \leq x_u^* \leq x_v^* \quad \text{or} \quad x_v^* \leq x_u^* \leq 0.$$

*That is, the more risk averse utility adopts the more conservative portfolio.*

*Proof:* We prove only the case  $x_u^* \geq 0$ . The other follows *mutatis mutandis*. Write  $u = G \circ v$ , where  $G$  is strictly increasing and concave. Then  $(\star)$  becomes

$$\mathbf{E} G'(v(w + x_u^* \mathbf{q})) v'(w + x_u^* \mathbf{q}) \mathbf{q} = 0.$$

Since  $G$  is concave,  $G'$  is nonincreasing, and thus so is  $G' \circ v$ . By Lemma 1,

$$G'(v(w + x_u^* \mathbf{q})) \mathbf{q} \leq G'(v(w)) \mathbf{q}.$$

Since  $v' > 0$ , we have

$$G'(v(w + x_u^* \mathbf{q})) v'(w + x_u^* \mathbf{q}) \mathbf{q} \leq G'(v(w)) v'(w + x_u^* \mathbf{q}) \mathbf{q},$$

and taking expectations yields

$$\underbrace{\mathbf{E} G'(v(w + x_u^* \mathbf{q})) v'(w + x_u^* \mathbf{q}) \mathbf{q}}_{=0 \text{ by } (\star)} \leq G'(v(w)) \mathbf{E} v'(w + x_u^* \mathbf{q}) \mathbf{q}.$$

That is,

$$\mathbf{E} v'(w + x_u^* \mathbf{q}) \mathbf{q} \geq 0.$$

But the first order condition for  $x_v^*$  is

$$\mathbf{E} v'(w + x_v^* \mathbf{q}) \mathbf{q} = 0.$$

Now set  $h(x) = \mathbf{E} v'(w + x \mathbf{q}) \mathbf{q}$ . Then  $h(x_u^*) \geq 0 = h(x_v^*)$ . But  $h'(x_v^*) = \mathbf{E} v''(w + x_v^* \mathbf{q}) \mathbf{q}^2 \leq 0$  by the second order condition for  $x_v^*$ . If  $u$  and  $v$  are close enough so that  $h'(x) \leq 0$  on the interval between  $x_v^*$  and  $x_u^*$ , then  $x_u^* \leq x_v^*$ . (If  $v$  is concave, then  $h' \leq 0$  and no closeness assumption is needed.) ■

## References

- [1] Hildreth, C. 1974. Expected utility of uncertain ventures. *Journal of the American Statistical Association* 69(345):9–17.