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Private Information and Trade: An Introduction

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Private information is information possessed by an agent that is either unknowable or unverifiable by outsiders. A good example is one's own preference relation. By its very nature, an agent is a monopolist with respect to his private information, and this can lead to a different kind of inefficiency.

1 Bilateral price bargaining

Consider the problem of trading a discrete object. The **seller** initially owns the object and values it at s. The **buyer** has value b. These values are not knowable to the other agent. If the buyer's value exceeds the seller's value, $b > s$, then they can trade at an intermediate price p and both will be better off. A mechanism for trading must determine whether to trade and at what price. Moreover it can rely only on information provided (perhaps indirectly) by the buyer and seller.

One particularly useful approach to modeling mechanisms like these was proposed by John Harsanyi [4]. His approach is called the **Bayesian game** model. In this model the buyer and seller act as if nature moves first and assigns the buyer and seller their values by drawing them from known probability distributions. A **strategy** is then a function from each trader's information (his value) to an action, depending on the rules of exchange. (Think of nature as dealing them a hand from a deck of cards. Traders should be prepared to act on whatever hand they possess.) A **Bayes–Nash equilibrium** is simply a Nash equilibrium of a Bayesian game.

We shall refer to actions in this game as **bids**. But a strategy is a **bidding function**. The bidding functions will be denoted $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$, where $\boldsymbol{\beta}(b)$ is the bid made by a buyer with value b and $\boldsymbol{\sigma}(s)$ is the bid made by a seller with value s.

There are some interpretational problems with this model. Why should a given buyer who knows his value formulate a plan for what to do in the event he has a different value? Much ink has been spilled over his question, but I will ignore it for now.

For concreteness we shall assume that the buyer believes that nature chooses s from a cumulative distribution function G, and the seller believes b is drawn from distribution F. That is, the buyer believes that $\text{Prob}\,[s \leqslant t] = G(t)$ and the seller believes that $\text{Prob}\,[b \leqslant t] = F(t)$. To simplify things, let's suppose that the distributions F and G have common support $[0, 1]$, satisfy $F(0) = G(0) = 0$, $F(1) = G(1) = 1$, and are continuously differentiable. This ensures that they have densities, so calculations are simpler. Let's also assume that the buyer and seller are risk neutral, and that prices and values are commensurable, and that b and s are stochastically independent.

2 Split-the-difference pricing

We now analyze one particular **pricing mechanism** for trading, namely the sealed bid auction with split-the-difference payoffs. This may have first been analyzed by Chatterjee and Samuelson [1]. (This exposition owes a lot to Samuelson [6]. A more detailed treatment may be found in Gibbons [3, Chapter 3].) In this mechanism, each trader submits a written **bid**. If the buyer's bid β is higher than the seller's bid σ, then the trade is made at a price $p = \frac{\beta+\sigma}{2}$. Otherwise, the traders leave, never to return. (It is important that both leave, otherwise the information that was revealed by their bids may induce them to want to trade. Allowing this to happen will affect their bids.)

We are interested in characterizing the equilibrium bidding functions β and σ, where $\beta(b)$ is the bid made by a buyer with value b and $\sigma(s)$ is the bid made by a seller with value s.

2.1 Payoffs

The payoff to the buyer with value b who makes no payment and does not acquire the object is zero, while if he acquires the object at a price p, his payoff is $b - p$. Similarly if the seller retains the object and collects no payment, his payoff is s, while if he surrenders the object for price p his payoff is simply p.

From the buyer's point of view, if he has value b and makes bid β, then his expected payoff is

$$\pi^B(b,\beta) = 0 \times \text{Prob}\,[\sigma > \beta] + \int_0^\beta \left(b - \frac{\beta+\sigma}{2} \right) (\text{density of } \sigma)\, d\sigma.$$

The seller's expected payoff from biddinng σ, when his values is s is

$$\pi^S(s,\sigma) = s \times \text{Prob}\,[\sigma > \beta] + \int_\sigma^\infty \left(\frac{\beta+\sigma}{2} \right) (\text{density of } \beta)\, d\beta.$$

2.2 Density of bids

Now what are the densities of σ and β? Suppose the seller uses bidding function $\boldsymbol{\sigma}$, and assume that $\boldsymbol{\sigma}$ is an increasing differentiable function. (This seems reasonable, and indeed we shall find an equilibrium in which $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$ are increasing differentiable functions, but there may be "pathological" equilibria.) Then for any t, the probability that $\sigma \leqslant t$ is just $G(\boldsymbol{\sigma}^{-1}(t))$. So let φ denote $\boldsymbol{\sigma}^{-1}$.

$$\varphi = \boldsymbol{\sigma}^{-1}$$

Then

$$\operatorname{Prob}[\sigma \leqslant t] = \operatorname{Prob}[\boldsymbol{\sigma}(s) \leqslant t] = \operatorname{Prob}[s \leqslant \boldsymbol{\sigma}^{-1}(t)] = G\big(\boldsymbol{\sigma}^{-1}(t)\big) = G\big(\varphi(t)\big).$$

So the density of the seller's bids σ is

$$\frac{d}{dt}\operatorname{Prob}[\sigma \leqslant t] = G'\big(\varphi(t)\big)\varphi'(t).$$

From the seller's point of view, the cumulative distribution of buyer's bids is

$$\operatorname{Prob}[\beta \leqslant t] = F\big(\boldsymbol{\beta}^{-1}(t)\big) = F\big(\psi(t)\big),$$

where $\boldsymbol{\beta}$ is the buyer's bidding function and

$$\psi = \boldsymbol{\beta}^{-1},$$

(assuming β is increasing). This has density

$$F'\big(\psi(t)\big)\psi'(t).$$

2.3 Buyer's optimization problem

Thus the buyer's expected payoff from bidding β when his value is b is

$$\int_0^\beta \left(b - \frac{\beta+\sigma}{2}\right) G'\big(\varphi(\sigma)\big)\varphi'(\sigma)\,d\sigma.$$

Now a wise buyer will choose β to maximize this payoff. The first order condition for this is:

$$(b-\beta)G'\big(\varphi(\beta)\big)\varphi'(\beta) + \int_0^\beta \left(-\tfrac{1}{2}\right) G'\big(\varphi(\sigma)\big)\varphi'(\sigma)\,d\sigma = 0.$$

(See my notes on differentiating an integral.) We can rewrite this as

$$(b-\beta)G'\big(\varphi(\beta)\big)\varphi'(\beta) - \tfrac{1}{2}G\big(\varphi(\beta)\big) = 0. \tag{1}$$

2.4 Seller's optimization problem

The seller wants to choose σ to maximize his expected payoff. Recalling that he keeps the object if $\beta < \sigma$, which happens with probability $F\big(\psi(\sigma)\big)$. Thus the expected payoff from bidding σ is

$$sF\big(\psi(\sigma)\big) + \int_\sigma^1 \frac{\beta+\sigma}{2} F'(\psi(\beta))\psi'(\beta)\,d\beta.$$

The first order condition for a maximum with respect to σ is

$$sF'\big(\psi(\sigma)\big)\psi'(\sigma) - \sigma F'\big(\psi(\sigma)\big)\psi'(\sigma) + \int_\sigma^1 \tfrac{1}{2}F'\big(\psi(\beta)\big)\psi'(\beta)\,d\beta = 0$$

or

$$(s-\sigma)F'\big(\psi(\sigma)\big)\psi'(\sigma) + \tfrac{1}{2}\left[1 - F\big(\psi(\sigma)\big)\right] = 0. \tag{2}$$

2.5 Equilibrium

Now we impose the following (Bayes-Nash) equilibrium condition. In (1), $b = \psi(\beta)$ and in (2), $s = \varphi(\sigma)$. That is, the bidding functions are optimal, given the other trader's bidding function. Thus we can rewrite the first-order conditions as

$$\begin{aligned}\big(\psi(\beta) - \beta\big) G'(\varphi(\beta))\varphi'(\beta) - \tfrac{1}{2}G(\varphi(\beta)) &= 0\\ \big(\varphi(\sigma) - \sigma\big) F'(\psi(\sigma))\psi'(\sigma) - \tfrac{1}{2}F(\psi(\sigma)) &= -\tfrac{1}{2}.\end{aligned}$$

Since this must be true for every β and σ we actually have a pair of differential equations that characterize the equilibrium. In general this system is hard to solve, and I don't know how to do it.

2.6 Solution for a particular distribution

But for the special case of the uniform distribution $F(t) = G(t) = t$ on $[0, 1]$, (using the more traditional x for the dummy variable) the system of differential equations reduces to

$$\big(\psi(x) - x\big)\varphi'(x) - \tfrac{1}{2}\varphi(x) = 0$$
$$\big(\varphi(x) - x\big)\psi'(x) - \tfrac{1}{2}\psi(x) = -\tfrac{1}{2}.$$

Since this must be true for all x, inspection suggests that φ, ψ should be linear. (Otherwise, act like a physicist and replace ψ, φ by Taylor's series.)

So let

$$\begin{aligned}\psi(x) &= \alpha x + \delta \qquad \alpha > 0\\ \varphi(x) &= \gamma x + \eta \qquad \gamma > 0.\end{aligned}$$

Thus we must have

$$\begin{aligned}(\alpha x + \delta - x)\gamma - \tfrac{1}{2}(\gamma x + \eta) &= 0\\ (\gamma x + \eta - x)\alpha - \tfrac{1}{2}(\alpha x + \delta) &= -\tfrac{1}{2}.\end{aligned}$$

Regrouping,

$$\begin{aligned}(\alpha\gamma - \tfrac{3}{2}\gamma)x + (\delta\gamma - \tfrac{1}{2}\eta) &= 0\\ (\alpha\gamma - \tfrac{3}{2}\alpha)x + (\alpha\eta - \tfrac{1}{2}\delta + \tfrac{1}{2}) &= 0.\end{aligned}$$

Since this must hold for all x, the four coefficients must all be zero. This gives us four linear equations in the four unknowns $\alpha, \gamma, \delta, \eta$. The solution is:

$$\begin{aligned}\alpha &= \tfrac{3}{2}\\ \delta &= -\tfrac{1}{8}\\ \gamma &= \tfrac{3}{2}\\ \eta &= -\tfrac{3}{8}.\end{aligned}$$

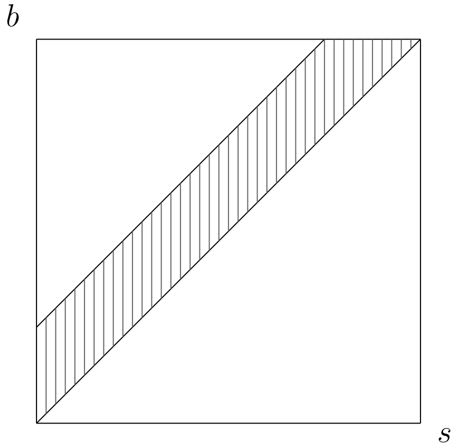

Figure 1. $s < b \leqslant s + \frac{1}{4}$

Recalling that $\varphi = \boldsymbol{\sigma}^{-1}$ and $\psi = \boldsymbol{\beta}^{-1}$ we have

$$b = \tfrac{3}{2}\beta - \tfrac{1}{8} \quad \text{so} \quad \boldsymbol{\beta}(b) = \tfrac{2}{3}b + \tfrac{1}{12}$$

$$s = \tfrac{3}{2}\sigma - \tfrac{3}{8} \quad \text{so} \quad \boldsymbol{\sigma}(s) = \tfrac{2}{3}s + \tfrac{1}{4}.$$

2.7 Efficiency of split-the-difference

Now is this efficient? Efficiency demands that trade should take place whenever $b > s$. Trade actually takes place whenever $\boldsymbol{\beta}(b) > \boldsymbol{\sigma}(s)$ or

$$\tfrac{2}{3}b + \tfrac{1}{12} > \tfrac{2}{3}s + \tfrac{1}{4}$$

or

$$b > s + \tfrac{1}{4}.$$

Thus the outcome of the mechanism is inefficient whenever $s < b \leqslant s + \frac{1}{4}$. See Figure 1. This happens with probability $\frac{7}{32}$.

3 A More General Result

The fact that the split-the-difference rule is inefficient does not mean that there is no other (cleverer) rule that is efficient. Nevertheless there isn't, as was shown by Myerson and Satterthwaite [5]. To see why, we first need to make an important observation.

3.1 The revelation principle

A **mechanism** for trading sets up a Bayesian game by specifying an action set for each trader and an outcome function. Let B be the buyer's set of actions. (In the split-the-difference rule, B is just the set of possible bids.) Let S denote the seller's action set. The mechanism specifies two functions:

$$\begin{aligned} t(\beta,\sigma) &= \begin{cases} 1 & \text{if buyer gets object} \\ 0 & \text{if seller keeps object} \end{cases} \\ p(\beta,\sigma) &= \text{payment from buyer to seller,} \end{aligned}$$

where $\beta \in B$ is the buyer's chosen action and $\sigma \in S$ is the seller's. Now let $\boldsymbol{\beta}^* \colon b \mapsto \boldsymbol{\beta}^*(b)$ denote the buyer's Bayes–Nash equilibrium strategy. That is, a buyer with value b chooses action $\boldsymbol{\beta}^*(b)$ in equilibrium. Similarly, $\boldsymbol{\sigma}^* \colon s \mapsto \boldsymbol{\sigma}^*(s)$ is the seller's equilibrium strategy. When the values are b and s, the buyer acquires the object if and only if $t\bigl(\boldsymbol{\beta}^*(b), \boldsymbol{\sigma}^*(s)\bigr) = 1$ and pays $p\bigl(\boldsymbol{\beta}^*(b), \boldsymbol{\sigma}^*(s)\bigr)$.

Consider another mechanism where $\hat{B}$ is the set of buyer's values and $\hat{S}$ is the set of seller's values. Define the outcome functions by

$$\begin{aligned} \hat{t}(b,s) &= t\bigl(\boldsymbol{\beta}^*(b), \boldsymbol{\sigma}^*(s)\bigr) \\ \hat{p}(b,s) &= p\bigl(\boldsymbol{\beta}^*(b), \boldsymbol{\sigma}^*(s)\bigr). \end{aligned}$$

It follows from the definition of equilibrium that an equilibrium strategy for the buyer is to choose action b when his value is b, and for the seller to choose s when his value is s. In other words, the first mechanism is equivalent to the a mechanism in which actions correspond to values and the equilibrium action choice is to choose the true value. This observation is known as the **revelation principle**

3.2 Revelation mechanisms

Because of the revelation principle the only mechanisms that need to be considered are the **incentive compatible direct revelation mechanisms**. That is, mechanisms where strategies are values, and the equilibrium bidding functions are truthful, that is, $\boldsymbol{\beta}(b) = b$ and $\boldsymbol{\sigma}(s) = s$. Note that these bidding functions are strictly increasing and continuously differentiable.

For the sake of concreteness again take $B = S = [0,1]$, and assume

$$F(0) = G(0) = 0 \qquad \text{and} \qquad F(1) = G(1) = 1.$$

Let

$$\pi^B(b,\beta) = \int_0^1 \bigl[b \cdot t(\beta,s) - p(\beta,s)\bigr] G'(s)\,ds$$

be the buyer's expected payoff when his value is b and he bids β, and the seller bids his true value. Similarly let

$$\pi^S(s,\sigma) = \int_0^1 \big[\big(1 - t(b,\sigma)\big)s + p(b,\sigma)\big] F'(b)\,db$$

be the seller's expected payoff when his value is s and he bids σ and the buyer bids his true value.

By incentive compatibility, bidding you value is optimal, so

$$\pi^B(b,b) = \max_\beta \pi^B(b,\beta),$$

and denote this value by $V_B(b)$. Likewise

$$\pi^S(s,s) = \max_\sigma \pi^S(s,\sigma) = V_S(s).$$

Define

$$\bar{t}(b) = \int_0^1 t(b,s)G'(s)\,ds, \qquad \text{and} \qquad \bar{p}(b) = \int_0^1 p(b,s)G'(s)\,ds.$$

That is, $\bar{t}(b)$ is the probability that the buyer receives the object when his type is b, and $\bar{p}(b)$ is his expected payment (given that the seller is bidding his value). Then

$$\pi^B(b,\beta) = b\bar{t}(\beta) - \bar{p}(\beta).$$

So incentive compatibility for the buyer can be written as

$$V_B(b) = \pi^B(b,b) = b\bar{t}(b) - \bar{p}(b) \geqq b\bar{t}(b') - \bar{p}(b') = b\bar{t}(b') - [b'\bar{t}(b') - V_B(b')]\,,$$

or

$$V_B(b) \geqslant V_B(b') + \bar{t}(b')(b' - b). \tag{3}$$

This implies in fact

$$V_B(b) = \sup_{b'} V_B(b') + b'\bar{t}(b') - \bar{t}(b')b.$$

Now the function $b \mapsto V_B(b') + b'\bar{t}(b') - \bar{t}(b')b$ is an affine function of b, so V_B is convex, being the pointwise supremum of affine functions. Moreover, by (3), we see that for every point b', $\bar{t}(b')$ is a supergradient of V_B at b'. Thus for all but countably many values of b, the function V_B is differentiable and

$$V'(b) = \bar{t}(b).$$

In particular, since convex functions are integrals of their supergradients, we also know that

$$\begin{aligned} V_B(b) &= V_B(0) + \int_0^b \bar{t}(x)\,dx. \\ &= V_B(0) + \int_0^b \int_0^1 t(x,s)G'(s)\,ds\,dx \end{aligned} \tag{4}$$

A similar argument shows that

$$V_S(s) = V_S(1) - \int_s^1 \int_0^1 t(b,x) F'(b)\, db\, dx. \tag{5}$$

3.3 Inefficiency is inevitable when trade is voluntary

We now show that in general, voluntary participation and efficiency are incompatible. To do this, we need to make sure that the informational problem is nontrivial, in that we are not ex ante sure whether trade should occur. A sufficient condition is the following nondegeneracy condition:

$$\int_0^1 G(t)\big(1 - F(t)\big)\, dt > 0. \tag{6}$$

What this condition does is guarantee that there is a set of positive measure of values of t satisfying $\operatorname{Prob}[s \leqslant t] > 0$ and $\operatorname{Prob}[b > t] > 0$. This implies that there is positive probability that trade is optimal.

Traders will voluntarily participate only if their expected payoff is at least as great as not participating. That is,

$$V_B(b) \geqslant 0 \quad \text{and} \quad V_S(s) \geqslant s \quad \text{for all } b, s.$$

Now let us see what efficiency demands. An efficient outcome demands that

$$t(b,s) = 1 \iff b > s, \tag{7}$$

or

$$t(b,s) = \mathbf{1}_{b>s}.$$

Now by (4), (5), and (7) we have

$$\begin{aligned} V_B(b) &= V_B(0) + \int_0^b \int_0^1 t(x,s) G'(s)\, ds\, dx \\ &= V_B(0) + \int_0^b \int_0^1 \mathbf{1}_{x>s} G'(s)\, ds\, dx \\ &= V_B(0) + \int_0^b G(x)\, dx. \end{aligned}$$

Similarly

$$V_S(s) = V_S(1) - \int_s^1 F(b)\, db$$

Taking expectations,

$$EV_B = V_B(0) + \int_0^1 \int_0^b G(s)\, ds\, dF(b) \quad \text{and} \quad EV_S = V_S(1) - \int_0^1 \int_s^1 F(b)\, db\, dG(s).$$

Integrating by parts,

$$\int_0^1 \left\{ \int_0^b G(s)\,ds \right\} dF(b) = \int_0^1 G(s)\,ds - \int_0^1 G(b)F(b)\,db = \int_0^1 G(x)\bigl(1 - F(x)\bigr)\,dx$$

$$\int_0^1 \left\{ \int_s^1 F(b)\,db \right\} dG(s) = 0 + \int_0^1 G(x)F(x)\,dx.$$

Therefore

$$EV_B + EV_S = \left\{ V_B(0) + \int_0^1 G(x)\bigl(1 - F(x)\bigr)\,dx \right\} + \left\{ V_S(1) - \int_0^1 G(x)F(x)\,dx \right\}. \tag{8}$$

Now consider what happens when the buyer's value is b and the seller's value is s. The sum of the ex post payoffs is

$$b\mathbf{1}_{b>s} - p(b,s) + s(1 - \mathbf{1}_{b>s}) + p(b,s) = \max\{b,s\}.$$

Taking expectations with respect to both b and s,

$$\begin{aligned}
E(V_B + V_S) &= E\max\{b,s\} \\
&= \int_0^1 \int_0^1 \max\{b,s\} F'(b)G'(s)\,db\,ds \\
&= \int_0^1 \int_0^b bF'(b)G'(s)\,ds\,db + \int_0^1 \int_0^s sF'(b)G'(s)\,db\,ds \\
&= \int_0^1 bF'(b)G(b)\,db + \int_0^1 sF(s)G'(s)\,ds \\
&= \int_0^1 x\bigl[F'(x)G(x) + G'(x)F(x)\bigr]\,dx \\
&= xF(x)G(x)\Big|_0^1 - \int_0^1 F(x)G(x)\,dx \\
&= 1 - \int_0^1 F(x)G(x)\,dx \qquad (9)
\end{aligned}$$

Equating the two expressions (8) and (9) for $EV_B + EV_S$ gives

$$V_B(0) + \int_0^1 G(x)\bigl(1 - F(x)\bigr)\,dx + V_S(1) - \int_0^1 G(x)F(x)\,dx = 1 - \int_0^1 F(x)G(x)\,dx$$

so

$$V_B(0) + V_S(1) = 1 - \int_0^1 G(x)\bigl(1 - F(x)\bigr)\,dx$$

but by the voluntary participation constraints, $V_S(1) \geqslant 1$ and $V_B(0) \geqslant 0$, so

$$1 - \int_0^1 G(x)\bigl(1 - F(x)\bigr)\,dx \geqslant 1,$$

which contradicts nondegeneracy (6).

Thus voluntary participation and efficiency are inconsistent. See [2, § 7.4.4, pp. 275–278] for a similar argument in a slightly different context.

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