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Comparative Statics, the Old-Fashioned Way

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"Comparative statics" analysis tells us how equilibrium values of endogenous variables $x_1, \ldots, x_n$ (the things we want to solve for) change as a function of the exogenous parameters $p_1, \ldots, p_m$. (As such it is hardly unique to economics.) Typically we can write the equilibrium conditions of our model as the zero of a system of equations in the endogenous variables and the exogenous parameters:

$$\begin{aligned} F^1(x_1, \ldots, x_n; p_1, \ldots, p_m) &= 0 \\ &\vdots \\ F^n(x_1, \ldots, x_n; p_1, \ldots, p_m) &= 0 \end{aligned} \tag{1}$$

This implicitly defines x as a function of p, which we will explicitly denote $x = \xi(p)$, or

$$(x_1, \ldots, x_n) = \big(\xi^1(p_1, \ldots, p_m), \ldots, \xi^n(p_1, \ldots, p_m)\big).$$

This explicit function, if it exists, satisfies the implicit definition

$$F\big(\xi(p); p\big) = 0 \tag{2}$$

for at least a rectangle of values of p. The Implicit Function Theorem (see, e.g., [1, Theorem 7-6, p. 147] or [3, Theorem 9.28, p. 224]) tells us when such an explicit function exists. (Basically, an explicit function exists whenever it is possible to solve for all its partial derivatives.)

Setting $G(p) = F\big(\xi(p); p\big)$, and differentiating G^i with respect to p_j, yields, by equation (2),

$$\sum_k \frac{\partial F^i}{\partial x_k} \frac{\partial \xi^k}{\partial p_j} + \frac{\partial F^i}{\partial p_j} = 0 \tag{3}$$

for each $i = 1, \ldots, n$, $j = 1, \ldots, m$. In matrix terms we have

$$\begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi^1}{\partial p_1} & \cdots & \frac{\partial \xi^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial \xi^n}{\partial p_1} & \cdots & \frac{\partial \xi^n}{\partial p_m} \end{bmatrix} + \begin{bmatrix} \frac{\partial F^1}{\partial p_1} & \cdots & \frac{\partial F^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial p_1} & \cdots & \frac{\partial F^n}{\partial p_m} \end{bmatrix} = 0. \tag{4}$$

Provided $\begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{bmatrix}$ has an inverse (the hypothesis of the Implicit Function Theorem) we can solve this:

$$\begin{bmatrix} \frac{\partial \xi^1}{\partial p_1} & \cdots & \frac{\partial \xi^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial \xi^n}{\partial p_1} & \cdots & \frac{\partial \xi^n}{\partial p_m} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial F^1}{\partial p_1} & \cdots & \frac{\partial F^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial p_1} & \cdots & \frac{\partial F^n}{\partial p_m} \end{bmatrix} \tag{5}$$

The old-fashioned derivation (see, e.g., Samuelson's *Foundations* [4, pp. 10–14]) of this same result runs like this: "Totally differentiate" the ith row of equation (1) to get

$$\sum_k \frac{\partial F^i}{\partial x_k}\,dx_k + \sum_\ell \frac{\partial F^i}{\partial p_\ell}\,dp_\ell = 0 \tag{6}$$

for all i. Now set all dp_ℓ's equal to zero except p_j, and divide by dp_j to get

$$\sum_k \frac{\partial F^i}{\partial x_k}\frac{dx_k}{dp_j} + \frac{\partial F^i}{\partial p_j} = 0 \tag{7}$$

for all i and j, which is the same as equation (3). For further information on total differentials and how to manipulate them, see [1, Chapter 6].

Using Cramer's Rule (e.g. [2, pp. 93–94]), we see then that

$$\frac{dx_i}{dp_j} = \frac{\partial \xi^i}{\partial p_j} = -\frac{\begin{vmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_{i-1}} & \frac{\partial F^1}{\partial p_j} & \frac{\partial F^1}{\partial x_{i+1}} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_{i-1}} & \frac{\partial F^n}{\partial p_j} & \frac{\partial F^n}{\partial x_{i+1}} & \cdots & \frac{\partial F^n}{\partial x_n} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{vmatrix}}. \tag{8}$$

Or, letting Δ denote the determinant of $\begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{bmatrix}$, and letting $\Delta_{i,j}$ denote the determinant of the matrix formed by deleting its i-th row and j-th column, we have

$$\frac{\partial \xi^i}{\partial p_j} = -\sum_{k=1}^n (-1)^{i+k}\frac{\partial F^k}{\partial p_j}\frac{\Delta_{k,i}}{\Delta}. \tag{9}$$

Lagrangeans

In this section we consider the special case where the equilibrium conditions are the first order conditions of a maximization problem. Specifically we consider the problem of maximizing $f(x;p)$ with respect to x subject to the constraint $g(x;p) = 0$. If the assumptions of the Lagrange Multiplier Theorem are satisfied, then the first order conditions are

$$\begin{aligned} \frac{\partial f(x_1,\ldots,x_n;p_1,\ldots,p_m)}{\partial x_1} + \lambda\frac{\partial g(x_1,\ldots,x_n;p_1,\ldots,p_m)}{\partial x_1} &= 0 \\ &\vdots \\ \frac{\partial g(x_1,\ldots,x_n;p_1,\ldots,p_m)}{\partial x_n} + \lambda\frac{\partial g(x_1,\ldots,x_n;p_1,\ldots,p_m)}{\partial x_n} &= 0 \\ g(x_1,\ldots,x_n;p_1,\ldots,p_m) &= 0. \end{aligned} \tag{10}$$

We shall assume that the strong second order conditions are satisfied, that is,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} + \lambda \frac{\partial^2 g}{\partial x_i \partial x_j}\right) v_i v_j < 0 \tag{11}$$

for every $v = (v_1, \ldots, v_i) \neq 0$ satisfying

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x_i} vi = 0. \tag{12}$$

(The necessary condition is that (11) holds with weak inequality.) The strong second order conditions are equivlaent to the following condition on the NW principal minors of the bordered Hessian.

$$(-1)^k \begin{vmatrix} \dfrac{\partial^2(f+\lambda g)}{\partial x_1^2} & \cdots & \dfrac{\partial^2(f+\lambda g)}{\partial x_1 \partial x_k} & \dfrac{\partial g}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2(f+\lambda g)}{\partial x_k \partial x_1} & \cdots & \dfrac{\partial^2(f+\lambda g)}{\partial x_k^2} & \dfrac{\partial g}{\partial x_k} \\ \dfrac{\partial g}{\partial x_1} & \cdots & \dfrac{\partial g}{\partial x_k} & 0 \end{vmatrix} > 0, \qquad k = 2, \ldots, n.$$

In particular, for $k = n$, this implies that the matrix

$$\begin{bmatrix} \dfrac{\partial^2(f+\lambda g)}{\partial x_1^2} & \cdots & \dfrac{\partial^2(f+\lambda g)}{\partial x_1 \partial x_n} & \dfrac{\partial g}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2(f+\lambda g)}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2(f+\lambda g)}{\partial x_n^2} & \dfrac{\partial g}{\partial x_n} \\ \dfrac{\partial g}{\partial x_1} & \cdots & \dfrac{\partial g}{\partial x_n} & 0 \end{bmatrix} \tag{13}$$

is nonsingular, and hence invertible.

Now return attention to the system of equations (10). For $i = 1, \ldots, n$ define

$$F^i(x_1, \ldots, x_n, \lambda; p_1, \ldots, p_m) = \frac{\partial (f + \lambda g)(x, p)}{\partial x_i}$$

and define

$$F^{n+1}(x_1, \ldots, x_n, \lambda; p_1, \ldots, p_m) = g(x, p).$$

The first order conditions take the form $F(x, \lambda; p) = 0$.

Now

$$\frac{\partial F^i}{\partial x_j} = \frac{\partial^2(f+\lambda g)}{\partial x_i \partial x_j} \qquad i = 1, \ldots, n,\ j = 1, \ldots, n,$$

$$\frac{\partial F^i}{\partial \lambda} = \frac{\partial g}{\partial x_i} \qquad i = 1, \ldots, n,$$

$$\frac{\partial F^{n+1}}{\partial x_j} = \frac{\partial g}{\partial x_i} \qquad j = 1, \ldots, n,$$

and

$$\frac{\partial F^{n+1}}{\partial \lambda} = 0.$$

In other words,

$$\begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} & \frac{\partial F^1}{\partial \lambda} \\ \vdots & & \vdots & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} & \frac{\partial F^n}{\partial \lambda} \\ \frac{\partial F^{n+1}}{\partial x_1} & \cdots & \frac{\partial F^{n+1}}{\partial x_n} & \frac{\partial F^{n+1}}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 (f+\lambda g)}{\partial x_1^2} & \cdots & \frac{\partial^2 (f+\lambda g)}{\partial x_1 \partial x_n} & \frac{\partial g}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 (f+\lambda g)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 (f+\lambda g)}{\partial x_n^2} & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} & 0 \end{bmatrix}$$

is nonsingular, and hence invertible.

Thus the strong second order conditions imply the Jacobian conditions for the Implicit Function Theorem, which guarantees the differentiability of x and λ as functions of the parameters, and equationmatrixsoln becomes

$$\begin{bmatrix} \frac{\partial x^1}{\partial p_1} & \cdots & \frac{\partial x^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial x^n}{\partial p_1} & \cdots & \frac{\partial x^n}{\partial p_m} \\ \frac{\partial \lambda}{\partial p_1} & \cdots & \frac{\partial \lambda}{\partial p_m} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 (f+\lambda g)}{\partial x_1^2} & \cdots & \frac{\partial^2 (f+\lambda g)}{\partial x_1 \partial x_n} & \frac{\partial g}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 (f+\lambda g)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 (f+\lambda g)}{\partial x_n^2} & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 (f+\lambda g)}{\partial x_1 \partial p_1} & \cdots & \frac{\partial^2 (f+\lambda g)}{\partial x_1 \partial p_m} \\ \vdots & & \vdots \\ \frac{\partial^2 (f+\lambda g)}{\partial x_n \partial p_1} & \cdots & \frac{\partial^2 (f+\lambda g)}{\partial x_n \partial p_m} \\ \frac{\partial g}{\partial p_1} & \cdots & \frac{\partial g}{\partial p_m} \end{bmatrix} \tag{14}$$

Utility maximization subject to a budget constraint

Now apply this to the case where $f(x;pn,m) = u(x)$ and $g(x;p,m) = m - p \cdot x$. Then $\frac{\partial (f+\lambda g)}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i$, so $\frac{\partial^2 (f+\lambda g)}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $\frac{\partial^2 (f+\lambda g)}{\partial x_i \partial p_j} = -\lambda \delta_{ij}$, and $\frac{\partial^2 (f+\lambda g)}{\partial x_i \partial m} = 0$; $\frac{\partial g}{\partial x_i} = -p_i$ and $\frac{\partial g}{\partial m} = 1$. Thus (14) becomes

$$\begin{bmatrix} \frac{\partial x^1}{\partial p_1} & \cdots & \frac{\partial x^1}{\partial p_n} & \frac{\partial x^1}{\partial m} \\ \vdots & & \vdots & \vdots \\ \frac{\partial x^n}{\partial p_1} & \cdots & \frac{\partial x^n}{\partial p_n} & \frac{\partial x^n}{\partial m} \\ \frac{\partial \lambda}{\partial p_1} & \cdots & \frac{\partial \lambda}{\partial p_n} & \frac{\partial \lambda}{\partial m} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} & -p_1 \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 u}{\partial x_n^2} & -p_n \\ -p_1 & \cdots & -p_n & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ -x_1 & \cdots & \cdots & -x_n & 1 \end{bmatrix}$$

Using the first order condition $\dfrac{\partial u}{\partial x_i} = \lambda p_i$ we have

$$\begin{bmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \cdots & \dfrac{\partial^2 u}{\partial x_1 \partial x_n} & -p_1 \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \dfrac{\partial^2 u}{\partial x_n^2} & -p_n \\ -p_1 & \cdots & -p_n & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \cdots & \dfrac{\partial^2 u}{\partial x_1 \partial x_n} & -\frac{1}{\lambda}\dfrac{\partial u}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \dfrac{\partial^2 u}{\partial x_n^2} & -\frac{1}{\lambda}\dfrac{\partial u}{\partial x_n} \\ -\frac{1}{\lambda}\dfrac{\partial u}{\partial x_1} & \cdots & -\frac{1}{\lambda}\dfrac{\partial u}{\partial x_n} & 0 \end{bmatrix}$$

Since

$$\begin{vmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \cdots & \dfrac{\partial^2 u}{\partial x_1 \partial x_n} & -\frac{1}{\lambda}\dfrac{\partial u}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \dfrac{\partial^2 u}{\partial x_n^2} & -\frac{1}{\lambda}\dfrac{\partial u}{\partial x_n} \\ -\frac{1}{\lambda}\dfrac{\partial u}{\partial x_1} & \cdots & -\frac{1}{\lambda}\dfrac{\partial u}{\partial x_n} & 0 \end{vmatrix} = \frac{1}{(-\lambda)^2} \begin{vmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \cdots & \dfrac{\partial^2 u}{\partial x_1 \partial x_n} & \dfrac{\partial u}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \dfrac{\partial^2 u}{\partial x_n^2} & \dfrac{\partial u}{\partial x_n} \\ \dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{\partial u}{\partial x_n} & 0 \end{vmatrix},$$

the strong second order conditions reduce to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} v_i v_j < 0$$

for every $v = (v_1, \ldots, v_i) \neq 0$ satisfying

$$\sum_{i=1}^{n} p_i v_i = 0,$$

or

$$(-1)^k \begin{vmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \cdots & \dfrac{\partial^2 u}{\partial x_1 \partial x_k} & \dfrac{\partial u}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2 u}{\partial x_k \partial x_1} & \cdots & \dfrac{\partial^2 u}{\partial x_k^2} & \dfrac{\partial u}{\partial x_k} \\ \dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{\partial u}{\partial x_k} & 0 \end{vmatrix} > 0, \qquad k = 2, \ldots, n.$$

References

[1] Apostol, T. M. 1957. *Mathematical analysis: A modern approach to advanced calculus*. Addison-Wesley series in mathematics. Reading, Massachusetts: Addison Wesley.

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[4] Samuelson, P. A. 1965. *Foundations of economic analysis*. New York: Athenaeum. Reprint of edition published by Harvard University Press, 1947.