

The economics of first-order linear differential equations

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The following theorem is a standard statement of the solution to a first order linear differential equation. I took it from Apostol [1, Theorems 8.2 and 8.3, pp. 309–310].

1 Theorem (First order linear differential equation) Assume P, Q are continuous on the open interval I . Let $a \in I, b \in \mathbf{R}$.

Then there is one and only one function $y = f(x)$ that satisfies the initial value problem

$$y' + P(x)y = Q(x)$$

with $f(a) = b$. It is given by

$$f(x) = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)} dt$$

where

$$A(x) = \int_a^x P(t) dt.$$

The theorem appears a bit mysterious in this form, but I can give it an economic interpretation that makes it obvious (at least to me). The first thing we will do is change the variable on which y depends from x to time, t . Interpret $y(t)$ as the value of a savings account at time t . At each point of time it earns an instantaneous rate of return $r(t)$. Moreover, we add a “flow” of additional savings to the account at the rate $s(t)$. (Negative values of s indicate a flow of withdrawals.) Thus the rate of change of the value of the account is

$$y'(t) = r(t)y(t) + s(t). \tag{1}$$

Moreover, let's rewrite the initial condition as $y(t_o) = y_0$. This yields the following translation.

2 Theorem (First order linear differential equation) Assume r, s are continuous on the open interval I . Let $t_o \in I, y_0 \in \mathbf{R}$.

Then there is one and only one function $y = f(t)$ that satisfies the initial value problem

$$y' = r(t)y + s(t)$$

with $f(t_o) = y_0$. It is given by

$$f(t) = [y_0 + S(t)] e^{\bar{r}(t)(t-t_o)}$$

where

$$\bar{r}(t) = \frac{1}{t-t_o} \int_{t_o}^t r(\tau) d\tau$$

and

$$S(t) = \int_{t_o}^t s(\tau) e^{-\bar{r}(\tau)(\tau-t_o)} d\tau.$$

But this version is obviously true!

Proof: We rely on the following well-known (easily proved) result: $\lim_{n \rightarrow \infty} (1 + (r/n))^{nt} = e^{rt}$, which is true for all r . That is, the result of compounding interest on a dollar continuously over t periods is e^{rt} dollars.

Case 1: $s = 0$. I claim that if $s(t) = 0$ for all t , then the value of the account at time t is given by

$$y(t) = y_0 e^{\bar{r}(t)(t-t_0)}, \quad (2)$$

where¹

$$\bar{r}(t) = \frac{1}{t-t_0} \int_{t_0}^t r(\tau) d\tau \text{ is the average interest rate over the interval } [t_0, t].$$

Case 2: General s . But in general, the additional savings $s(t)$ is not zero. In order to deal with the general case, we use the incredibly useful notion of present value.

The **present value** at time t_0 of \$1 at time t is $e^{-\bar{r}(t)(t-t_0)}$.

For if you invest $e^{-\bar{r}(t)(t-t_0)}$ at t_0 , you will have $e^{-\bar{r}(t)(t-t_0)} e^{\bar{r}(t)(t-t_0)} = 1$ dollar at time t .

The present value $S(t)$ of all the additional savings up to time t is thus

$$S(t) = \int_{t_0}^t s(\tau) e^{-\bar{r}(\tau)(\tau-t_0)} d\tau.$$

But at time t the future value is

$$S(t) e^{\bar{r}(t)(t-t_0)}.$$

Thus the total value of the savings account at time t is given by

$$y(t) = (y_0 + S(t)) e^{\bar{r}(t)(t-t_0)}.$$

■

References

- [1] T. M. Apostol. 1967. *Calculus*, 2d. ed., volume 1. Waltham, Massachusetts: Blaisdell.

¹We can verify this by showing that y given by (2) solves (1).

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt} y_0 e^{\bar{r}(t)(t-t_0)} = y_0 e^{\bar{r}(t)(t-t_0)} \frac{d}{dt} (\bar{r}(t)(t-t_0)) \\ &= y_0 e^{\bar{r}(t)(t-t_0)} \frac{d}{dt} \int_{t_0}^t r(\tau) d\tau = y_0 e^{\bar{r}(t)(t-t_0)} r(t) \\ &= r(t) y(t), \end{aligned}$$

which is (1) with $s = 0$. (Or it can be proven directly.)