CALIFORNIA INSTITUTE OF TECHNOLOGY
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Preference and Demand Examples

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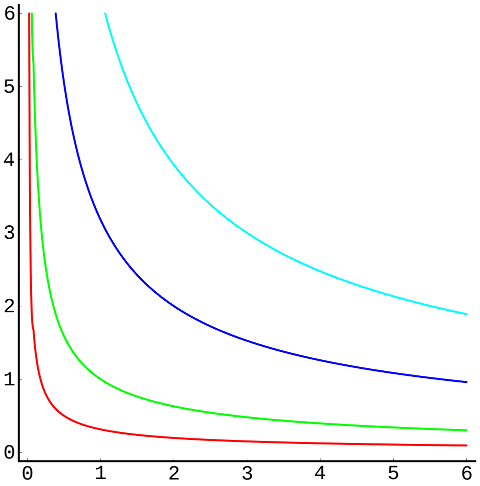
1 Cobb–Douglas preferences I: Logarithmic form

$$u_{\mathrm{I}}(x_1,\ldots,x_n)=\sum_{i=1}^{n}\alpha_i\ln x_i$$

where $\alpha_i>0$, $i=1,\ldots,n$, and $\sum_{i=1}^{n}\alpha_i=1$.



Representative contours of $\frac{2}{5}\ln x_1+\frac{3}{5}\ln x_2$.

Lagrangean:

$$\sum_{i=1}^{n}\alpha_i\ln x_i+\lambda\left(m-\sum_{i=1}^{n}p_ix_i\right)$$

First order conditions, using the binding constraint $m=\sum_{i=1}^{n}p_ix_i$:

$$\frac{\partial L}{\partial x_i}=\frac{\alpha_i}{x_i^*}-\lambda^*p_i=0\qquad i=1,\ldots,n.$$

So

$$\alpha_i=\lambda^*p_ix_i^*\qquad i=1,\ldots,n.\tag{1}$$

Summing over i yields

$$1=\lambda^*m.$$

as $\sum_{i=1}^{n}\alpha_i=1$, so (1) becomes

$$p_ix_i^*=\alpha_im,$$

that is, α_i is the fraction of income spent on good i, so the demand function is

$$\boxed{x_i^*(p,m) = \alpha_i \frac{m}{p_i}.}$$

Thus the indirect utility function is

$$\boxed{v(p,m) = \sum_{i=1}^n \alpha_i \ln\left(\alpha_i \frac{m}{p_i}\right) = \ln m - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln \alpha_i. \qquad (2)}$$

Roy's identity asserts that

$$\boxed{-\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial m}} = -\frac{-\frac{\alpha_i}{p_i}}{\frac{1}{m}} = \alpha_i \frac{m}{p_i} = x_i^*(p,m),}$$

which checks out okay.

Recall that the expenditure function e gives the level of income m needed to achieve a given level of utility υ. It therefore satisfies $v\big(p, e(p,\upsilon)\big) = \upsilon$, so we can compute the expenditure function by solving (2) for m in terms of v, and changing the symbol for m to e and the symbol for v to υ. (Note the distinction between the Roman letter, vee v, and the Greek letter ypsilon, υ.) So rewrite (2) to get

$$\upsilon = \ln e - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln \alpha_i,$$

rearranging gives

$$\ln e = \upsilon + \sum_{i=1}^n \alpha_i \ln p_i - \sum_{i=1}^n \alpha_i \ln \alpha_i,$$

so exponentiating gives

$$\boxed{e(p,\upsilon) = \exp(\upsilon) \frac{\prod_{i=1}^n p_i^{\alpha_i}}{\prod_{i=1}^n \alpha_i^{\alpha_i}}.}$$

The Hicksian compensated demands are the derivatives of the expenditure function, so

$$\boxed{\tilde{x}_j(p,\upsilon) = \frac{\alpha_j}{p_j} \exp(\upsilon) \frac{\prod_{i=1}^n p_i^{\alpha_i}}{\prod_{i=1}^n \alpha_i^{\alpha_i}}.}$$

2 Cobb–Douglas preferences II: Multiplicative form

We can subject the above utility to the increasing transformation $u_{\mathrm{II}} = \exp(u_{\mathrm{I}})$, or:

$$u_{\mathrm{II}}(x_1,\dots,x_n) = \prod_{i=1}^n x_i^{\alpha_i}$$

where $\alpha_i > 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^n \alpha_i = 1$. This should give the same demand, but the indirect utility and expenditure function will be transformed. Let's check.

Lagrangean:

$$\prod_{i=1}^n x_i^{\alpha_i} + \lambda \left(m - \sum_{i=1}^n p_i x_i \right)$$

First order conditions, using the binding constraint $m = \sum_{i=1}^n p_i x_i$:

$$\frac{\partial L}{\partial x_i} = \alpha_i \frac{\prod_{i=1}^n x_i^{*\alpha_i}}{x_i^*} - \lambda^* p_i = 0 \qquad i = 1, \ldots, n.$$

So letting $u^* = \prod_{i=1}^n x_i^{*\alpha_i}$,

$$\alpha_i u^* = \lambda^* p_i x_i^* \qquad i = 1, \ldots, n. \tag{3}$$

Summing over i yields

$$u^* = \lambda^* m.$$

as $\sum_{i=1}^n \alpha_i = 1$, so (3) becomes

$$\alpha_i \lambda^* m = \lambda^* p_i x_i^*,$$

or

$$p_i x_i^* = \alpha_i m,$$

that is, α_i is the fraction of income spent on good i, so the demand function is

$$\boxed{x_i^*(p, m) = \frac{\alpha_i m}{p_i}.}$$

Thus the indirect utility function is

$$\boxed{v(p, m) = m \prod_{i=1}^n \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i}.} \tag{4}$$

Thus

$$\frac{\partial v}{\partial p_j} = \alpha_j \left\{ \frac{m \prod_{i=1}^n \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i}}{\frac{\alpha_j}{p_j}} \right\} \frac{-\alpha_j}{p_j^2} = \frac{-\alpha_j}{p_j} m \prod_{i=1}^n \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i}.$$

Roy's identity asserts that

$$\boxed{-\frac{\frac{\partial v}{\partial p_j}}{\frac{\partial v}{\partial m}} = -\frac{\frac{-\alpha_j}{p_j} m \prod_{i=1}^n \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i}}{\prod_{i=1}^n \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i}} = \frac{\alpha_j m}{p_j} = x_j^*(p, m),}$$

which checks out okay.

We can compute the expenditure function by solving (4) for m in terms of v. Changing the symbol for m to e and the symbol for v to υ, rewrite (4) as

$$\upsilon = e \prod_{i=1}^n \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i}.$$

Rearranging gives

$$\boxed{e(p,v) = v\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}.}$$

If we wish, we can rewrite this as

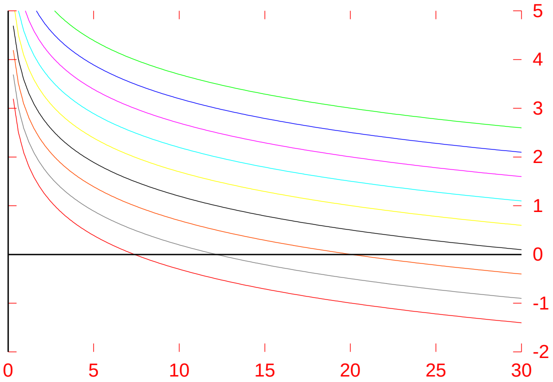
$$e(p,v) = v\frac{u(p)}{u(\alpha)},$$

where $u = u_{\mathrm{II}}$ is the Cobb–Douglas utility in multiplicative form, and $\alpha = (\alpha_1, \dots, \alpha_n)$.

3 Logarithmic quasi-linear preferences

$$u(y, x_1, \dots, x_n) = y + \beta\sum_{i=1}^{n}\alpha_i \ln x_i$$

where $\beta, \alpha_i > 0$, $i = 1, \dots, n$, and $\sum_{i=1}^{n}\alpha_i = 1$.



Representative contours of $y + \ln x$.

Note that each indifference curve is a vertical translate of every other curve, and that each intersects the x-axis.

For reasons that will become clear, let us make y the numéraire ($p_y = 1$). Then the Lagrangean is

$$y + \beta\sum_{i=1}^{n}\alpha_i \ln x_i + \lambda\left(m - y - \sum_{i=1}^{n}p_i x_i\right)$$

First order conditions, using the binding constraint $m = y + \sum_{i=1}^{n}p_i x_i$:

$$1 - \lambda^* \leqslant 0$$

with $\lambda^* = 1$ if $y^* > 0$, and

$$\beta\frac{\alpha_i}{x_i^*} - \lambda^* p_i = 0 \qquad i = 1, \dots, n.$$

So assuming $y^* > 0$, this gives

$$p_i x_i^*(p,m) = \alpha_i\beta.$$

In other words, the amount spent on good i is independent of prices and income. Thus

$$y^*(p,m) = m - \beta.$$

Note that this only works for $m \geqslant \beta$, which corresponds to $y^* \geqslant 0$.

If $m < \beta$, then $\lambda^* > 1$, and the remaining first order conditions become

$$\beta\frac{\alpha_i}{x_i^*} - \lambda^* p_i = 0 \qquad i = 1,\ldots,n,$$

which by the same reasoning as in problem (a) gives

$$x_i^*(p,m) = \alpha_i\frac{m}{p_i}.$$

Putting this all together yields

$$\begin{aligned} y^*(p,m) &= \begin{cases} m-\beta & m \geqslant \beta \\ 0 & m < \beta \end{cases} \\ x_i^*(p,m) &= \begin{cases} \alpha_i\dfrac{\beta}{p_i} & m \geqslant \beta \\ \alpha_i\dfrac{m}{p_i} & m < \beta \end{cases} \end{aligned}$$

This leads to the indirect utility

$$v(p,m) = \begin{cases} m - \beta + \beta\left(\ln\beta - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln\alpha_i\right) & m \geqslant \beta \\ \beta\left(\ln m - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln\alpha_i\right) & m < \beta \end{cases} \tag{5}$$

Solving (5) for $m = e$ in terms of v gives

$$e(p,v) = v + \beta - \beta\left(\ln\beta - \sum_{i=1}^n \alpha_i\ln p_i + \sum_{i=1}^n \alpha_i \ln\alpha_i\right)$$

for $v \geqslant \beta\left(\ln\beta - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i\ln\alpha_i\right)$, and

$$e(p,v) = \exp(v/\beta)\frac{\prod_{i=1}^n p_i^{\alpha_i}}{\prod_{i=1}^n \alpha_i^{\alpha_i}}$$

otherwise.

Oops, let's check Roy's identity:

$$-\frac{\dfrac{\partial v}{\partial p_i}}{\dfrac{\partial v}{\partial m}} = -\frac{-\dfrac{\alpha_i\beta}{p_i}}{1} = \begin{cases} \alpha_i\dfrac{\beta}{p_i} & m \geqslant \beta \\ = -\dfrac{-\dfrac{\alpha_i\beta}{p_i}}{\dfrac{\beta}{m}} = \alpha_i\dfrac{m}{p_i} & m < \beta. \end{cases} = x_i^*(p,m).$$

4 Linear preferences

$$u(x_1,\dots,x_n)=\sum_{i=1}^{n}\alpha_i x_i$$

where $\alpha_i \geqslant 0$, $i=1,\dots,n$, and $\sum_{i=1}^{n}\alpha_i=1$. (Hint: Remember Kuhn–Tucker.)

The Lagrangean is

$$\sum_{i=1}^{n}\alpha_i x_i+\lambda\left(m-\sum_{i=1}^{n}p_i x_i\right)$$

The first-order conditions are

$$\alpha_i-\lambda^* p_i \leqslant 0 \qquad i=1,\dots,n,$$

so $\lambda^*=\max_i \frac{\alpha_i}{p_i}$, and $\frac{\alpha_j}{p_j}<\lambda^*$ implies $x_j^*=0$. So for now assume that i^* is the unique maximizer of $\frac{\alpha_i}{p_i}$. Then

$$\boxed{x_j^*(p,m)=\begin{cases}\dfrac{m}{p_{i^*}} & j=i^*\\ 0 & \text{otherwise.}\end{cases}}$$

When i^* is not unique, there is no unique solution, but convex combinations of the above are all valid demands. That is,

$$\boxed{x^*(p,m)=\text{convex hull of }\left\{\frac{m}{p_j}e^j:\frac{\alpha_j}{p_j}\geqslant\frac{\alpha_i}{p_i},\ i=1,\dots,n\right\},}$$

where e^j is the j^{th} unit coordinate vector.

The indirect utility is thus

$$\boxed{v(p,m)=\alpha_{i^*}x_{i^*}^*=m\frac{\alpha_{i^*}}{p_{i^*}}=m\cdot\max_i\frac{\alpha_i}{p_i}=\frac{m}{\min\limits_i\dfrac{p_i}{\alpha_i}}.}$$

Roy's identity:

$$\boxed{\begin{aligned}-\frac{\dfrac{\partial v}{\partial p_{i^*}}}{\dfrac{\partial v}{\partial m}}&=-\frac{-\dfrac{m\alpha_{i^*}}{(p_{i^*})^2}}{\dfrac{\alpha_{i^*}}{p_{i^*}}}=\frac{m}{p_{i^*}}=x_{i^*}^*(p,m).\\ x_j^*(p,m)&=-\frac{\dfrac{\partial v}{\partial p_j}}{\dfrac{\partial v}{\partial m}}=-\frac{0}{\dfrac{\alpha_{i^*}}{p_{i^*}}}=0 \qquad j\neq i^*\end{aligned}}$$

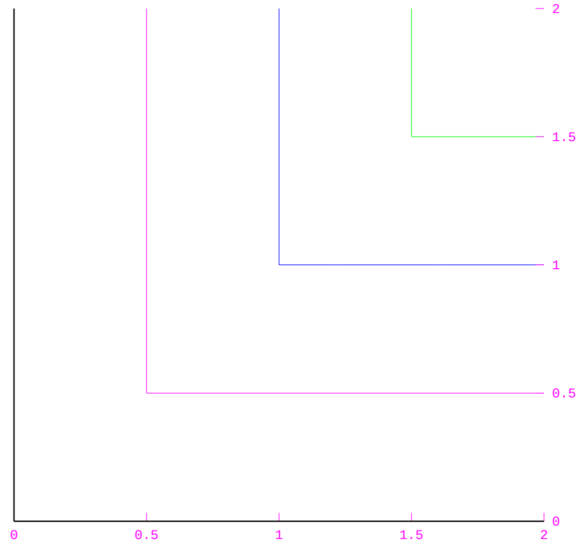
And the expenditure function satisfies

$$\boxed{e(p,v)=v\min_i\frac{p_i}{\alpha_i}}$$

5 Leontieff fixed-proportion preferences

$$u(x_1,\dots,x_n) = \min\{\alpha_1 x_1,\dots,\alpha_n x_n\}$$

where $\alpha_i \geqslant 0$, $i = 1,\dots,n$. (Hint: Calculus is worthless here.)



Representative contours of $\min\{x_1, x_2\}$.

It is easy to see that $\alpha_1 x_1^*(p,m) = \cdots = \alpha_n x_n^*(p,m)$, denote this common value by c. Then $p_i x_i^*(p,m) = c\frac{p_i}{\alpha_i}$, and summing over i gives $m = c\sum_{i=1}^n \frac{p_i}{\alpha_i}$, so

$$\boxed{x_i^*(p,m) = \frac{m}{\alpha_i \sum_{j=1}^n \dfrac{p_j}{\alpha_j}}}$$

The indirect utility is then

$$\boxed{v(p,m) = \frac{m}{\sum_{j=1}^n \dfrac{p_j}{\alpha_j}}. \qquad (6)}$$

Roy's identity:

$$\boxed{-\frac{\dfrac{\partial v}{\partial p_i}}{\dfrac{\partial v}{\partial m}} = -\frac{\dfrac{-\dfrac{m}{\alpha_i}}{\left(\sum_{j=1}^n \dfrac{p_j}{\alpha_j}\right)^2}}{\dfrac{1}{\sum_{j=1}^n \dfrac{p_j}{\alpha_j}}} = \frac{m}{\alpha_i \sum_{j=1}^n \dfrac{p_j}{\alpha_j}} = x_i^*(p,m).}$$

And by (6) the expenditure function satisfies

$$\boxed{e(p,\upsilon) = \upsilon \sum_{j=1}^{n} \frac{p_j}{\alpha_j}.}$$

So the Hicksian compensated demands are:

$$\boxed{\tilde{x}_i(p,\upsilon) = \frac{\upsilon}{\alpha_i}.}$$