CALIFORNIA INSTITUTE OF TECHNOLOGY
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Crib Sheet on Demand Theory

KC Border

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Assume $p \gg 0$, $m > 0$, u is continuous and locally nonsatiated on $\boldsymbol{R}^n_+$, and u is C^2, $u' \gg 0$ and strongly quasiconcave (its Hessian is negative definite on the subspace orthogonal to the gradient) on $\boldsymbol{R}^n_{++}$.

Utility Maximization	**Expenditure Minimization**
$\underset{x}{\text{maximize}}\, u(x)$ subject to $m - p \cdot x = 0$	$\underset{x}{\text{minimize}}\, p \cdot x$ subject to $u(x) - \upsilon = 0$

Solution

Ordinary (Walrasian) Demand	Hicksian Compensated Demand
$x^*(p, m)$	$\hat{x}(p, \upsilon)$
x^* is homogeneous of degree zero in (p, m).	$\hat{x}$ is homogeneous of degree 1 in p.

Value function

Indirect Utility	Expenditure Function
$v(p, m) = u\big(x^*(p, m)\big)$	$e(p, \upsilon) = p \cdot \hat{x}(p, \upsilon)$
v is quasi-convex in (p, m), homogeneous of degree zero in (p, m).	e is concave in p, and homogeneous of degree 1 in p.

Statement of Equivalence

$x^*(p, m) = \hat{x}\big(p, v(p, m)\big)$ $m = e\big(p, v(p, m)\big)$	$\hat{x}(p, \upsilon) = x^*\big(p, e(p, \upsilon)\big)$ $\upsilon = v\big(p, e(p, \upsilon)\big)$

Utility Maximization	**Expendidure Minimization**
Lagrangean	
$\mathcal{L}(x,\lambda;p,m) = u(x) + \lambda(m - p\cdot x)$	$\mathcal{L}(x,\mu;p,\upsilon) = p\cdot x - \mu(u(x) - \upsilon)$
Partials with respect to parameters	
$\frac{\partial \mathcal{L}(x,\lambda;p,m)}{\partial p_j} = -\lambda x_j$	$\frac{\partial \mathcal{L}(x,\mu;p,\upsilon)}{\partial p_j} = x_j$
$\frac{\partial \mathcal{L}(x,\lambda;p,m)}{\partial m} = \lambda$	$\frac{\partial \mathcal{L}(x,\mu;p,\upsilon)}{\partial \upsilon} = \mu$
Envelope Theorem	
$\frac{\partial v(p,m)}{\partial p_j} = -\lambda^*(p,m)x_j^*(p,m)$	$\frac{\partial e(p,\upsilon)}{\partial p_j} = \hat{x}_j(p,\upsilon)$
$\frac{\partial v(p,m)}{m} = \lambda^*(p,m)$	$\frac{\partial e(p,\upsilon)}{\partial \upsilon} = \hat{\mu}(p,\upsilon)$
Roy's Law	Hotelling/Shephard's Lemma
$x_j^*(p,m) = -\frac{\frac{\partial v(p,m)}{\partial p_j}}{\frac{\partial v(p,m)}{\partial m}}$	$\hat{x}_j(p,\upsilon) = \frac{\partial e(p,\upsilon)}{\partial p_j}$

The Slutsky equation

From the equivalence

$$\hat{x}(p,\upsilon) = x^*(p,e(p,\upsilon))$$

differentiating with respect to p_j yields

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial x_i^*(p,e(p,\upsilon))}{\partial p_j} + \frac{\partial x_i^*(p,e(p,\upsilon))}{\partial m}\frac{\partial e(p,\upsilon)}{\partial p_j}.$$

But $\frac{\partial e(p,\upsilon)}{\partial p_j} = \hat{x}_j(p,\upsilon) = x_j^*(p, e(p,\upsilon))$. Set $m = e(p,\upsilon)$, and write

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial x_i^*(p,m)}{\partial p_j} + x_j^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m}$$

which implies the **Slutsky equation**

$$\boxed{\frac{\partial x_i^*(p,m)}{\partial p_j} = \frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} - x_j^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m},}$$

where $\upsilon = v(p,m)$, which decomposes the effect of a price change into its **substitution effect** and **income effect**.

But

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial^2 e(p,\upsilon)}{\partial p_i \partial p_j},$$

so since e is concave in p, its Hessian is negative semidefinite (and symmetric), so the matrix

$$\left[\frac{\partial x_i^*(p,m)}{\partial p_j} + x_j^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m}\right] \text{ is negative semidefinite and symmetric.}$$

Consequently the diagonal terms satisfy

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_i} = \frac{\partial x_i^*(p,m)}{\partial p_i} + x_i^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m} \leqslant 0,$$

and we have the unusual **reciprocity** relation

$$\frac{\partial x_i^*(p,m)}{\partial p_j} + x_j^*(p,m)\frac{\partial x_i^*(p,m)}{\partial m} = \frac{\partial x_j^*(p,m)}{\partial p_i} + x_i^*(p,m)\frac{\partial x_j^*(p,m)}{\partial m}.$$