

Monotonicity and local nonsatiation

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The classical consumption set is \mathbf{R}_+^n . The Euclidean metric on \mathbf{R}_+^n is denoted d . The partial order \geq on \mathbf{R}_+^n is defined by $x \geq y$ if $x_i \geq y_i$ for $i = 1, \dots, n$. The partial order \gg is defined by $x \gg y$ if $x_i > y_i$ for $i = 1, \dots, n$. The partial order $>$ is defined by $x > y$ if $x \geq y$ and $x \neq y$.

A **regular preference** is a reflexive, transitive, total binary relation \succsim on \mathbf{R}_+^n . The asymmetric part of \succsim is denoted \succ . The preference \succsim is **locally nonsatiated** if for each $x \in \mathbf{R}_+^n$ and $\varepsilon > 0$, there is some $y \in \mathbf{R}_+^n$ such that $d(x, y) < \varepsilon$ and $y \succ x$. The preference relation \succsim is **monotonic** if $x \gg y$ implies $x \succ y$. Clearly every monotonic preference relation is locally nonsatiated. A preference relation is **upper semicontinuous** if for each x , the upper contour set $\{y \in \mathbf{R}_+^n : y \succsim x\}$ is closed, or equivalently the strict lower contour set $\{y \in \mathbf{R}_+^n : x \succ y\}$ is relatively open in \mathbf{R}^n . The preference \succsim is **convex** if for each y , the upper contour set $\{x : x \succsim y\}$ is convex.

Given a price vector $p \in \mathbf{R}_{++}^n$ and income level $m > 0$, the **budget set** $\beta(p, m)$ is the compact convex set $\{x \in \mathbf{R}_+^n : p \cdot x \leq m\}$. It is a fact that if \succsim is upper semicontinuous, then every compact set has a \succsim -greatest element; and if \succsim is locally nonsatiated, then every \succsim -greatest element x^* in $\beta(p, m)$ satisfies $p \cdot x^* = m$.

While there are clearly preference relations that are locally nonsatiated but not monotonic, from the point of view of competitive demand theory, local nonsatiation is no more general than monotonicity. By this I mean that if \succsim is a locally nonsatiated upper semicontinuous preference, then there is a monotonic preference \succsim^* that generates the same demand.

To simplify the explicit description of such a monotonic relation, let us introduce the following notation. Given a set A ,

$$x \succsim A \quad \text{means} \quad (\forall z \in A) [x \succsim z]$$

and let

$$D(x) = \{y \in \mathbf{R}_+^n : 0 \leq y \leq x\}.$$

Note that $D(x)$ is nonempty for $x \in \mathbf{R}_+^n$ as it contains x .

1 Proposition *Let \succsim be a locally nonsatiated and upper semicontinuous regular preference on \mathbf{R}_+^n . Then the binary relation \succsim^* on \mathbf{R}_+^n defined by*

$$x \succsim^* y \quad \text{if there exists } v \geq 0 \text{ such that } x \geq v \succsim D(y)$$

is a monotonic upper semicontinuous preference relation that generates the same demand for $(p, m) \in \mathbf{R}_{++}^n \times \mathbf{R}_+$. Moreover, if \succsim is convex, then \succsim^ is convex.*

Proof: Note that for each y the set $D(y)$ is compact, and since \succ is upper semicontinuous, the set $\mu(y)$ of \succ -greatest element of $D(y)$ is nonempty. So $x \not\preceq^* y$ if and only if there exist $v \leq x$ and $u \in \mu(y)$ with $v \succ u$.

First we show that \preceq^* is

- i. reflexive: Let $u \in \mu(x)$. Then $x \geq u \succ D(x)$, so $x \preceq^* x$.
- ii. transitive: Assume $x \not\preceq^* y$ and $y \not\preceq^* z$. Then there exist v_y, v_z, u_y , and u_z such that $x \geq v_y \succ u_y \in \mu(y)$ and $y \geq v_z \succ u_z \in \mu(z)$. Since $y \geq v_z$ and $u_y \succ D(y)$ we have $u_y \succ v_z$. Thus $x \geq v_y \succ u_y \succ v_z \succ D(z)$, so $x \not\preceq^* z$.
- iii. total: If $x \not\preceq^* y$, then by definition, for every $0 \leq u \leq x$, there is some $0 \leq v \leq y$ with $v \succ u$. Let $u^* \in \mu(x)$, and let $v^* \leq y$ satisfy $v^* \succ u^*$. Thus $y \geq v^* \succ u^* \succ D(x)$. Thus $y \preceq^* x$.

This also proves that \succ^* , the asymmetric part of \preceq^* , satisfies

$$y \succ^* x \text{ if and only if } (\exists v) [y \geq v \succ D(x)].$$

Next we show that \preceq^* is monotonic, that is, if $x \gg y$, then $x \preceq^* y$. So assume $x \gg y$ and let $z \in \mu(y) \subset D(y)$, so $x \gg z$. Then for $\varepsilon > 0$ small enough, $d(v, z) < \varepsilon$ implies $x \gg v$. By local nonsatiation, at least one such v satisfies $v \succ z \succ D(y)$. Thus $x \gg z \succ D(y)$, so $x \preceq^* y$.

To see that \preceq^* is upper semicontinuous, I shall prove that if $y \preceq^* x$, then there is an $\varepsilon > 0$ such that $d(x, x') < \varepsilon$ implies $y \preceq^* x'$ too. So assume $y \preceq^* x$. Then there exists $v \leq y$ such that $v \succ D(x)$. Since $D(x)$ is compact, I claim there is some $\varepsilon > 0$ such that $v \succ N_\varepsilon(D(x))$.¹ Then if $d(x, x') < \varepsilon$, we have $D(x') \subset N_\varepsilon(D(x))$ too, so $v \succ D(x')$. But this implies $y \preceq^* x'$.

Finally we show that for $p \gg 0$, a point x^* is \succ -greatest in $\beta(p, m)$ if and only if x^* is also \preceq^* -greatest.

Assume first that x^* is \succ -greatest in $\beta(p, m)$. Let $y \in \beta(p, m)$. Then $D(y) \subset \beta(p, m)$, so $x^* \geq x^* \succ D(y)$. Thus $x^* \preceq^* y$. Therefore x^* is \preceq^* -greatest in $\beta(p, m)$.

¹Here $N_\varepsilon(A)$ denotes the ε -neighborhood of A , that is, $\{x : (\exists y \in A) [d(x, y) < \varepsilon]\}$. It is an open set being the union of the open balls of radius ε centered on points of A . To see why this claim is true, let F denote the closed upper contour set $\{u : u \succ v\}$. Then the distance function $d(z, F) = \inf\{d(z, u) : u \in F\}$ is (Lipschitz) continuous and so achieves its minimum over the compact set $D(x)$. Since F and $D(x)$ are disjoint closed sets this minimum is strictly greater than zero. Choose $\varepsilon > 0$ less than this minimum.

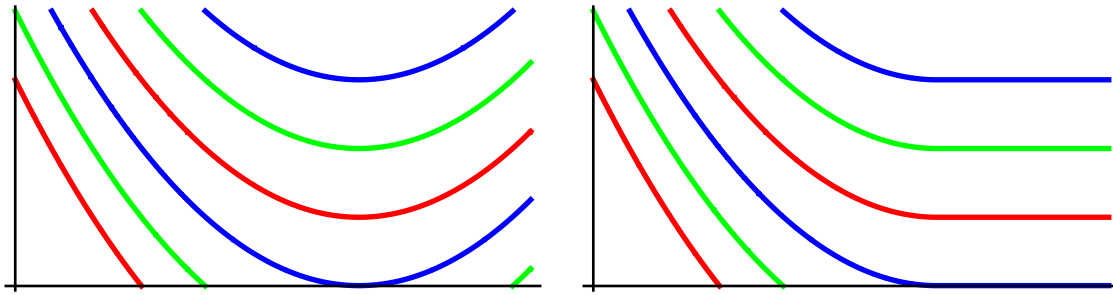


Figure 1. Indifference curves for locally nonsatiated utility $u(x, y) = y - (1 - x)^2$ and monotone utility with same demand.

Now assume that x^* is \succ^* -greatest in $\beta(p, m)$, and let \bar{x} be \succ -greatest. Since $x^* \not\succeq \bar{x}$ there is some $u \preceq x^*$ with $u \succ D(\bar{x})$. In particular, $u \succ \bar{x} \in D(\bar{x})$, so u too is \succ -greatest. But by local nonsatiation $p \cdot u = m$, so $u \preceq x^* \in \beta(p, m)$ implies $u = x^*$, so x^* is also \succ -greatest.

To see that \succ^* is convex if \succ is convex, let $x, x' \succ^* y$, where $x \succeq v \succ D(y)$ and let $x' \succeq v' \succ D(y)$. Then $(1 - \lambda)x + \lambda x' \succeq (1 - \lambda)v + \lambda v'$ and assuming \succ is convex, $(1 - \lambda)v + \lambda v' \succ D(y)$. Thus $(1 - \lambda)x + \lambda x' \succ^* y$. ■

Finally we mention that if u is an upper semicontinuous locally nonsatiated function, then its **monotonic hull** v , defined by $v(x) = \max\{u(y) : 0 \preceq y \preceq x\}$, is the smallest monotonic function that dominates u . The Berge maximum theorem implies that it is upper semicontinuous. If u is a utility that represents \succ , then its monotonic hull represents \succ^* .

2 Example Consider the quasilinear utility function for two goods x and y defined by

$$u(x, y) = y - (1 - x)^2$$

which gives a linear demand function for x . It is locally nonsatiated but not monotone. It has the property that the demand for x never exceeds 1. It has the same demand behavior as the monotone utility

$$v(x, y) = \begin{cases} y - (1 - x)^2 & x \leq 1 \\ y & x \geq 1 \end{cases}$$

which is its monotonic hull. See Figure 1. □