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WARP and the Slutsky matrix

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1 Samuelson's Weak Axiom of Revealed Preference

The Weak Axiom of Revealed Preference asserts that if you demand x when y is in the budget set, it is because you prefer x to y. Therefore you should never demand y when x is in the budget set. (This of course implicitly assumes a unique utility maximizer, or strict quasiconcavity of the utility.) Paul Samuelson [3, 4, 5, 6] showed that this observation alone is enough to deduce the negative semidefiniteness of the matrix of Slutsky substitution terms.

1 Definition (Samuelson's Weak Axiom of Revealed Preference) *Let $X \subset \boldsymbol{R}^{\mathrm{n}}$ be the consumption set. For an ordinary demand function $\xi \colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++} \to X$, define the binary relation S on X by*

$$x \; S \; y \quad \text{if} \quad (\exists (p,w)) \; [x = \xi(p,w) \;\&\; y \neq x \;\&\; p \cdot y \leqslant w].$$

*That is, x is demanded when y is in the budget set but not demanded, so x is **revealed preferred** to y. The demand function ξ obeys **Samuelson's Weak Axiom of Revealed Preference (SWARP)** if S is an asymmetric relation. That is, if for every $x, y \in X$,*

$$x \; S \; y \implies \neg y \; S \; x.$$

That is, if x is revealed preferred to y, then y is never revealed preferred to x.

The demand function ξ satisfies the **budget exhaustion condition** if for all (p, w),

$$p \cdot \xi(p,w) = w.$$

Under the budget exhaustion condition, we can rewrite SWARP in the form that Samuelson used. Let x^0 and x^1 belong to the range of ξ. That is, let

$$x^0 = \xi(p^0, w^0) = \xi(p^0, p^0 \cdot x^0) \quad \text{and} \quad x^1 = \xi(p^1, w^1) = \xi(p^1, p^1 \cdot x^1).$$

Then $p^1 \cdot x^0 \leqslant p^1 \cdot x^1$ and $x^0 \neq x^1$ imply $x^1 \; S \; x^0$; while $x^0 \neq x^1$ and $\neg x^0 \; S \; x^1$ imply $p^0 \cdot x^1 > p^0 \cdot x^0$. Thus, we can write SWARP in Samuelson's form:[1]

$$x^0 \neq x^1 \text{ and } p^1 \cdot x^0 \leqslant p^1 \cdot x^1 \quad \implies \quad p^0 \cdot x^1 > p^0 \cdot x^0.$$

[1] It may appear that this condition is weaker then than the one stated above, since it applies only to x_0 and x_1 in the range of ξ, whereas the condition above applies to all x and y in X, which may be larger than the range of ξ. However, any violation of SWARP as stated above would involve x and y with $x \; S \; y$ and $y \; S \; x$, which can only happen if both x and y belong to the range of ξ. Thus the definitions are equivalent.

2 Slutsky compensated demand

This leads us to define the **Slutsky compensated demand** s in terms of the ordinary demand function ξ via

$$s(p,\bar{x}) = \xi(p, p \cdot \bar{x})$$

where $\bar{x} \in X$ can be thought of as an initial endowment that determines the value of income w. Another interpretation is that if $\bar{x} = \xi(\bar{p}, \bar{w})$, then $s(p,\bar{x})$ is the demand $\xi(p,w)$ where w has been adjusted (compensated) so that consumption $\bar{x}$ is still just affordable at price vector p.

Note that

$$\frac{\partial s_i(p,\bar{x})}{\partial p_j} = \frac{\partial \xi_i(p, p \cdot \bar{x})}{\partial p_j} + \bar{x}_j \frac{\partial \xi_i(p, p \cdot \bar{x})}{\partial w}.$$

In particular, by setting $\bar{x} = \xi(p,w)$ we may define the **Slutsky substitution term**

$$\begin{aligned}\sigma_{i,j}(p,w) &= \frac{\partial s_i\big(p, \xi(p,w)\big)}{\partial p_j} \\ &= \frac{\partial \xi_i(p,w)}{\partial p_j} + \xi_j(p,w) \frac{\partial \xi_i(p,w)}{\partial w}.\end{aligned}$$

The following important lemma may be found in in Samuelson [6, equation (70), p. 109] or Mas-Colell, Whinston, and Green [2, Proposition 2.F.1, pp. 30–33].

2 Lemma *Let ξ satisfy the budget exhaustion condition and SWARP. Let*

$$x^0 = \xi(p^0, w^0) \quad \text{and} \quad x^1 = \xi(p^1, p^1 \cdot x^0).$$

Then

$$(p^1 - p^0) \cdot (x^1 - x^0) \leqslant 0,$$

with equality if and only if $x^1 = x^0$.

Proof: If $x^1 = x^0$, then the conclusion is true as an equality. So assume $x^1 \neq x^0$.

By budget exhaustion

$$p^1 \cdot x^1 = p^1 \cdot x^0. \tag{1}$$

Since $x^1 \neq x^0$, this says that $x^1 \; S \; x^0$. So by SWARP, we have $\neg x^0 \; S \; x^1$, that is,

$$p^0 \cdot x^1 > w^0 = p^0 \cdot x^0, \tag{2}$$

where the second equality follows from budget exhaustion. Subtracting inequality (2) from equality (1) gives

$$(p^1 - p^0) \cdot x^1 < (p^1 - p^0) \cdot x^0,$$

which proves the conclusion of the lemma. ∎

3 Theorem *Let* $\xi \colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++} \to \boldsymbol{R}^{\mathrm{n}}_{+}$ *be differentiable and satisfy the budget exhaustion condition and SWARP. Then for every* $(p, w) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++}$*, and every* $v \in \boldsymbol{R}^{\mathrm{n}}$*,*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j}(p, w) v_i v_j \leqslant 0.$$

That is, the matrix of Slutsky substitution terms is negative semidefinite.[2]

Proof: Fix $(p, w) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++}$ and $v \in \boldsymbol{R}^{\mathrm{n}}$. By homogeneity of degree 2 of the quadratic form in v, without loss of generality we may scale v so that $p \pm v \gg 0$.

Define the function x on $[-1, 1]$ via

$$x(t) = s\big(p + tv, \xi(p, w)\big). \tag{3}$$

Note that this is differentiable, and $x(0) = \xi(p, w)$.

By Lemma 2 (with $p + tv$ playing the rôle of p^1 and p playing the rôle of p^0),

$$(p + tv - p) \cdot \big(x(t) - x(0)\big) = tv \cdot \big(x(t) - x(0)\big) \leqslant 0.$$

For nonzero t, dividing by $t^2 > 0$ gives

$$v \cdot \frac{x(t) - x(0)}{t} \leqslant 0.$$

Taking limits as $t \to 0$ gives

$$v \cdot x'(0) \leqslant 0. \tag{4}$$

By the Chain Rule applied to (3),

$$x_i'(t) = \sum_{j=1}^{n} \frac{\partial s_i\big(p + tv, \xi(p, w)\big)}{\partial p_j} v_j. \tag{5}$$

Evaluating (5) at $t = 0$ yields

$$\begin{aligned} x_i'(0) &= \sum_{j=1}^{n} \frac{\partial s_i\big(p, \xi(p, w)\big)}{\partial p_j} v_j \\ &= \sum_{j=1}^{n} \sigma_{i,j}(p, w) v_j, \end{aligned}$$

where the second equality is just the definition of $\sigma_{i,j}(p, w)$. Combining this with (4) gives

$$0 \geqslant v \cdot x'(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j}(p, w) v_i v_j,$$

which completes the proof. ∎

This proof is Kihlstrom, Mas-Colell, and Sonnenschein's [1] more modern rewriting of Samuelson's argument.

[2] Most authors, myself included, usually reserve the term "negative semidefinite" for *symmetric* matrices. In this instance, I won't insist on it.

References

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