

Duality Approach to Demand Properties

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The constrained maximization problem is

$$\text{maximize } u(x) \quad \text{subject to } m - p \cdot x = 0.$$

We assume there is a unique interior maximizer. The Lagrangean for this problem is

$$L(x, \lambda; p, m) = u(x) + \lambda(m - p \cdot x).$$

The gradient of the constraint is $-p \neq 0$, so the Lagrange Multiplier Theorem applies. Let $x^*(p, m)$ be the solution with Lagrange multiplier $\lambda^*(p, m)$. The first-order conditions are

$$u_i(x^*) - \lambda^* p_i = 0 \quad i = 1, \dots, n.$$

Since $p \gg 0$ and each $u_i > 0$ by assumption U.2, we have

$$\lambda^* > 0.$$

Define the **indirect utility function** v to be the value function for this problem, that is,

$$v(p, m) = u(x^*(p, m)).$$

Then by the Envelope Theorem,

$$\frac{\partial v(p, m)}{\partial m} = \frac{\partial L}{\partial m} = \lambda^*(p, m) \quad \text{and} \quad \frac{\partial v(p, m)}{\partial p_j} = \frac{\partial L}{\partial p_j} = -\lambda^*(p, m)x_j^*(p, m).$$

Together these imply **Roy's Identity**, namely:

$$x_j^*(p, m) = -\frac{\frac{\partial v(p, m)}{\partial p_j}}{\frac{\partial v(p, m)}{\partial m}}.$$

Compare this with the expenditure minimization problem:

$$\text{minimize } p \cdot x \quad \text{subject to } u(x) - v \leq 0.$$

The Lagrangean for this problem is:

$$p \cdot x - \mu(u(x) - v).$$

Let $\hat{x}(p, v)$ solve the problem and let $\hat{\mu}(p, v)$ be the Lagrange multiplier. The function \hat{x} is also known as the **Hicksian compensated demand function**. Define the **expenditure function** e to be the value function for this problem, that is,

$$e(p, v) = p \cdot \hat{x}(p, v).$$

Then by the Envelope Theorem,

$$\frac{\partial e(p, v)}{\partial v} = \hat{\mu}(p, v) \quad \text{and} \quad \frac{\partial e(p, v)}{\partial p_j} = \hat{x}^j(p, v).$$

Moreover, by the Support Function Theorem, e is concave in p . Thus e is twice differentiable in p almost everywhere, and where it is differentiable, then

$$\begin{bmatrix} \frac{\partial^2 e(p, v)}{\partial^2 p_1} & \cdots & \frac{\partial^2 e(p, v)}{\partial p_n \partial p_1} \\ \vdots & & \vdots \\ \frac{\partial^2 e(p, v)}{\partial p_1 \partial p_n} & \cdots & \frac{\partial^2 e(p, v)}{\partial^2 p_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{x}^1}{\partial p_1} & \cdots & \frac{\partial \hat{x}^1}{\partial p_n} \\ \vdots & & \vdots \\ \frac{\partial \hat{x}^n}{\partial p_1} & \cdots & \frac{\partial \hat{x}^n}{\partial p_n} \end{bmatrix}$$

is symmetric and negative semidefinite. In particular then,

$$\frac{\partial \hat{x}^j}{\partial p_j} \leq 0$$

and

$$\frac{\partial \hat{x}^j}{\partial p_i} = \frac{\partial \hat{x}^i}{\partial p_j}$$

for $i, j = 1, \dots, n$,

What is the relationship between this and ordinary demand? From the equivalence of expenditure minimization and utility maximization we have

$$x^{*i}(p, e(p, v)) = \hat{x}^i(p, v),$$

which implies

$$\frac{\partial x^{*i}}{\partial p_j} + \frac{\partial x^{*i}}{\partial m} \frac{\partial e}{\partial p_j} = \frac{\partial \hat{x}^i}{\partial p_j}.$$

Rearranging,

$$\frac{\partial x^{*i}}{\partial p_j} = \frac{\partial \hat{x}^i}{\partial p_j} - \frac{\partial x^{*i}}{m} \frac{\partial e}{\partial p_j}.$$

Now use $\frac{\partial e}{\partial p_j} = \hat{x}^j$ and $\hat{x}^j(p, v) = x^{*j}(p, m)$ where $m = e(p, v)$ to conclude

$$\frac{\partial x^{*i}(p, m)}{\partial p_j} = \frac{\partial \hat{x}^i(p, v(p, m))}{\partial p_j} - x^{*j}(p, m) \frac{\partial x^{*i}(p, m)}{\partial m}.$$

This is the famous **Slutsky equation**. Or rearranging another way we find

$$\begin{aligned} \begin{bmatrix} \frac{\partial^2 e}{\partial p_1^2} & \cdots & \frac{\partial^2 e}{\partial p_n \partial p_1} \\ \vdots & & \vdots \\ \frac{\partial^2 e}{\partial p_1 \partial p_n} & \cdots & \frac{\partial e}{\partial p_n} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \hat{x}^1}{\partial p_1} & \cdots & \frac{\partial \hat{x}^1}{\partial p_n} \\ \vdots & & \vdots \\ \frac{\partial \hat{x}^n}{\partial p_1} & \cdots & \frac{\partial \hat{x}^n}{\partial p_n} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial x^{*1}}{\partial p_1} + x^{*1} \frac{\partial x^{*1}}{\partial m} & \cdots & \frac{\partial x^{*1}}{\partial p_n} + x^{*n} \frac{\partial x^{*1}}{\partial m} \\ \vdots & & \vdots \\ \frac{\partial x^{*n}}{\partial p_1} + x^{*1} \frac{\partial x^{*n}}{\partial m} & \cdots & \frac{\partial x^{*n}}{\partial p_j} + x^{*n} \frac{\partial x^{*n}}{\partial m} \end{bmatrix} \end{aligned}$$

is symmetric and negative semidefinite.

The problem with Hicksian compensated demand is that the utility level v is in principle not observable. This problem can be solved by considering the **Slutsky compensated demand**, which solves the following problem.

$$\underset{x}{\text{maximize}} \quad u(x) \quad \text{subject to} \quad p \cdot x \leq p \cdot \omega.$$

These demands are also called **offer curves**. Let $\tilde{x}(p, \omega)$ denote the solution to this problem. These demands are in principle observable. Note that

$$\tilde{x}(p, \omega) = x^*(p, p \cdot \omega).$$

Thus

$$\frac{\partial \tilde{x}^i(p, \omega)}{\partial p_j} = \frac{\partial x^{*i}(p, p \cdot \omega)}{\partial p_j} + \frac{\partial x^{*i}(p, p \cdot \omega)}{\partial m} \omega_j.$$

Thus, fixing (\bar{p}, \bar{m}) and setting $\bar{x} = x^*(\bar{p}, \bar{m})$, and $\bar{v} = u(\bar{x})$, we have

$$\frac{\partial \tilde{x}^i(\bar{p}, \bar{x})}{\partial p_j} = \frac{\partial x^{*i}(\bar{p}, \bar{m})}{\partial p_j} + \frac{\partial x^{*i}(\bar{p}, \bar{m})}{\partial m} \bar{x}_j = \frac{\partial \hat{x}^i(\bar{p}, \bar{v})}{\partial p_j}.$$