

Examples of Cost and Production Functions

These notes show how you can use the first order conditions for cost minimization to actually solve for cost functions. The basic steps are these: (1) Solve each of the first order conditions for x_i in terms of the parameters (wages and output level) and the Lagrange multiplier. (2) Substitute these expressions for x_i into the production function and solve for the Lagrange multiplier. (3) Substitute this expression for the Lagrange multiplier into the expression for x_i . (4) Multiply each x_i by its wage, and sum to get the cost.

Admittedly all the examples here are specially chosen to be amenable to this approach. One thing these examples make clear is it there is often a duality between families of cost and production functions. For instance, the cost function associated with a Leontieff production function is linear, while the cost function associated with a linear production function is Leontieff.

In all cases, assume $y > 0$ and $w \gg 0$.

1 von Neumann–Leontieff Production Function

In this constant returns to scale production function, the inputs must be used in exactly the right proportions or the excess is wasted.

$$y = \min \left\{ \frac{x_1}{\alpha_1}, \dots, \frac{x_n}{\alpha_n} \right\},$$

where each $\alpha_i > 0$. Note that the production function fails to be differentiable at any interesting point, so you cannot actually apply the Lagrange multiplier theorem. Nevertheless, simple reasoning shows that the conditional factor demands must satisfy $y = \frac{x_i}{\alpha_i}$ for each i , so

$$\hat{x}_i(y, w) = \alpha_i y.$$

So the cost function is

$$c(y, w) = y \sum_{i=1}^n w_i \alpha_i.$$

Note that the cost function is quite smooth even though the production function is not. This is to be expected since for $w \gg 0$ the cost minimizing input combination is unique. (Recall the support function theorem).

And just as predicted by the support function theorem,

$$\frac{\partial c(y, w)}{\partial w_i} = \alpha_i y = \hat{x}_i(y, w).$$

2 Linear Production Function

With this constant returns to scale production function, all inputs are perfect substitutes for each other (provided units are chosen properly).

$$y = \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

where each $\alpha_i > 0$, $i = 1, \dots, n$.

The Lagrangean for the cost minimization problem is

$$\sum_{i=1}^n w_i x_i - \lambda \left(\sum_{i=1}^n \alpha_i x_i - y \right)$$

and the naïve first order conditions are

$$\frac{\partial L}{\partial x_i} = w_i - \lambda \alpha_i = 0 \quad i = 1, \dots, n,$$

which taken at face value imply $\frac{w_1}{\alpha_1} = \cdots = \frac{w_n}{\alpha_n}$, which is unlikely since these are all exogenous. This is a red flag that signals that the nonnegativity constraints are binding and that you need to examine the Kuhn–Tucker first order conditions. They are

$$w_i - \lambda \alpha_i \geq 0, \quad i = 1, \dots, n,$$

and

$$x_i > 0 \implies w_i - \lambda \alpha_i = 0 \quad \text{and} \quad w_i - \lambda \alpha_i > 0 \implies x_i = 0.$$

In addition, $\lambda \geq 0$ and $\lambda (\sum_{i=1}^n \alpha_i x_i - y) = 0$.

Thus

$$\frac{w_i}{\alpha_i} \geq \lambda \quad i = 1, \dots, n.$$

The question is, can we have strict inequality for each i ? The answer is no, as that would imply $x_i = 0$ for each i and the output would be zero, not $y > 0$. So the solution must satisfy

$$\hat{\lambda} = \min_i \frac{w_i}{\alpha_i}.$$

Let i^* satisfy $\hat{\lambda} = \frac{w_{i^*}}{\alpha_{i^*}}$. That is, i^* is a factor that maximizes “bang per buck.” The conditional factor demand given by:

$$\hat{x}_i = \begin{cases} \frac{y}{\alpha_i} & i = i^* \\ 0 & i \neq i^* \end{cases}$$

minimizes cost, and the cost function is

$$c(y, w) = y \cdot \min \left\{ \frac{w_1}{\alpha_1}, \dots, \frac{w_n}{\alpha_n} \right\}.$$

This is the cost function even if i^* is not unique, but when there is more than one such i^* , the conditional factor demand is no longer a unique input vector, but rather a set of cost minimizing input vectors. In fact, the set of cost minimizing input vectors is the convex set:

$$\text{co} \left\{ \frac{y}{\alpha_i} e^i : \frac{w_i}{\alpha_i} = \hat{\lambda} = \min_j \frac{w_j}{\alpha_j} \right\}.$$

Note that even though the production function is very smooth, the cost function fails to be differentiable (for $n \geq 2$). This is to be expected since the bordered Hessian of the production function is given by

$$\begin{bmatrix} f_{11} & \dots & f_{1n} & f_1 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ f_{n1} & \dots & f_{nn} & f_n \\ f_1 & \dots & f_n & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & \alpha_1 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \alpha_n \\ \alpha_1 & \dots & \alpha_n & 0 \end{bmatrix},$$

which is singular for $n \geq 2$. (It has rank 2.)

3 Cobb–Douglas Production Function

This production function is given by

$$y = \gamma x_1^{\alpha_1} \dots x_n^{\alpha_n},$$

where each $\alpha_i > 0$, $i = 1, \dots, n$. It is homogeneous of degree

$$\alpha = \sum_{i=1}^n \alpha_i.$$

Lagrangian:

$$w \cdot x - \lambda(\gamma x_1^{\alpha_1} \dots x_n^{\alpha_n} - y)$$

First order conditions, using the binding constraint $y = \gamma x_1^{\alpha_1} \dots x_n^{\alpha_n}$:

$$\frac{\partial L}{\partial x_i} = w_i - \lambda \alpha_i \frac{y}{x_i} = 0 \quad i = 1, \dots, n.$$

So

$$x_i = \lambda \alpha_i \frac{y}{w_i} \quad i = 1, \dots, n. \tag{1}$$

But $y = \gamma x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, so

$$y = \gamma \prod_{i=1}^n \left(\lambda \alpha_i \frac{y}{w_i} \right)^{\alpha_i} = \gamma \lambda^\alpha y^\alpha \prod_{i=1}^n \left(\frac{\alpha_i}{w_i} \right)^{\alpha_i}.$$

Solving this for λ gives

$$\begin{aligned} \hat{\lambda} &= \left[\gamma y^{\alpha-1} \prod_{i=1}^n \left(\frac{\alpha_i}{w_i} \right)^{\alpha_i} \right]^{-1/\alpha} \\ &= \gamma^{-1/\alpha} y^{(1-\alpha)/\alpha} \left(\prod_{i=1}^n \alpha_i^{-\alpha_i/\alpha} \right) \left(\prod_{i=1}^n w_i^{\alpha_i/\alpha} \right) \end{aligned}$$

To simplify notation a bit, set

$$\begin{aligned} \beta_i &= \frac{\alpha_i}{\alpha}, \\ b &= \gamma^{\frac{1}{\alpha}} \cdot \prod_i \alpha_i^{-\beta_i}, \end{aligned}$$

so

$$\hat{\lambda} = b y^{(1-\alpha)/\alpha} \prod_{i=1}^n w_i^{\beta_i}.$$

Substituting this for λ in (1) gives the conditional factor demands

$$\begin{aligned} \hat{x}_j(y, w) &= b y^{(1-\alpha)/\alpha} \prod_{i=1}^n w_i^{\beta_i} \alpha_j \frac{y}{w_j} \\ &= \frac{\alpha_j}{w_j} b y^{1/\alpha} \prod_{i=1}^n w_i^{\beta_i}, \end{aligned}$$

for $j = 1, \dots, n$. So the cost function is

$$c(y, w) = \alpha b y^{1/\alpha} \prod_{i=1}^n w_i^{\beta_i},$$

which is a Cobb–Douglas function of w with constant returns to scale.

Note that

$$\frac{\partial c(y, w)}{\partial y} = b y^{(1-\alpha)/\alpha} \prod_{i=1}^n w_i^{\beta_i} = \hat{\lambda},$$

and

$$\frac{\partial c(y, w)}{\partial w_j} = \alpha \frac{\beta_j}{w_j} b y^{(1-\alpha)/\alpha} \prod_{i=1}^n w_i^{\beta_i} = \hat{x}_j(y, w).$$

4 Generalized Arrow–Chenery–Minhaus–Solow Production Function

$$y = \gamma \left(\sum_{i=1}^n \alpha_i x_i^\rho \right)^{\frac{1}{\rho}}.$$

where $\alpha_i > 0$, $\rho < 1$, $\rho \neq 0$. This production function exhibits constant returns to scale.

Trust me, and rewrite the constraint as $\gamma^\rho \sum_i \alpha_i x_i^\rho - y^\rho = 0$. The Lagrangean is

$$\sum_{i=1}^n w_i x_i - \lambda \left(\gamma^\rho \sum_{i=1}^n \alpha_i x_i^\rho - y^\rho \right).$$

The first order conditions are:

$$w_i - \lambda \gamma^\rho \alpha_i \rho x_i^{\rho-1} = 0,$$

so

$$x_i^{1-\rho} = \lambda \alpha_i \rho \gamma^\rho / w_i.$$

or

$$x_i = (\lambda \alpha_i \rho \gamma^\rho / w_i)^{1/(1-\rho)}. \quad (2)$$

To solve for λ , we need to substitute this back into the production function, which I'll do in steps.

$$\begin{aligned} x_i &= (\lambda \alpha_i \rho \gamma^\rho / w_i)^{1/(1-\rho)} \\ x_i^\rho &= (\lambda \alpha_i \rho \gamma^\rho / w_i)^{\rho/(1-\rho)} \\ \alpha_i x_i^\rho &= \alpha_i (\lambda \alpha_i \rho \gamma^\rho / w_i)^{\rho/(1-\rho)} \\ \sum_{i=1}^n \alpha_i x_i^\rho &= (\lambda \rho \gamma^\rho)^{\rho/(1-\rho)} \sum_{i=1}^n \alpha_i (\alpha_i / w_i)^{\rho/(1-\rho)} \\ \gamma^\rho \sum_{i=1}^n \alpha_i x_i^\rho &= \gamma^\rho (\lambda \rho \gamma^\rho)^{\rho/(1-\rho)} \sum_{i=1}^n \alpha_i (\alpha_i / w_i)^{\rho/(1-\rho)} \\ y^\rho &= \gamma^\rho (\lambda \rho \gamma^\rho)^{\rho/(1-\rho)} \sum_{i=1}^n \alpha_i (\alpha_i / w_i)^{\rho/(1-\rho)} \\ y &= \gamma (\lambda \rho \gamma^\rho)^{1/(1-\rho)} \left(\sum_{i=1}^n \alpha_i (\alpha_i / w_i)^{\rho/(1-\rho)} \right)^{1/\rho} \\ &= (\lambda \rho \gamma)^{1/(1-\rho)} \left(\sum_{i=1}^n \alpha_i^{\frac{1}{1-\rho}} w_i^{\frac{\rho}{1-\rho}} \right)^{1/\rho} \end{aligned}$$

Solving for $\hat{\lambda}$:

$$\hat{\lambda}^{\frac{1}{1-\rho}} = y \cdot (\gamma\rho)^{\frac{-1}{1-\rho}} \left(\sum_i \alpha_i^{\frac{1}{1-\rho}} w_i^{\frac{\rho}{\rho-1}} \right)^{\frac{-1}{\rho}}. \quad (3)$$

Substituting (3) into (2) gives

$$\begin{aligned} \hat{x}_i(y, w) &= y \cdot (\gamma\rho)^{\frac{-1}{1-\rho}} \underbrace{\left(\sum_k \alpha_k^{\frac{1}{1-\rho}} w_k^{\frac{\rho}{\rho-1}} \right)^{\frac{-1}{\rho}}}_{\lambda^{\frac{1}{1-\rho}}} (\rho\gamma^{\rho}\alpha_i)^{\frac{1}{1-\rho}} w_i^{\frac{\rho-1}{\rho}} \\ &= y \cdot \gamma^{-1} \alpha_i^{\frac{1}{1-\rho}} w_i^{\frac{\rho-1}{\rho}} \left(\sum_k \alpha_k^{\frac{1}{1-\rho}} w_k^{\frac{\rho}{\rho-1}} \right)^{\frac{-1}{\rho}}. \end{aligned} \quad (4)$$

Thus

$$\begin{aligned} w_i \hat{x}_i &= y \cdot \gamma^{-1} \cdot \alpha_i^{\frac{1}{1-\rho}} w_i^{\frac{\rho}{\rho-1}} \cdot \left(\sum_k \alpha_k^{\frac{1}{1-\rho}} w_k^{\frac{\rho}{\rho-1}} \right)^{\frac{-1}{\rho}} \\ \sum_i w_i \hat{x}_i &= y \cdot \gamma^{-1} \cdot \left(\sum_i \alpha_i^{\frac{1}{1-\rho}} w_i^{\frac{\rho}{\rho-1}} \right) \cdot \left(\sum_k \alpha_k^{\frac{1}{1-\rho}} w_k^{\frac{\rho}{\rho-1}} \right)^{\frac{-1}{\rho}} \\ \sum_i w_i \hat{x}_i &= y \cdot \gamma^{-1} \cdot \left(\sum_i \alpha_i^{\frac{1}{1-\rho}} w_i^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}. \end{aligned}$$

Or setting $\beta_i = \alpha_i^{\frac{1}{1-\rho}}$ and $\sigma = \frac{\rho}{\rho-1}$ gives the cost function as

$$c(y, w) = y\gamma^{-1} \left(\sum_i \beta_i w_i^{\sigma} \right)^{\frac{1}{\sigma}}.$$

This has the form of an ACMS function with parameter σ instead of ρ .

You can verify that $\frac{\partial c(y, w)}{\partial w_i} = \hat{x}_i(y, w)$ using (4). But note that $\frac{\partial c(y, w)}{\partial y} \neq \hat{\lambda}$. Why not?