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Introduction to Correspondences

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In these particular notes, all spaces are metric spaces. The set of extended real numbers, $\boldsymbol{R} \cup \{\infty, -\infty\}$, is denoted $\boldsymbol{R}^\sharp$.

1 Correspondences

There are many instances when a set of points depends on a parameter. For instance:

- In economics, the budget set $\beta(p, m) = \{x \in \boldsymbol{R}^{\mathrm{n}}_+ : p \cdot x \leqslant m\}$ is the set of commodity vectors x that can be bought with income m at the vector of prices p. This set depends on the parameter vector $(p, m) \in \boldsymbol{R}^{\mathrm{n}}_+ \times \boldsymbol{R}_+$.
- In a metric space (X, d), the open ball $B_\varepsilon(x) = \{y \in X : d(y, x) < \varepsilon\}$ is a set that depends on the parameter vector $(x, \varepsilon) \in X \times \boldsymbol{R}_{++}$.
- For a family of constrained optimization problems

  $$\text{maximize } f(x) \text{ subject to } g(x) = \alpha,$$

  the constraint set $\{x \in X : g(x) = \alpha\}$ depends on the parameter α.

In order to capture this dependence we would like to have a notion of a set-valued function. The seemingly obvious idea a function $f \colon X \to 2^Y$ from a set X in to the set of subsets of Y may not be the best choice. The problem comes when we try to imagine its graph, which is a subset of $X \times 2^Y$. A simpler idea is what we call a correspondence, which is just another name for a binary relation from X to Y.

Definition 1 *A* ***correspondence*** φ *from* X *to* Y *associates to each point in* X *a subset* $\varphi(x)$ *of* Y*. We write this as* $\varphi\colon X \twoheadrightarrow Y$*. For a correspondence* $\varphi\colon X \twoheadrightarrow Y$*, let* $\operatorname{gr}\varphi$ *denote the* ***graph*** *of* φ*, which we define to be*

$$\operatorname{gr}\varphi = \{(x,y) \in X \times Y : y \in \varphi(x)\}.$$

Let $\varphi\colon X \twoheadrightarrow Y$*, and let* $F \subset X$*. The* ***image*** $\varphi(F)$ *of* F *under* φ *is defined by*

$$\varphi(F) = \bigcup_{x\in F} \varphi(x).$$

The graph of a correspondence as we have defined it is perhaps possible to visualize if X and Y are subsets of the line or plane.

2 Inverse images

For correspondences there are two useful notions of inverse.

Definition 2 *The* ***upper*** *(or* ***strong****)* ***inverse*** *of* E *under* φ*, denoted* $\varphi^u[E]$*, is defined by*

$$\varphi^u[E] = \{x \in X : \varphi(x) \subset E\}.$$

The ***lower*** *(or* ***weak****)* ***inverse*** *of* E *under* φ*, denoted* $\varphi^\ell[E]$*, is defined by*

$$\varphi^\ell[E] = \{x \in X : \varphi(x) \cap E \neq \varnothing\}.$$

For a single y *in* Y*, define*

$$\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\}.$$

Note that $\varphi^{-1}(y) = \varphi^\ell[\{y\}]$*.*

Berge [5] used the notation φ^+ and φ^- for the upper and lower inverses respectively, but that notation is anathema my friend Roko, who was a lattice theorist.

3 Notions of continuity for functions

Let (X,d) and (Y,ρ) be metric spaces, and let $f\colon X \to Y$ be a *function*. (For a review of the important concepts relating to metric spaces see my notes at `http://www.hss.caltech.edu/~kcb/Notes/MetricSpaces.pdf`.) The two standard, equivalent definitions of continuity are these.

Definition 3 (Topological definition of continuity) *The function f is* ***continuous at x*** *if whenever G is a neighborhood of $f(x)$, then $f^{-1}[G]$ is a neighborhood of x. That is, for every open set G with $f(x) \in G$, there exists an open set U with $x \in U$ such that*

$$z \in U \implies f(z) \in G.$$

Note that this definition is topological, that is it uses only the notion of neighborhoods and open sets and does not mention the metrics. The next definition makes explicit use of the metrics.

Definition 4 (Metric definition of continuity) *The function f is* ***continuous at x*** *if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that*

$$d(x,z) < \delta \implies \rho\big(f(x), f(z)\big) < \varepsilon.$$

The following proposition is a simple consequence of the definition of an open set in a metric space, but let's walk through it to see where problems arise when trying to define continuity for correspondences.

Proposition 5 *A function f is continuous at x in the sense of Definition 3 if and only if it is in the sense of Definition 4.*

Proof: ($\implies$) Assume f is continuous at x in the sense of Definition 3, and let $\varepsilon > 0$ be given. Then $G = B_\varepsilon\big(f(x)\big) = \{y : \rho\big(y, f(x)\big) < \varepsilon\}$ is an open set with $f(x) \in G$, so there is some open set U containing x such that $z \in U \implies f(z) \in G$. By definition of openness of U, there is a $\delta > 0$ such that $d(z,x) < \delta \implies z \in U$. Thus

$$d(z,x) < \delta \implies \rho\big(f(z), f(x)\big) < \varepsilon,$$

that is, f is continuous at x in the sense of Definition 4.

($\impliedby$) Assume f is continuous at x in the sense of Definition 4, and let G be an open set containing $f(x)$. By definition of openness of G, there is some $\varepsilon > 0$ such that $\rho\big(y, f(x)\big) < \varepsilon \implies y \in G$. Then there is a $\delta > 0$ such that $d(x,z) < \delta \implies \rho\big(f(z), f(x)\big) < \varepsilon$. That is, letting U denote the open set $B_\delta(x)$, we have

$$z \in U \implies f(z) \in G,$$

so f is continuous at x in the sense of Definition 3. ∎

4 Notions of continuity for correspondences

To try to apply Definition 3 or Definition 4 to correspondences we first have to confront the problem that there are two natural notions of inverse image, so there are two natural versions of each definition. But there is another problem, namely, the topological and metric versions are not equivalent. But first we have to figure out the appropriate statement of the metric ε-δ definition. Recall that $d(x, A)$ is defined to be $\inf_{z \in A} d(x, z)$.

Definition 6 *Let A be set in a metric space (X, d). The **ε-neighborhood** $N_\varepsilon(A)$ of A is defined by*

$$N_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x) = \{x \in X : d(x, A) < \varepsilon\}.$$

Note that as union of open balls $B_\varepsilon(x)$, the set $N_\varepsilon(A)$ is open. If X is also a vector space with translation-invariant metric d, then we also have $N_\varepsilon(A) = A + B_\varepsilon(0)$.

Now let's start with the upper inverse. The ε-δ definition of upper hemicontinuity is this:

Definition 7 (Metric upper hemicontinuity) *A correspondence $\varphi\colon X \twoheadrightarrow Y$ satisfies **Property U_M at x** if the upper inverse of an ε-neighborhood of $\varphi(x)$ is a neighborhood of x. In other words, φ satisfies Property U_M at x if for every $\varepsilon > 0$, there is $\delta > 0$ such that*

$$d(z, x) < \delta \implies \varphi(z) \subset N_\varepsilon\big(\varphi(x)\big).$$

We say that φ satisfies Property U_M if it satisfies Property U_M at each $x \in X$.

The topological version is this:

Definition 8 (Topological upper hemicontinuity) *A correspondence $\varphi\colon X \twoheadrightarrow Y$ satisfies **Property U_T at x** if the upper inverse of an open neighborhood of $\varphi(x)$ is a neighborhood of x. In other words, φ satisfies Property U_T at x if for every open set G with $\varphi(x) \subset G$, there is an open set U such that*

$$z \in U \implies \varphi(z) \subset G.$$

We say that φ satisfies Property U_T if it satisfies Property U_T at each $x \in X$.

What is the relationship between these properties?

Proposition 9 *If $\varphi\colon X \twoheadrightarrow Y$ satisfies Property U_T at x, then it satisfies Property U_M at x.*

If $\varphi\colon X \twoheadrightarrow Y$ satisfies Property U_M at x and if $\varphi(x)$ is compact, then it satisfies Property U_T at x.

The first half of the proposition is straightforward and similar to Proposition 5, the second half relies on the following lemma, the proof of which is straightforward.

Lemma 10 *Let G be an open subset of a metric space (X, d) and let K be a nonempty compact subset of G. Then there is an $\varepsilon > 0$ such that*

$$K \subset N_\varepsilon(K) \subset G.$$

Thus Property U_T is stronger than Property U_M for correspondences that do not have compact values, and the next example shows that it may be so strong as to rule out many natural correspondences.

Example 11 The correspondence $\varphi\colon \boldsymbol{R}^{\mathrm{m}}_{+} \twoheadrightarrow \boldsymbol{R}^{\mathrm{m}}$ defined by

$$\varphi(x) = \{y \in \boldsymbol{R}^{\mathrm{m}} : y \leqq x\}$$

is satisfies Property U_M, but does not satisfy Property U_T.

To see that this violates Property U_T, let $n = 2$ and let

$$G = \{(x, y) \in \boldsymbol{R}^2 : y < 1/|x| \text{ if } x < 0\}.$$

Note that this include the entire right half-plane as well as $\varphi\big((0,0)\big)$. But for any $\delta > 0$, the set $\varphi\big((\delta,\delta)\big)$ contains $(x, \delta/2)$, which does not belong to G if $x < 0$ and $|x| > 2/\delta$. Thus φ violates Property U_T.

I'll leave it to you to show that φ does satisfy Property U_M.

This example shows that it might be impossible for a correspondence satisfy Property U_T if is not compact-valued. (Although constant correspondences are the useful exception.)

□

So either Property U_T or U_M can be used as the basis for a definition of continuity. In the past I've preferred U_T (e.g., [1, 6]), but now I may wish to join other authors, e.g., Jean-Pierre Aubin [2], who prefer U_M.

Definition 12 *A correspondence* $\varphi\colon X \twoheadrightarrow Y$ *is* ***upper hemicontinuous (uhc) at x*** *if whenever* x *is in the upper inverse of an open set, so is a neighborhood of* x*; and* φ *is* ***lower hemicontinuous (lhc) at x*** *if whenever* x *is in the lower inverse of an open set so is a neighborhood of* x*.*

In other words, φ *is uhc at* x *if for any open set* $G \subset Y$*, if*

$$\varphi(x) \subset G,$$

then there is an open set $U \subset X$ *containing* x *such that*

$$z \in U \implies \varphi(z) \subset G.$$

And φ *is lhc at* x *if for any open set* $G \subset Y$*, if*

$$\varphi(x) \cap G \neq \varnothing,$$

then there is an open set $U \subset X$ *containing* x *such that*

$$z \in U \implies \varphi(z) \cap G \neq \varnothing.$$

The correspondence $\varphi\colon X \twoheadrightarrow Y$ *is* ***upper hemicontinuous*** *if it is upper hemicontinuous at every* $x \in X$*. The correspondence* $\varphi\colon X \twoheadrightarrow Y$ *is* ***lower hemicontinuous*** *if it is lower hemicontinuous at every* $x \in X$*. Thus* φ *is upper hemicontinuous if the upper inverses of open sets are open and* φ *is lower hemicontinuous if the lower inverses of open sets are open.*

A correspondence is ***continuous*** *if it is both upper and lower hemicontinuous.*

Warning! As I mentioned above, my definition of upper hemicontinuity is not standard. In particular, Berge [4, 5] requires in addition that φ have compact values in order to be called upper hemicontinuous. See Moore [9] for a catalog of related definitions. It is true that the most interesting results apply to compact-valued correspondences (see, e.g., Example 23 below), but it seems to me (and others, e.g., Phelps [10, Definition 7.2, p. 102]) to make upper hemicontinuity and compact values separate properties.

If $\varphi\colon X \twoheadrightarrow Y$ is singleton-valued its graph cannot be distinguished from the graph of an ordinary function from X to Y and we may sometimes identify the two. In this case the upper and lower inverses of a set coincide and agree with the inverse regarded as a function. Either form of hemicontinuity is equivalent to continuity as a function. The term "semicontinuity" has sometimes been used to mean hemicontinuity (indeed Phelps uses it), but this usage can lead to confusion when discussing real-valued singleton correspondences. A semicontinuous real-valued function is not a hemicontinuous correspondence unless it is also continuous.

Definition 13 *The correspondence $\varphi \colon E \twoheadrightarrow F$ is **closed at x** if whenever $x_n \to x$, $y_n \in \varphi(x_n)$, and $y_n \to y$, then $y \in \varphi(x)$. A correspondence is **closed** if it is closed at every point of its domain, that is, if its graph is closed.*

Fact 14 *If φ is closed at x, then $\varphi(x)$ is a closed set.*

Proof: Let y_n be a sequence in $\varphi(x)$ with $y_n \to y$. Define $x_n = x$ for all n, so $x_n \to x$ and $y_n \in \varphi(x_n)$. Since φ is closed at x, we have $y \in \varphi(x)$. This shows that $\varphi(x)$ is a closed set. ∎

Example 15 (Closedness vs. Upper Hemicontinuity) In general, a correspondence may be closed without being upper hemicontinuous, and vice versa.

Define $\varphi \colon \boldsymbol{R} \twoheadrightarrow \boldsymbol{R}$ via

$$\varphi(x) = \begin{cases} \{\frac{1}{x}\} & x \neq 0 \\ \{0\} & x = 0. \end{cases}$$

Then φ is closed but not upper hemicontinuous.

Define $\mu \colon \boldsymbol{R} \twoheadrightarrow \boldsymbol{R}$ via $\mu(x) = (0, 1)$. Then μ is upper hemicontinuous in my sense, but not closed.[1] □

However, a closed-valued upper hemicontinuous correspondence is a closed correspondence, and if the range space Y is compact, then a closed correspondence is upper hemicontinuous.

Proposition 16 *Let $\varphi \colon X \twoheadrightarrow Y$ be a upper hemicontinuous at x and assume that $\varphi(x)$ is a closed set. Then φ is closed at x.*

Moreover, if Y is compact, and φ is closed at x, then φ is upper hemicontinuous at x.

[1] Again, under Berge's definition, an upper hemicontinuous correspondence is automatically closed.

Proof: Let φ be upper hemicontinuous at x and assume $\varphi(x)$ is closed. Let $x_n \to x$, $y_n \in \varphi(x_n)$, and $y_n \to y$. Suppose by way of contradiction that $y \notin \varphi(x)$. Since $\varphi(x)$ is closed, there are disjoint open sets G and U in Y with $\varphi(x) \subset G$ and $y \in U$. Since φ is upper hemicontinuous at x, the upper inverse $\varphi^{\mathrm{u}}[G]$ is an open neighborhood of x. Since $x_n \to x$, there is some N such that $n \geqslant N$ implies $x_n \in \varphi^{\mathrm{u}}[G]$ or $\varphi(x_n) \subset G$. In particular, $y_n \in \varphi(x_n) \subset G$, so $y_n \notin U$. Thus $y_n \not\to y$, a contradiction. This establishes that φ is closed at x.

Now assume that Y is compact and φ is closed at x. Assume by way of contradiction that there is some open set G with $\varphi(x) \subset G$ but $\varphi^{\mathrm{u}}[G]$ is not a neighborhood of x. This means that for every n there is some x_n in $B_{1/n}(x)$, the ball of radius $1/n$ centered at x, such that $\varphi(x_n) \not\subset G$. Thus there exists $y_n \in \varphi(x_n)$ with $y_n \notin G$. Since Y is compact, the sequence y_n has a subsequence converging to some $y \in Y$. Along this subsequence we have $x_n \to x$, $y_n \in \varphi(x_n)$, $y_n \to y$ so since φ is closed at x we must have $y \in \varphi(x) \subset G$. But no $y_n \in G$, which contradicts $y_n \to y$. This contradiction establishes that for every open set G with $\varphi(x) \subset G$ it must be the case that $\varphi^{\mathrm{u}}[G]$ is a neighborhood of x. That is, φ is upper hemicontinuous at x. ∎

Remark 17 We can weaken the assumption that the range space Y is compact. What we need is for each x there is a neighborhood whose image is included in some compact set. So let us say that φ is **locally bounded at *x*** if there is a neighborhood U of x in X and a compact set $K \subset Y$ such that $\varphi(U) \subset K$. Then the argument above proves the following result.

Corollary 18 *Let $\varphi\colon X \twoheadrightarrow Y$ be locally bounded and closed at x. Then φ is upper hemicontinuous at x.*

Corollary 19 *Let $\varphi\colon X \twoheadrightarrow Y$.*

1. *If φ is upper hemicontinuous and closed-valued, then φ is closed.*

2. *If φ is closed and locally bounded everywhere, then φ is upper hemicontinuous.*

Proposition 20 (Hemicontinuity and sequences) *Let $\varphi\colon X \twoheadrightarrow Y$.*

1. *If φ is nonempty compact-valued, then φ is upper hemicontinuous at x if and only if for every sequence $x_n \to x$ and $y_n \in \varphi(x_n)$ there is a convergent subsequence of y_n with limit in $\varphi(x)$.*

2. *Then φ is lower hemicontinuous at x if and only if $x_n \to x$ and $y \in \varphi(x)$ imply that there is a sequence $y_n \in \varphi(x_n)$ with $y_n \to y$.*

Proof: (1.) ($\implies$) Suppose φ is upper hemicontinuous at x, $x_n \to x$ and $y_n \in \varphi(x_n)$. Since φ is compact-valued, $\varphi(x)$ has a bounded neighborhood U. Since φ is upper hemicontinuous, there is a neighborhood V of x such that $\varphi(V) \subset U$. Thus y_n is eventually in U, thus bounded, and so has a convergent subsequence. Since compact sets are closed, this limit belongs to $\varphi(x)$.

($\impliedby$) Now suppose that for every sequence $x_n \to x$, $y_n \in \varphi(x_n)$, there is a subsequence of y_n with limit in $\varphi(x)$. Suppose φ is not upper hemicontinuous; then there is a neighborhood U of x and a sequence $z_n \to x$ with $y_n \in \varphi(z_n)$ and $y_n \notin U$. Such a sequence y_n can have no subsequence with limit in $\varphi(x)$, a contradiction.

(2.) ($\implies$) Assume φ is lower hemicontinuous at x and that $x_n \to x$ and $y \in \varphi(x)$. Then for each natural number k, the set $G_k = \{z \in Y : d(z, y) < 1/k\}$ is an open set with $y \in G_k \cap \varphi(x) \neq \varnothing$. Since φ is lower hemicontinuous at x, the lower inverse $\varphi^\ell[G_k]$ is a neighborhood of x. Since $x_n \to x$, there is some $n(k)$ such that for all $n \geqslant n(k)$, we have $x_n \in \varphi^\ell[G_k]$, that is, $\varphi(x_n) \cap G_k \neq \varnothing$. Without loss of generality we may take $n(k+1) > n(k)$ for all k. For $n < n(1)$ pick any $y_n \in \varphi(x_n)$. For $n(k) \leqslant n < n(k+1)$ pick $y_n \in \varphi(x_n) \cap G_k$. Then $n \geqslant n(k)$ implies $d(y_n, y) < 1/k$, so $y_n \to y$ as desired.

($\impliedby$) Assume that whenever $x_n \to x$ and $y \in \varphi(x)$, there is a sequence $y_n \in \varphi(x_n)$ with $y_n \to y$. Let G be an open subset of Y with $\varphi(\bar{x}) \cap G \neq \varnothing$, and let $\bar{y} \in \varphi(\bar{x}) \cap G$. Let $U = \varphi^\ell[G]$. We need to show that U is a neighborhood of $\bar{x}$. Suppose by way of contradiction that U is not a neighborhood of $\bar{x}$. Then for each natural number n, the ball $B_{1/n}(\bar{x})$ is not a subset of U, so pick $x_n \in B_{1/n}(\bar{x}) \cap U^c$. Then $x_n \to \bar{x}$, but $\varphi(x_n) \cap G = \varnothing$ for each n. On the other hand, by hypothesis there are $y_n \in \varphi(x_n)$ with $y_n \to \bar{y}$. Thus there is some n_0 such that $n \geqslant n_0$ implies $y_n \in G$. But then for $n \geqslant n_0$, we have $y_n \in \varphi_n \cap G = \varnothing$, a contradiction. Therefore $U = \varphi^\ell[G]$ is a neighborhood of $\bar{x}$. ∎

Proposition 21 *Let $\varphi, \mu \colon X \twoheadrightarrow Y$, and define $(\varphi \cap \mu) \colon X \twoheadrightarrow Y$ pointwise by $(\varphi \cap \mu)(x) = \varphi(x) \cap \mu(x)$. Suppose $\varphi(x) \cap \mu(x) \neq \varnothing$.*

1. *If φ and μ are upper hemicontinuous at x and closed-valued, then the correspondence $(\varphi \cap \mu)$ is upper hemicontinuous at x.*

2. *If μ is closed at x and φ is upper hemicontinuous at x and $\varphi(x)$ is compact, then $(\varphi \cap \mu)$ is upper hemicontinuous at x.*

Proof: Let U be an open neighborhood of $\varphi(x) \cap \mu(x)$. Put $C = \varphi(x) \cap U^c$.

1. Note that C is closed and $\mu(x) \cap C = \varnothing$. Thus there are disjoint open sets V_1 and V_2 with $\mu(x) \subset V_1$ and $C \subset V_2$. Since μ is upper hemicontinuous at x, there is a neighborhood W_1 of x with $\mu(W_1) \subset V_1 \subset V_2^c$. Now $\varphi(x) \subset U \cup V_2$, which is open and so x has a neighborhood W_2 with $\varphi(W_2) \subset U \cup V_2$, as φ is upper hemicontinuous at x. Put $W = W_1 \cap W_2$. Then for $z \in W$, $\varphi(z) \cap \mu(z) \subset V_2^c \cap (U \cup V_2) \subset U$. Thus $(\varphi \cap \mu)$ is upper hemicontinuous at x.

2. Note that in this case C is compact and $\mu(x) \cap C = \varnothing$. Since μ is closed at x, if $y \notin \mu(x)$ then we cannot have $y_n \to y$, where $y_n \in \mu(x_n)$ and $x_n \to x$. Thus there is a neighborhood U_y of y and W_y of x with $\mu(W_y) \subset U_y^c$. Since C is compact, we can write $C \subset V_2 = U_{y^1} \cup \cdots \cup U_{y_n}$; so setting $W_1 = W_{y^1} \cap \cdots \cap W_{y_n}$, we have $\mu(W_1) \subset V_2^c$. The rest of the proof is as in part 1. ∎

5 Exercises and examples

Example 22 Every constant correspondence, $\varphi(x) = A$ for all x, is both upper and lower hemicontinuous. □

Example 23 Let $\boldsymbol{R}^{\mathrm{m}}_{+} = \{x \in \boldsymbol{R}^{\mathrm{m}} : x \geqq 0\}$. The compact-valued correspondence $\varphi\colon \boldsymbol{R}^{\mathrm{m}}_{+} \twoheadrightarrow \boldsymbol{R}^{\mathrm{m}}_{+}$ defined by

$$\varphi(x) = \{y \in \boldsymbol{R}^{\mathrm{m}}_{+} : y \leqq x\}$$

is continuous.

I will prove that φ is upper hemicontinuous. Let G be an open subset of $\boldsymbol{R}^{\mathrm{m}}_{+}$ and assume $x \in \varphi^{\mathrm{u}}[G]$. We need to find a $\delta > 0$ so that if $d(x, x') < \delta$, then $\varphi(x') \subset G$.

The set $F = G^c$ is closed, and the distance function $d(y, F) = \inf_{z \in F} d(y, z)$ is continuous. It therefore achieves a minimum value m on the compact set $\varphi(x)$. Since $\varphi(x)$ and F are disjoint closed sets the minimum value m satisfies $m > 0$. Set $\delta = m/2$. Then the neighborhood $N_\delta\bigl(\varphi(x)\bigr) = \bigcup_{z \in \varphi(x)} B_\delta(z)$ of $\varphi(x)$ is disjoint from F, which is to say it is included in G.

Now note that for any $x' \in B_\delta(x)$, every $z' \in \varphi(x')$ lies within δ of some point $z \in \varphi(x)$ (namely $z = z' + x - x'$), so $z' \in G$. Thus $\varphi(x') \subset G$.

The lower hemicontinuity of φ is straightforward. □

Exercise 24 *If the graph of φ is open, then φ is lower hemicontinuous.*

Exercise 25 *Let $f\colon X \to \boldsymbol{R}$ be continuous. What can we say about these correspondences:*

$$
\begin{aligned}
x &\twoheadrightarrow \{y \in X : f(y) \geqslant f(x)\} \\
\alpha &\twoheadrightarrow \{y \in X : f(y) \geqslant \alpha\} \\
x &\twoheadrightarrow \{y \in X : f(y) > f(x)\} \\
\alpha &\twoheadrightarrow \{y \in X : f(y) > \alpha\} \\
x &\twoheadrightarrow \{y \in X : f(y) = f(x)\} \\
\alpha &\twoheadrightarrow \{y \in X : f(y) = \alpha\}.
\end{aligned}
$$

What if f is only upper semicontinuous?

Exercise 26 *Let $f, g\colon X \to \boldsymbol{R}$ be continuous and assume that $f(x) \geqslant g(x)$ for all x. Show that*

$$x \twoheadrightarrow \{\alpha \in \boldsymbol{R} : f(x) \geqslant \alpha \geqslant g(x)\}$$

is continuous.

Example 27 The classical budget space is

$$\mathcal{B} = \{(p, m) \in \boldsymbol{R}^{\mathrm{n}} \times \boldsymbol{R} : p \gg 0,\ m > 0\}.$$

Define $\mathcal{B}_0 = \{(p, m) \in \boldsymbol{R}^{\mathrm{n}} \times \boldsymbol{R} : p \geqq 0,\ m \geqslant 0\}$. The budget correspondence

$$\beta(p, m) = \{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : p \cdot x \leqslant m\}$$

is compact-valued and continuous on $\mathcal{B}$. It is closed, but not lower hemicontinuous on $\mathcal{B}_0$.

Upper hemicontinuity on $\mathcal{B}$ follows from the fact that the graph of β is clearly closed, and it is easy to see that β is locally bounded on $\mathcal{B}$.

For lower hemicontinuity at a point $(\bar{p}, \bar{m}) \in \mathcal{B}$, assume G is open and let $\bar{x} \in \beta(\bar{p}, \bar{m}) \cap G$.

There are two cases: Case 1 is that $\bar{x} \neq 0$. In this case, pick some $\hat{x} \neq \bar{x}$ satisfying $\hat{x} \leqq \bar{x}$ and $\hat{x} \in G$. Then $\bar{p} \cdot \hat{x} < \bar{m}$, since $\bar{p} \gg 0$. Thus $U = \{(p, m) \in \mathcal{B} : p \cdot \hat{x} < m\}$ is an open neighborhood of $(\bar{p}, \bar{m})$ such that for every $(p, m) \in U$, we have $\hat{x} \in \beta(p, m) \bigcap G \neq \varnothing$, which shows that β is lhc at $(\bar{p}, \bar{m})$. Case 2 is that $\bar{x} = 0$. In this case, $U = \mathcal{B}$ is an open neighborhood of $(\bar{p}, \bar{m})$ satisfying $\bar{x} \in \beta(p, m) \bigcap G \neq \varnothing$, for all $(p, m) \in U$. This proves that β is lhc at $\bar{p}, \bar{m})$.

Now if $\bar{m} = 0$ and some $\bar{p}_j = 0$, then $\bar{x} = e^j$ (the j^{th} unit coordinate vector) we have $\bar{x} \in \beta(\bar{p}, \bar{m})$. Now consider the sequence $(p_n, m_n) \to (\bar{p}, \bar{m})$ defined by $p_n = \bar{p} + (1/n)e^j$ and $m_n = 1/n^2$. If $x \in \beta(p_n, m_n)$, then $x_j \leqslant 1/n$, so if $x_n \in \beta(p_n, m_n)$ we cannot have $x_n \to \bar{x}$, so by Propostion 20, β is not lhc at $(\bar{p}, \bar{m})$. □

6 The Berge maximum theorem

One of the most useful and powerful theorems employed in mathematical economics and game theory is the "maximum theorem" due to Berge [4]. It deals with the continuity of the solution and value of a parametrized family of constrained optimization problems. We start with a set P of *parameter* or *state* values, and a set X of *controls* or *actions* or *choice variables*. To each parameter value p, there is a *constraint set* or *feasible set* $\varphi(p)$ of controls,

$$\varphi\colon P \twoheadrightarrow X.$$

The goal is to maximize an *objective function* f over the feasible set. We allow f to depend on both the parameter and the control. The domain of f need not be all of $P \times X$, but it must include the graph of φ,

$$f\colon \operatorname{gr}\varphi \to \boldsymbol{R}.$$

We define the *optimal value function* $V\colon P \to \boldsymbol{R}^\sharp$ by

$$V(p) = \sup\{f(p, x) : x \in \varphi(p)\}.$$

Note that we allow this to be extended real-valued (the supremum may be ∞). And recall that $\sup \varnothing = -\infty$.

Definition 28 *Also recall that an extended real-valued function $f\colon Z \to \boldsymbol{R}^\sharp$ on a metric space is* ***upper semicontinuous*** *if for every $\alpha \in \boldsymbol{R}$, the upper contour set $\{z \in Z : f(z) \geqslant \alpha\}$ is closed (or equivalently $\{z \in Z : f(z) < \alpha\}$ is open). The function f is* ***lower semicontinuous*** *if for every $\alpha \in \boldsymbol{R}$, the lower contour set $\{z \in Z : f(z) \leqslant \alpha\}$ is closed (or equivalently $\{z \in Z : f(z) > \alpha\}$ is open).*

We can localize the concept and say that f is upper semicontinuous at z if $f(z) < \alpha$ implies that there is a neighborhood U of z such that $x \in U$ implies $f(x) < \alpha$. Lower semicontinuity at a point is defined similarly.

You should prove the following.

Fact 29 *A real-valued function is continuous if and only if it is both upper and lower semicontinuous.*

Proposition 30 (Lower semicontinuity of the optimal value function) *Let $\varphi\colon P \twoheadrightarrow X$ be lower hemicontinuous at p_0. Let $f\colon \operatorname{gr}\varphi \to \boldsymbol{R}$ be lower semicontinuous. Then the optimal value function V is lower semicontinuous at p_0.*

Proof: We need to prove that if $V(p_0) > \alpha$, then there is a neighborhood W of p_0 such that $p \in W$ implies $V(p) > \alpha$. So assume that $V(p_0) > \alpha$. Then there is some $x_0 \in \varphi(p_0)$ satisfying

$$f(p_0, x_0) > \alpha.$$

Since f is lower semicontinuous on $\operatorname{gr}\varphi$, there is an open neighborhood $U \times V$ of (p_0, x_0) such that for all $(p, x) \in \operatorname{gr}\varphi \cap U \times V$ we have $f(p, x) > \alpha$. Since φ is lower hemicontinuous, $W = \varphi^{\ell}[V] \cap U$ is a neighborhood of p_0. By the definition of lower inverse, for each $p \in W$ there is some $x \in \varphi(p) \cap V$, which implies $f(p, x) > \alpha$. Therefore $V(p) = \sup_{x \in \varphi(p)} f(x, p) > \alpha$ too. ∎

Proposition 31 (Upper semicontinuity of the optimal value function) *Assume the feasibility correspondence $\varphi\colon P \twoheadrightarrow X$ has nonempty compact values. Let $f\colon \operatorname{gr}\varphi \to \boldsymbol{R}$ be upper semicontinuous. Then the optimal value function V is actually a maximum,*

$$V(p) = \max\{f(x, p) : x \in \varphi(p)\}.$$

If φ is upper hemicontinuous at p_0, then V is upper semicontinuous at p_0.

Proof: We need to show that if $V(p_0) < \alpha$, then there is a neighborhood W of p_0 such that for all $p \in W$ we have $V(p) < \alpha$. Since $V(p_0) < \alpha$, for every $x \in \varphi(p_0)$ we have $f(p_0, x) < \alpha$. Since f is upper semicontinuous, for each $x \in \varphi(p_0)$, there is a neighborhood $W_x \times U_x$ of (p_0, x) such that for every $(p, z) \in W_x \times U_x$ we have $f(p, z) < \alpha$. Since $\varphi(p)$ is compact, there are finitely many $x_1, \ldots, x_k$ such that $U = \bigcup_{j=1}^{k} U_{x_j}$ includes $\varphi(p_0)$. Put $W = \bigcap_{j=1}^{k} W_{x_j}$. Then for any $(p,x) \in W \times U$ we have $f(p,x) < \alpha$. Moreover $\varphi(p_0) \subset U$, so $\varphi^{\mathrm{u}}[U]$ is a neighborhood of p_0. Now for any $p \in W \cap \varphi^{\mathrm{u}}[U]$ and any $x \in \varphi(p)$ we have $x \in U$ so $f(p,x) < \alpha$ which implies $V(p) < \alpha$. That is, V is upper semicontinuous at p_0. ∎

Berge Maximum Theorem *Let $\varphi\colon P \twoheadrightarrow X$ be a compact-valued correspondence. Let $f\colon \operatorname{gr}\varphi \to \boldsymbol{R}$ be continuous. Define the "argmax" correspondence $\mu\colon P \twoheadrightarrow X$ by*

$$\mu(p) = \{x \in \varphi(p) : x \text{ maximizes } f(p,\cdot) \text{ on } \varphi(p)\},$$

and the optimal value function $V\colon P \to \boldsymbol{R}$ by

$$V(p) = f(p,x) \qquad \text{for any } x \in \mu(p).$$

If φ is continuous at p, then μ is closed and upper hemicontinuous at p and V is continuous at p. Furthermore, μ is compact-valued.

Proof: Given the previous Propositions 30 and 31, all that remains is to show that μ is closed at p, for then $\mu = \varphi \cap \mu$ and Proposition 21 (2) implies that μ is upper hemicontinuous at p. Let $p_n \to p$, $x_n \in \mu(p_n)$, $x_n \to x$. We wish to show $x \in \mu(p)$. Since φ is upper hemicontinuous and compact-valued, Proposition 20 (1) implies that indeed $x \in \varphi(p)$. Suppose $x \notin \mu(p)$. Then there is $z \in \varphi(p)$ with $f(p,z) > f(p,x)$. Since φ is lower hemicontinuous at p, by Proposition 20 there is a sequence $z_n \to z$ with $z_n \in \varphi(p_n)$. Since $z_n \to z$, $x_n \to x$, and $f(p,z) > f(p,x)$, the continuity of f implies that eventually $f(p_n,z_n) > f(p_n,x_n)$, contradicting $x_n \in \mu(p_n)$. ∎

7 More about correspondences

Proposition 32 (Upper hemicontinuous image of a compact set) *Let $\varphi\colon K \twoheadrightarrow Y$ be upper hemicontinuous and compact-valued and let K be compact. Then $\varphi(K)$ is compact.*

Proof: Let $\{U_\alpha\}$ be an open covering of $\varphi(K)$. Since $\varphi(x)$ is compact, there is a finite subcover $U_x^1, \ldots, U_x^{n_x}$, of $\varphi(x)$. Put $V_x = U_x^1 \cup \cdots \cup U_x^{n_x}$. Then since φ is upper hemicontinuous, $\varphi^{\mathrm{u}}[V_x]$ is open and contains x. Hence K is covered by a finite number of $\varphi^{\mathrm{u}}[V_x]$s and the corresponding U_x^is are a finite cover of $\varphi(K)$. ∎

Proposition 33 (Sums of correspondences) *Let* $\varphi_i \colon X \twoheadrightarrow \boldsymbol{R}^{\mathrm{m}}$, $i = 1, \ldots, k$.

(a) *If each* φ_i *is upper hemicontinuous at* x *and compact-valued, then*

$$\sum_i \varphi_i \colon z \mapsto\!\!\!\rightarrow \sum_i \varphi_i(z)$$

is upper hemicontinuous at x *and compact-valued.*

(b) *If each* φ_i *is lower hemicontinuous at* x*, then the sum* $\sum_i \varphi_i$ *is lower hemicontinuous at* x.

Proof: Exercise. Hint: Use Proposition 20(1). ∎

Proposition 34 (Products of correspondences) *Let* $\gamma_i \colon E \twoheadrightarrow Y_i$, $i = 1, \ldots, k$.

(a) *If each* γ_i *is upper hemicontinuous at* x *and compact-valued, then*

$$\prod_i \gamma_i \colon z \mapsto\!\!\!\rightarrow \prod_i \gamma_i(z)$$

is upper hemicontinuous at x *and compact-valued.*

(b) *If each* γ_i *is lower hemicontinuous at* x*, then the product* $\prod_i \gamma_i$ *is lower hemicontinuous at* x.

(c) *If each* γ_i *is closed at* x*, then* $\prod_i \gamma_i$ *is closed at* x.

Proof: Exercise. ∎

References

[1] Aliprantis, C. D. and K. C. Border. 2006. *Infinite dimensional analysis: A hitchhiker's guide*, 3d. ed. Berlin: Springer–Verlag.

[2] Aubin, J.-P. 1998. *Optima and equilibria: An introduction to nonlinear analysis*, 2d. ed. Number 140 in Graduate Texts in Mathematics. Springer–Verlag.

[3] Aubin, J.-P. and H. Frankowska. 1990. *Set-valued analysis*. Boston: Birkhäuser.

[4] Berge, C. 1959. *Espaces topologiques et fonctions multivoques*. Paris: Dunod.

[5] Berge, C. 1997. *Topological spaces*. Mineola, New York: Dover. Originally published in French by Dunod, Paris, 1962 as *Espaces topologiques, fonctions multivoques*. Reprint of the English translation by E. M. Patterson, originally published by Oliver and Boyd, Edinburgh and London, 1963.

[6] Border, K. C. 1985. *Fixed point theorems with applications to economics and game theory*. New York: Cambridge University Press.

[7] Hildenbrand, W. 1974. *Core and equilibria of a large economy*. Princeton: Princeton University Press.

[8] Klein, E. and A. C. Thompson. 1984. *Theory of correspondences: Including applications to mathematical economics*. New York: John Wiley and Sons.

[9] Moore, J. C. 1968. A note on point-set mappings. In J. P. Quirk and A. M. Zarley, eds., *Papers in Quantitative Economics, 1*, pages 129–137. University of Kansas Press, Lawrence, Kansas.

[10] Phelps, R. R. 1993. *Convex functions, monotone operators and differentiability*, 2d. ed. Number 1364 in Lecture Notes in Mathematics. Berlin: Springer–Verlag.