CALIFORNIA INSTITUTE OF TECHNOLOGY
Division of the Humanities and Social Sciences

Core of a replica economy

KC Border*
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Consider a pure exchange economy $\mathcal{E}$ with m consumers and ℓ goods. (Each consumption set is $\boldsymbol{R}^\ell_+$.) The endowment of consumer i is ω^i and his preference relation is $\succcurlyeq_i$.

A **coalition** is a nonempty subset of consumers. An allocation $(x^1, \ldots, x^m)$ is **blocked by coalition S** if there is a partial allocation $(\tilde{x}^i)_{i \in S}$ such that

1. $\sum_{i \in S} \tilde{x}^i = \sum_{i \in S} \omega^i$.
2. For each $i \in S$, $\tilde{x}^i \succ_i x^i$.

The allocation is **weakly blocked** if (2) is replaced by

2′. For each $i \in S$, $\tilde{x}^i \succcurlyeq_i x^i$, and for some $k \in S$, $\tilde{x}^k \succ_k x^k$.

The **core** of the economy is the set of unblocked allocations.

Lemma 1 *If each preference relation is continuous and strictly monotonic, an allocation is blocked if and only if it is weakly blocked.*

Theorem 2 *Assume each preference relation is locally nonsatiated. Then every Walrasian equilibrium allocation is in the core.*

Proof: Let $(\bar{x}^1, \ldots, \bar{x}^m, p)$ be a Walrasian equilibrium, and suppose by way of contradiction that the allocation $(\bar{x}^1, \ldots, \bar{x}^m)$ is blocked. Then there is a coalition S and $(\tilde{x}^i)_{i \in S}$ satisfying

$$\tilde{x}^i \succ_i \bar{x}^i$$

for each $i \in S$ and

$$\sum_{i \in S} \tilde{x}^i = \sum_{i \in S} \omega^i. \tag{1}$$

Since preferences are locally nonsatiated, in equilibrium, all income is spent so $p \cdot \bar{x}^i = p \cdot \omega^i$. Also, by utility maximization subject to the budget constraint, we have

$$\tilde{x}^i \succ_i \bar{x}^i \implies p \cdot \tilde{x}^i > p \cdot \bar{x}^i = p \cdot \omega^i$$

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for each $i \in S$. Summing over S yields

$$p \cdot \sum_{i \in S} \tilde{x}^i > p \cdot \sum_{i \in S} \bar{x}^i = p \cdot \sum_{i \in S} \omega^i,$$

which contradicts (1). ∎

Replica economies

The n^{th} replica $\mathcal{E}_n$ of $\mathcal{E}$ has $n \times m$ consumers, n of each of m **types**. Consumers of type i have the same endowment ω^i and the same preference relation $\succcurlyeq_i$.

Lemma 3 (Equal treatment property) *Assume preferences are strictly monotonic, strictly convex, and continuous. Then in the core of a replica economy, consumers of the same type receive the same consumption.*

That is, let $(x^{1,1}, \ldots, x^{1,n}, \ldots, x^{m,1}, \ldots, x^{m,n})$ *belong to the core of* $\mathcal{E}_n$. *Then for each type* i, *and each* $j, k = 1, \ldots, n$ *we have*

$$x^{i,j} = x^{i,k}.$$

Proof: Let $(x^{1,1}, \ldots, x^{1,n}, \ldots, x^{m,1}, \ldots, x^{m,n})$ belong to the core of $\mathcal{E}_n$. Since every consumer of type i has the same preference relation, they can all agree on which of them has the worst consumption allocation $x^{i,j}$. (They may be indifferent, in which case any of them qualifies as having the worst allocation.) Form a coalition S that has one consumer of each type, that consumer having the worst allocation for his type. Consider the partial allocation $(\tilde{x}^i)_{i \in S}$ (here we are indexing members of S solely by their type) defined by

$$\tilde{x}^i = \frac{\sum_{j=1}^n x^{i,j}}{n}$$

Now by definition of an allocation

$$\sum_{i=1}^m \sum_{j=1}^n x^{i,j} = \sum_{i=1}^m \sum_{j=1}^n \omega^{i,j} = \sum_{i=1}^m n\omega^i.$$

Dividing by n we get

$$\sum_{i=1}^m \tilde{x}^i = \sum_{i=1}^m \frac{\sum_{j=1}^n x^{i,j}}{n} = \sum_{i=1}^m \omega^i.$$

Now suppose by way of contradiction that for some type i, we have unequal treatment. Then by strict convexity of preference, $\tilde{x}^i = \frac{1}{n}\sum_{j=1}^n x^{i,j} \succ_i x^{i,j^*(i)}$, where $(i, j^*(i))$ is the worst off of type i. Then S weakly blocks via $(\tilde{x}^1, \ldots, \tilde{x}^m)$, a contradiction. Thus we must have equal treatment. ∎

Given equal treatment, we can treat every core allocation in a replica economy, as if it were an allocation the original economy. (This is not true of general allocations, since an allocation in $\mathcal{E}_n$ actually belongs to $\boldsymbol{R}^{mn\ell}$, not $\boldsymbol{R}^{m\ell}$.)

Theorem 4 (Limit of the core) *Assume preferences are strictly monotonic, continuous, and strictly convex. Suppose the allocation $(\bar{x}^1, \ldots, \bar{x}^m)$ belongs to the core of $\mathcal{E}_n$ for each n. Then there exists a nonzero price vector $p \in \boldsymbol{R}^\ell$ such that $(\bar{x}^1, \ldots, \bar{x}^m, p)$ is a Walrasian quasi-equilibrium.*

Proof: The proof is similar to the proof of the second welfare theorem, but involves the initial endowment. For each $i = 1, \ldots, m$ define

$$P_i = \{z \in \boldsymbol{R}^\ell : \omega^i + z \succ_i \bar{x}^i\}.$$

That is, P_i is the set of net trades from ω^i that make a consumer of type i better off than his core allocation $\bar{x}^i$. Define

$$P = \text{convex hull} \bigcup_{i=1}^m P_i.$$

That is, P is the set of all vectors of the form $\sum_{i=1}^m \alpha_i z^i$ where each $z^i \in P_i$, $\alpha_i \geqslant 0$, and $\sum_{i=1}^m \alpha_i = 1$.

I claim that $0 \notin P$. To see why, note that the continuity of preferences implies that each P_i is open, so that their sum is open, which in turn implies that the convex hull is open. So assume by way of contradiction that 0 belongs to P. Then there is some strictly negative vector $v \ll 0$ that also belongs to P. We can thus write $v = \sum_{i=1}^m \alpha_i z^i$ where each $z^i \in P_i$, $\alpha_i \geqslant 0$, and $\sum_{i=1}^m \alpha_i = 1$. Moreover, by perturbing the coefficients α_i slightly we can assume that they are all rational. (The mapping $(\beta_1, \ldots, \beta_m) \to \sum_{i=1}^m \beta_i z^i$ is continuous.) Putting all the coefficients over a common denominator n we get

$$0 \gg v = \sum_{i=1}^m \frac{k_i}{n} z^i. \tag{2}$$

Consider now a coalition S that has k_i members of each type i, and consider the partial equal treatment allocation where each consumer in S of type i

receives

$$\tilde{x}^i = \omega^i + z^i.$$

By construction, z^i belongs to P_i, so $\tilde{x}^i \succ_i \bar{x}^i$. I now need to show that this partial allocation is feasible for the coalition S. But

$$\sum_{i \in S} k_i \tilde{x}^i = \sum_{i \in S} k_i \omega^i + z^i = \sum_{i \in S} k_i \omega^i + \sum_{i \in S} k_i z^i = \sum_{i \in S} k_i \omega^i + nv,$$

where the last equality follows from (2). Since $nv \ll 0$, the partial allocation adds up to less that coalition S's endowment. Since preferences are monotone, this difference can be given to anyone in S. The upshot is that $(\tilde{x}^i)$ blocks the allocation $(\bar{x}^i)$ in the n-replica economy $\mathcal{E}_n$, a contradiction. Therefore

$$0 \notin P.$$

We now use the separating hyperplane theorem to find the existence of a nonzero $p \in \boldsymbol{R}^\ell$ such that $p \cdot z \geqslant 0$ for all $z \in P$. In particular, for each i, if $z \in P_i$, then $p \cdot z \geqslant 0$. Another way to say the same thing is

$$x \succ_i \bar{x}^i \quad \implies \quad p \cdot x \geqslant p \cdot \omega^i.$$

Since preferences are locally nonsatiated, if $x \succcurlyeq_i \bar{x}^i$ there is a sequence $x_n \to x$ with $x_n \succ_i x \succcurlyeq_i \bar{x}^i$. Thus $p \cdot x_n \geqslant p \cdot \omega^i$, so

$$x \succcurlyeq_i \bar{x}^i \implies p \cdot x \geqslant p \cdot \omega^i.$$

In particular, $p \cdot \bar{x}^i \geqslant p \cdot \omega^i$ for each i, and since $\sum_{i=1}^m \bar{x}^i = \sum_{i=1}^m \omega^i$, we conclude that for each i,

$$p \cdot \bar{x}^i = p \cdot \omega^i.$$

Thus, $p \cdot \bar{x}^i = p \cdot \omega^i$ and $x \succcurlyeq_i \bar{x}^i$ implies $p \cdot x \geqslant p \cdot \bar{x}^i$, which proves that we have a Walrasian quasi-equilibrium. ∎

Suggested reading

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