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A Polyhedral Cone Counterexample

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Abstract

This is an example of a pointed generating convex cone in $\boldsymbol{R}^4$ with 5 extreme rays, but whose dual cone has 6 extreme rays (and vice-versa).

Recall that a **ray** in a vector space is the set of nonnegative scalar multiples of a single nonzero point. A **cone** is a nonempty subset C of a vector space that is closed under multiplication by nonnegative scalars. A cone is **trivial** it it contains only 0. A nontrivial cone is the union of the rays generated by its nonzero points. A cone C is **generating** if $C - C$ is the entire vector space, or equivalently if it spans the space. A **convex cone** is a cone that is a convex set. A set in a vector space is a convex cone if and only if it is closed under nonnegative linear combinations. A convex cone is **pointed** if it includes no lines. A ray A is an **extreme ray** of the cone C if it is a subset of C and if points on A cannot be written as a linear combination of linearly independent points in C, that is, if $x \in A$, $x = y + z$, $y, z \in C$ together imply that y and z are dependent. A **finite cone** is the convex cone generated by finitely many nonzero points. A finite cone has finitely many extreme rays, and a pointed finite cone is the convex hull of its extreme rays. Finally, the **dual cone** C^* of a cone $C \subset \boldsymbol{R}^{\mathrm{m}}$ is defined by

$$C^* = \{p \in \boldsymbol{R}^{\mathrm{m}} : p \cdot y \leqslant 0 \text{ for all } y \in C\}.$$

For a finite cone C (actually any closed convex cone), $C^{**} = C$. We shall use the following characterization of extreme rays of C^*:

Weyl's Facet Lemma *Let C be a finite cone in $\boldsymbol{R}^{\mathrm{m}}$ generated by $a_1, \ldots, a_n$. Then a nonzero point $p \in C^* \subset \boldsymbol{R}^{\mathrm{m}}$ is on an extreme ray of C^* if and only if $\{a_i : p \cdot a_i = 0\}$ has rank $m - 1$.*

See, e.g., D. Gale [8, Theorem 2.16, p. 65] for a proof of this result. (Warning: He omits the requirement that p be nonzero from the statement, but not the proof.) Or see Theorems 11–12 in H. Weyl [13], which are stated in terms of facets of cones. Note that a consequence of this is that the dual cone of a finite cone is also a finite cone.

Example

Consider the finite convex cone C in $\boldsymbol{R}^4$ generated by the set $\mathcal{A} = \{a_1, \dots, a_5\}$ where

$$a_n = \begin{bmatrix} 1 \\ n \\ n^2 \\ n^3 \end{bmatrix}.$$

Let A be the 4×5 matrix A with columns in $\mathcal{A}$:

$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \end{bmatrix} \end{array}$$

Then the cone C is just

$$C = \{Ax : x \geqq 0\}.$$

It is easy to verify that every subset of $\{a_1, \dots, a_5\}$ of size four is linearly independent. Thus the cone C spans $\boldsymbol{R}^4$, or in other words, it is generating. It is also easy to see that C is pointed (that is, it contains no lines, only half-lines), as it is a subset of the nonnegative cone.

I claim that the dual cone C^*

$$C^* = \{p \in \boldsymbol{R}^4 : p \cdot y \leqslant 0 \text{ for all } y \in C\} = \{p \in \boldsymbol{R}^4 : p'A \leqq 0\}$$

is generated by the 6 points $p_1, \dots, p_6$ that make up the 6 columns of the 4×6 matrix

$$P = \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{matrix} \\ \begin{bmatrix} -60 & -30 & -10 & 6 & 12 & 20 \\ 47 & 31 & 17 & -11 & -19 & -29 \\ -12 & -10 & -8 & 6 & 8 & 10 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

That is, $C^* = \{Pz : z \geqq 0\}$. Moreover, I claim that the cone C has five extreme rays (generated by $a_1, \ldots, a_5$), and C^* has six extreme rays (generated by $p_1, \ldots, p_6$).

Proof

The cone C^* is the set of solutions p to the system of inequalities

$$
\begin{gathered}
p \cdot a_1 \leqslant 0 \\
\vdots \\
p \cdot a_5 \leqslant 0
\end{gathered}
$$

We shall use Weyl's Lemma to find the extreme rays of C^*. In our example $m = 4$ and $n = 5$. We shall use the "brute force" approach and look at *all* subsets of $\mathcal{A} = \{a_1, \ldots, a_5\}$ of rank 3. Since any four vectors belonging to $\mathcal{A}$ are linearly independent, a subset of $\mathcal{A}$ has rank 3 if and only if it has three elements. Fortunately there are only $\binom{5}{3} = 10$ of these subsets, so it is feasible to enumerate them by hand. Each subset B of size three determines a one-dimensional subspace in $\boldsymbol{R}^4$ (a line) consisting of vectors orthogonal to each element of B (the **orthogonal complement** of B). It is straightforward to solve for this subspace, and I have done so. Points p_i taken from each of these ten lines are used for the columns of the 4×10 matrix

$$
\hat{P} = \begin{array}{c} \begin{array}{cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \end{array} \\ \begin{bmatrix} -60 & -30 & -10 & 6 & 12 & 20 & -40 & -24 & -15 & -8 \\ 47 & 31 & 17 & -11 & -19 & -29 & 38 & 26 & 23 & 14 \\ -12 & -10 & -8 & 6 & 8 & 10 & -11 & -9 & -9 & -7 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}
$$

(Note that you have seen $p_1, \ldots, p_6$ before.) Now construct the 5×10 matrix whose elements are the inner products $p_j \cdot a_i$:

$$
A'\hat{P} = \begin{array}{c|cccccccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \\ \hline a_1 & -24 & -8 & 0 & 0 & 0 & 0 & -12 & -6 & 0 & 0 \\ a_2 & -6 & 0 & 0 & 0 & -2 & -6 & 0 & 0 & 3 & 0 \\ a_3 & 0 & 0 & -4 & 0 & 0 & -4 & 2 & 0 & 0 & -2 \\ a_4 & 0 & -2 & -6 & -6 & 0 & 0 & 0 & 0 & -3 & 0 \\ a_5 & 0 & 0 & 0 & -24 & -8 & 0 & 0 & 6 & 0 & 12 \end{array}
$$

For the first six columns, all the entries are nonpositive, so $p_1, \ldots, p_6$ each belong to C^*. However for columns 7 through 10, there are entries of both signs. This means that for $i = 7, \ldots, 10$, no nonzero multiple of p_j belongs to C^*.

Further inspection shows that

$$\begin{aligned}
\{a_i : p_1 \cdot a_i = 0\} &= \{a_3, a_4, a_5\} \\
\{a_i : p_2 \cdot a_i = 0\} &= \{a_2, a_3, a_5\} \\
\{a_i : p_3 \cdot a_i = 0\} &= \{a_1, a_2, a_5\} \\
\{a_i : p_4 \cdot a_i = 0\} &= \{a_1, a_2, a_3\} \\
\{a_i : p_5 \cdot a_i = 0\} &= \{a_1, a_3, a_4\} \\
\{a_i : p_6 \cdot a_i = 0\} &= \{a_1, a_4, a_5\} \\
\{a_i : p_7 \cdot a_i = 0\} &= \{a_2, a_4, a_5\} \\
\{a_i : p_8 \cdot a_i = 0\} &= \{a_2, a_3, a_4\} \\
\{a_i : p_9 \cdot a_i = 0\} &= \{a_1, a_3, a_5\} \\
\{a_i : p_{10} \cdot a_i = 0\} &= \{a_1, a_2, a_4\}
\end{aligned}$$

This accounts for all subsets of $\{a_1, \ldots, a_5\}$ of rank 3. So by Weyl's Facet Lemma, it shows that C^* is generated by $p_1, \ldots, p_6$, which lie on distinct extreme rays of C^*.

As an aside, you should verify that

$$\begin{aligned}
\{p_j : p_j \cdot a_1 = 0\} &= \{p_3, p_4, p_5, p_6\} && \text{has rank 3} \\
\{p_j : p_j \cdot a_2 = 0\} &= \{p_2, p_3, p_4\} && \text{has rank 3} \\
\{p_j : p_j \cdot a_3 = 0\} &= \{p_1, p_2, p_4, p_5\} && \text{has rank 3} \\
\{p_j : p_j \cdot a_4 = 0\} &= \{p_1, p_5, p_6\} && \text{has rank 3} \\
\{p_j : p_j \cdot a_5 = 0\} &= \{p_1, p_2, p_3, p_6\} && \text{has rank 3,}
\end{aligned}$$

confirming that $a_1, \ldots, a_5$ are on distinct extreme rays of $C^{**} = C$.

Notes on the example

The points $a_1, \ldots, a_5$ are multiples of five distinct nonzero points on the moment curve in $\boldsymbol{R}^4$. The **moment curve** is the set of points of the form (t, t^2, t^3, t^4), for $t \geqslant 0$. G. M. Ziegler [15, Example 0.6, pp. 10–13] describes a polytope based on the moment curve that suggested this example. I used T. Christof and A. Loebel's computer program PORTA [3, 4] to compute the dual cone and the facets of C. The program uses the **Fourier–Motzkin Elimination Algorithm** (see, e.g., G. M. Ziegler [15, § 1.2, pp. 32–39]) with extensions due to N. V. Chernikova [1, 2] to efficiently find the six extreme rays of C^*. That left me with only four subsets of rank 3 to find the orthogonal complement by hand. After finding two by hand, I used Mathematica 5.0 to compute $p_7, \ldots, p_{10}$ and all the inner products $p_j \cdot a_i$, and its `MatrixRank` function to double check the ranks. Feel free to check any of these computations by hand.

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