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Sums of sets, etc.*

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If E and F are subsets of $\boldsymbol{R}^{\mathrm{m}}$, define the **sum**

$$E + F = \{x + y : x \in E;\ y \in F\}.$$

More generally the sum $E_1 + \cdots + E_n$ is the set of vectors of the form $x^1 + \cdots + x^n$, where each $x^i \in E_i$.

The next result may be found for instance in [3]. It relies on the simple fact that

$$p \cdot (x^1 + \cdots + x^n) = p \cdot x^1 + \cdots + p \cdot x^n.$$

1 Lemma *Let $E_1, \ldots, E_n$ be sets in $\boldsymbol{R}^{\mathrm{m}}$, and put $E = E_1 + \cdots + E_n$. Let $x^i \in E_i$, $i = 1, \ldots, n$, and $x = x^1 + \cdots + x^n$. Then*

$$x \text{ maximizes } p \text{ over } E \quad \Longleftrightarrow \quad (x^i \text{ maximizes } p \text{ over } E_i \text{ for each } i = 1, \ldots, n)$$

Proof: ($\Longrightarrow$) Suppose by way of contradiction that for some j, $z \in E_j$ and $p \cdot z > p \cdot x^j$. Then $x' = x^1 + \cdots + x^{j-1} + z + x^{j+1} + \cdots + x^n \in E$, and $p \cdot x' > p \cdot x$, a contradiction.

($\Longleftarrow$) Let $z \in E$. Then $z = z^1 + \cdots + z^n$, where each $z^i \in E_i$. By hypothesis, $p \cdot z^i \leqslant p \cdot x^i$ for each i, so summing we have $p \cdot z = p \cdot (z^1 + \cdots + z^n) \leqslant p \cdot (x^1 + \ldots + x^n) = p \cdot x$, so x maximizes p over E. ∎

1 Is a sum of closed sets closed?

An important question is whether the sum of closed sets is itself closed. The next example shows that it is not automatic.

2 Example The sum $E + F$ may fail to be closed even if E and F are closed. For instance, set

$$E = \{(x, y) \in \boldsymbol{R}^2 : y \geqslant 1/x \text{ and } x > 0\} \quad \text{and} \quad F = \{(x, y) \in \boldsymbol{R}^2 : y \geqslant -1/x \text{ and } x < 0\}$$

Then E and F are closed, but

$$E + F = \{(x, y) \in \boldsymbol{R}^2 : y > 0\}$$

is not closed. □

To state sufficient conditions for the sum of closed sets to be closed we must make a fairly long digression.

*These notes are largely based on [1].

2 Asymptotic cones

A **cone** is a nonempty subset of $\boldsymbol{R}^{\mathrm{m}}$ closed under multiplication by nonnegative scalars. That is, C is a cone if whenever $x \in C$ and $\lambda \in \boldsymbol{R}_+$, then $\lambda x \in C$. A cone is **nontrivial** if it contains a point other than zero.

3 Definition *Let $E \subset \boldsymbol{R}^{\mathrm{m}}$. The **asymptotic cone** of E, denoted $\boldsymbol{A}E$ is the set of all possible limits z of sequences of the form $\{\lambda_n x^n\}$, where each $x^n \in E$, each $\lambda_n > 0$, and $\lambda_n \to 0$. Let us call such a sequence a **defining sequence for z**.*

This definition is equivalent to that in Debreu [2], and generalizes the notion of the recession cone of a convex set. This form of the definition was chosen because it makes most properties of asymptotic cones trivial consequences of the definition.

The **recession cone** 0^+F of a closed convex set F is the set of all directions in which F is unbounded, that is, $0^+F = \{z : \forall x \in F\ \forall \alpha \geqslant 0\ x + \alpha z \in F\}$. (See Rockafellar [4, Theorem 8.2].)

4 Lemma **(a)** *$\boldsymbol{A}E$ is indeed a cone.*

(b) *If $E \subset F$, then $\boldsymbol{A}E \subset \boldsymbol{A}F$.*

(c) *$\boldsymbol{A}(E + x) = \boldsymbol{A}E$ for any $x \in \boldsymbol{R}^{\mathrm{m}}$.*

(cc) *$0^+E \subset \boldsymbol{A}E$.*

(d) *$\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1 + E_2)$.*

(e) *$\boldsymbol{A}\prod_{i\in I} E_i \subset \prod_{i\in I} \boldsymbol{A}E_i$.*

(f) *$\boldsymbol{A}E$ is closed.*

(g) *If E is convex, then $\boldsymbol{A}E$ is convex.*

(h) *If E is closed and convex, then $\boldsymbol{A}E = 0^+E$. (The asymptotic cone really is a generalization of the recession cone.)*

(i) *If C is a cone, then $\boldsymbol{A}C = \operatorname{cl} C$.*

(j) *$\boldsymbol{A}\bigcap_{i\in I} E_i \subset \bigcap_{i\in I} \boldsymbol{A}E_i$. The reverse inclusion need not hold.*

(k) *If $E + F$ is convex, then $\boldsymbol{A}E + \boldsymbol{A}F \subset \boldsymbol{A}(E + F)$.*

(l) *A set $E \subset \boldsymbol{R}^{\mathrm{m}}$ is bounded if and only if $\boldsymbol{A}E = \{0\}$.*

Proof: Here are proofs of selected parts. The others are easy, and should be treated as an exercise.

(cc) $0^+E \subset \boldsymbol{A}E$.

Let $z \in 0^+E$. Then for any $n > 0$ and any $x \in E$, we have $x + nz \in E$. But $\frac{1}{n}(x + nz) \to z$, so $z \in \boldsymbol{A}E$.

(d) $\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1 + E_2)$.

For $x_2 \in E_2$, by definition $E_1 + x_2 \subset E_1 + E_2$, so by (b), $\boldsymbol{A}(E_1 + x_2) \subset \boldsymbol{A}(E_1 + E_2)$, so by (c), $\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1 + E_2)$.

(f) $\boldsymbol{A}E$ is closed.

Let x^n be a sequence in $\boldsymbol{A}E$ with $x^n \to x$. For each m there is a sequence $\lambda_{n,m}x^{n,m}$ with $\lim_m \lambda_{n,m}x^{n,m} = x^n$, $\lambda_{n,m} \to 0$ as $m \to \infty$, $x^{n,m} \in E$, and each $\lambda_{n,m} > 0$. Then for each k there is N_k such that for all $n \geqslant N_k$, $|x^n - x| < 1/k$, and M_k such that for all $m \geqslant M_k$, $|\lambda_{N_k,m}x^{N_k,m} - x^{N_k}| < 1/k$, and L_k such that for all $m \geqslant L_k$, $\lambda^{N_k,m} < 1/k$. Set $P_k = \max\{M_k, L_k\}$, $y^k = x^{N_k,P_k}$, and $\lambda_k = \lambda^{N_k,P_k}$. Then each $\lambda_k > 0$, $\lambda_k \to 0$ and $|\lambda_k y^k - x| < 2/k$, so $x \in \boldsymbol{A}E$.

(g) If E is convex, then $\boldsymbol{A}E$ is convex.

Let $x, y \in \boldsymbol{A}E$ and $\alpha \in [0,1]$. Since $\boldsymbol{A}E$ is a cone, $\alpha x \in \boldsymbol{A}E$ and $(1-\alpha)y \in \boldsymbol{A}E$. Thus there are defining sequences $\lambda_n x^n \to \alpha x$ and $\gamma_n y^n \to (1-\alpha)y$. Since E is convex, $z^n = \frac{\lambda_n}{\gamma_n+\lambda_n}x^n + \frac{\gamma_n}{\gamma_n+\lambda_n}y^n \in E$ for each n. Set $\delta_n = \gamma_n + \lambda_n > 0$. Then $\delta_n \to 0$ and $\delta_n z^n = \lambda_n x^n + \gamma_n y^n \to \alpha x + (1-\alpha)y$. Thus $\alpha x + (1-\alpha)y \in \boldsymbol{A}E$.

(h) If E is closed and convex, then $\boldsymbol{A}E = 0^+E$.

In light of (cc), it suffices to prove that $\boldsymbol{A}E \subset 0^+E$, so let $z \in \boldsymbol{A}E$, $x \in E$, and $\alpha \geqslant 0$. We wish to show that $x + \alpha z \in E$. By definition of $\boldsymbol{A}E$ there is a sequence $\lambda_n z^n \to z$ with $z^n \in E$, $\lambda_n > 0$, and $\lambda_n \to 0$. Then for n large enough $0 \leqslant \alpha\lambda_n < 1$, so $(1-\alpha\lambda_n)x + \alpha\lambda_n z^n \in E$ as E is convex. But $(1-\alpha\lambda_n)x + \alpha\lambda_n z^n \to x + \alpha z$. Since E is closed, $x + \alpha z \in E$.

(i) If C is a cone, then $\boldsymbol{A}C = \operatorname{cl} C$.

It is easy to show that $C \subset \boldsymbol{A}C$, as $\frac{1}{n}nx \to x$ is a defining sequence. Since $\boldsymbol{A}C$ is closed by (f), we have $\operatorname{cl} C \subset \boldsymbol{A}C$. On the other hand if $\lambda_n \geqslant 0$ and $x^n \in C$, then $\lambda_n x^n \in C$, as C is a cone, so $\boldsymbol{A}C \subset \operatorname{cl} C$.

(j) $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \bigcap_{i\in I} \boldsymbol{A}E_i$. The reverse inclusion need not hold.

By (b), $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \boldsymbol{A}E_j$ for each j, so $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \bigcap_{i\in I} \boldsymbol{A}E_i$.

For a failure of the reverse inclusion, consider the even nonnegative integers $E_1 = \{0, 2, 4, \ldots\}$ and the odd nonnegative integers $E_2 = \{1, 3, 5, \ldots\}$. Then $E_1 \cap E_2 = \varnothing$, so $\boldsymbol{A}(E_1 \cap E_2) = \varnothing$, but $\boldsymbol{A}E_1 = \boldsymbol{A}E_2 = \boldsymbol{A}E_1 \cap \boldsymbol{A}E_2 = \boldsymbol{R}_+$.

(k) If $E + F$ is convex, then $\boldsymbol{A}E + \boldsymbol{A}F \subset \boldsymbol{A}(E + F)$.

Let z belong to $\boldsymbol{A}E + \boldsymbol{A}F$. Then there exist defining sequences $\{\lambda_n x^n\} \subset E$ and $\{\alpha_n y^n\} \subset F$ with $\lambda_n x^n + \alpha_n y^n \to z$. Let $x' \in E$ and $y' \in F$. (If either E or F is empty, the result is trivial.) Then $\{\lambda_n(x^n + y')\} \subset E + F$ and $\{\alpha_n(x' + y^n)\} \subset E + F$, so

$$(\lambda_n + \alpha_n)\left(\frac{\lambda_n}{\lambda_n + \alpha_n}(x^n + y') + \frac{\alpha_n}{\lambda_n + \alpha_n}(x' + y^n)\right) \to z,$$

is a defining sequence for z in $E + F$.

(l) A set $E \subset \boldsymbol{R}^{\mathrm{m}}$ is bounded if and only if $\boldsymbol{A}E = \{0\}$.

If E is bounded, clearly $\boldsymbol{A}E = \{0\}$. If E is not bounded, let $\{x^n\}$ be an unbounded sequence in E. Then $\lambda_n = |x^n|^{-1} \to 0$ and $\{\lambda_n x^n\}$ is a sequence on the unit sphere, which is compact. Thus there is a subsequence converging to some x in the unit sphere. Such an x is a nonzero member of $\boldsymbol{A}E$.

∎

5 Example The asymptotic cone of a non-convex set need not be convex. Let $E = \{(x,y) \in \boldsymbol{R}^2 : y = \frac{1}{x},\ x > 0\}$. This hyperbola is not convex and its asymptotic cone is the union of the nonnegative x- and y-axes. But the asymptotic cone of a non-convex set may be convex. Just think of the integers in $\boldsymbol{R}^1$. □

6 Example In general, it need not be that $\boldsymbol{A}(E+F) \subset \boldsymbol{A}E + \boldsymbol{A}F$, even if E and F are closed and convex. For instance, let E be the set of points lying above a standard parabola:

$$E = \{(x,y) : y \geqslant x^2\}.$$

The asymptotic cone of E, which is the same as its recession cone, is just the y-axis above zero:

$$\boldsymbol{A}E = \{(0,y) : y \geqslant 0\}.$$

Now observe that $E + (-E) = \boldsymbol{R}^2$, so $\boldsymbol{A}\big(E + (-E)\big) = \boldsymbol{R}^2$. Thus

$$\boldsymbol{A}E + \boldsymbol{A}(-E) \subsetneq \boldsymbol{A}\big(E + (-E)\big).$$

□

3 When a sum of closed sets is closed

7 Theorem (Closure of the sum of sets) *Let $E, F \subset \boldsymbol{R}^{\mathrm{m}}$ be closed and nonempty. Suppose that $x \in \boldsymbol{A}E$, $y \in \boldsymbol{A}F$ and $x + y = 0$ together imply that $x = y = 0$. Then $E + F$ is closed, and $\boldsymbol{A}(E+F) \subset \boldsymbol{A}E + \boldsymbol{A}F$.*

Proof: Both parts of the theorem follow from the following claim:

> If $\{\lambda_n\}$ is a bounded sequence of real numbers with each $\lambda_n > 0$, $\{x^n\}$ is a sequence in E, and $\{y^n\}$ is a sequence in F, and if $\lambda_n(x^n + y^n)$ converges to some point, then there is a common subsequence along which both $\{\lambda_k x^k\}$ and $\{\lambda_k y^k\}$ converge.

Given this claim, I now show that $E + F$ is closed: Let $x^n + y^n \to z$ with $\{x^n\} \subset E$, $\{y^n\} \subset F$. By the claim above (with $\lambda_n = 1$ for all n) there is a common subsequence with $x^k \to x$ and $y^k \to y$. Since E and F are closed, $x \in E$ and $y \in F$. Therefore $z = x + y \in E + F$, so $E + F$ is closed.

To see that $\boldsymbol{A}(E+F) \subset \boldsymbol{A}E + \boldsymbol{A}F$, let $z \in \boldsymbol{A}(E+F)$. That is, z is the limit of a defining sequence $\{\lambda_n(x^n + y^n)\}$, where $x^n \in E$ and $y^n \in F$. Since $\lambda_n \to 0$, it is a bounded sequence. Thus by the claim there is a common convergent subsequence, and by definition $\lim_k \lambda_k x^k \in \boldsymbol{A}E$ and $\lim_k \lambda_k y^k \in \boldsymbol{A}F$, so $z \in \boldsymbol{A}E + \boldsymbol{A}F$.

We now turn to the proof of the claim. Let $\{\lambda_n\}$ be a bounded sequence of strictly positive real numbers, $\{x^n\}$ a sequence in E, and $\{y^n\}$ a sequence in F, and assume $\lim_n \lambda_n(x^n + y^n) = z$.

First observe that if say $\lambda_k x^k \to x$, then $\lambda_k y^k \to z - x$. So suppose by way of contradiction that neither $\{\lambda_n x^n\}$ nor $\{\lambda_n y^n\}$ has a convergent subsequence. Since every bounded sequence in $\boldsymbol{R}^{\mathrm{m}}$ has a convergent subsequence, we must have $|\lambda_n x^n| \to \infty$ and $|\lambda_n y^n| \to \infty$. In particular, since $\{\lambda_n\}$ is bounded, $|x^n| \to \infty$ and $|y^n| \to \infty$. Given this, for n large enough, $|x^n|$ and $|y^n|$ are nonzero. Since the unit sphere is compact, along some common subsequence, $\frac{x^k}{|x^k|} \to x$ and $\frac{y^k}{|y^k|} \to y$. Obviously $|x| = |y| = 1$, but since $\frac{1}{|x^k|} \to 0$, the limit point x belongs to $\boldsymbol{A}E$, and likewise $y \in \boldsymbol{A}F$.

Under the hypotheses of the theorem we cannot then have $x + y = 0$, since that would entail $x = y = 0$, which is at odds with $|x| = |y| = 1$. Therefore $x + y \neq 0$. But since x and y are on the unit sphere, this implies that $0 \notin \operatorname{co}\{x, y\}$. (Geometrically this says that if a chord is not a diameter, it does not pass through the center of the sphere. Algebraically, if $0 = \alpha x + (1-\alpha)y$ with $0 \leqslant \alpha \leqslant 1$ and $|x| = |y| = 1$, then $\alpha x = -(1-\alpha)y$, and taking norms on both sides gives $\alpha = 1 - \alpha$ $(= \frac{1}{2})$, so $x + y = 0$.)

Therefore by the separating hyperplane theorem, we can separate 0 from $\operatorname{co}\{x, y\}$. That is, there is a $p \neq 0$ and an $\varepsilon > 0$ such that $p \cdot x \geqslant \varepsilon$ and $p \cdot y \geqslant \varepsilon$. Now for large enough n, $p \cdot \frac{x^n}{|x^n|} > \frac{\varepsilon}{2}$ and $p \cdot \frac{y^n}{|y^n|} > \frac{\varepsilon}{2}$. Thus

$$\lambda_n p \cdot x^n = \lambda_n |x^n| p \cdot \frac{x^n}{|x^n|} > \lambda_n |x^n| \frac{\varepsilon}{2} \xrightarrow[n]{} \infty,$$

where the strict inequality uses $\lambda_n > 0$. A similar result holds for $\{y^n\}$. Thus $\lambda_n p \cdot (x^n + y^n) \to \infty$, but on the other hand $\lambda_n p \cdot (x^n + y^n) \to p \cdot z < \infty$. This contradiction establishes the claim. ∎

8 Definition *Let $C_1, \ldots, C_n$ be cones in $\boldsymbol{R}^{\mathrm{m}}$. We say that they are* ***positively semi-independent*** *if whenever $x^i \in C_i$ for each $i = 1, \ldots, n$,*

$$x_1 + \cdots + x_n = 0 \quad \Longrightarrow \quad x^1 = \cdots = x^n = 0.$$

Clearly, any subset of a set of semi-independent cones is also semi-independent.

9 Corollary *Let $E_i \subset \boldsymbol{R}^{\mathrm{m}}$, $i = 1, \ldots, n$, be closed and nonempty. If $\boldsymbol{A}E_i$, $i = 1, \ldots, n$, are positively semi-independent, then $\sum_{i=1}^n E_i$ is closed, and $\boldsymbol{A} \sum_{i=1}^n E_i \subset \sum_{i=1}^n \boldsymbol{A}E_i$.*

Proof: This follows from Theorem 7 by induction on n. ∎

10 Corollary *Let $E, F \subset \boldsymbol{R}^{\mathrm{m}}$ be closed and let F be compact. Then $E + F$ is closed.*

Proof: A compact set is bounded, so by Lemma 4(l) its asymptotic cone is $\{0\}$. Apply Theorem 7. ∎

4 When is an intersection of closed sets bounded?

11 Proposition *Let $E_i \subset \boldsymbol{R}^{\mathrm{m}}$, $i = 1, \ldots, n$, be nonempty. If $\bigcap_{i=1}^n \boldsymbol{A}E_i = \{0\}$, then $\bigcap_{i=1}^n E_i$ is bounded.*

Proof: By Lemma 4(l), $\bigcap_{i=1}^n E_i$ is bounded if and only if $\boldsymbol{A}\left(\bigcap_{i=1}^n E_i\right) = \{0\}$. But by Lemma 4(j), $\boldsymbol{A}\left(\bigcap_{i=1}^n E_i\right) \subset \bigcap_{i=1}^n \boldsymbol{A}E_i$, and the proposition follows. ∎

References

[1] K. C. Border. 1985. *Fixed point theorems with applications to economics and game theory*. New York: Cambridge University Press.

[2] G. Debreu. 1959. *Theory of value: An axiomatic analysis of economic equilibrium*. New Haven: Yale University Press.

[3] T. C. Koopmans. 1957. *Three essays on the state of economic science*. New York: McGraw-Hill.

[4] R. T. Rockafellar. 1970. *Convex analysis*. Princeton, NJ: Princeton University Press.