CALIFORNIA INSTITUTE OF TECHNOLOGY
Division of the Humanities and Social Sciences

Arrow's General (Im)Possibility Theorem

KC Border
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Let X be a nonempty set of *social alternatives* and let $\mathcal{P}$ denote the set of *preference relations* over X. That is, $\mathcal{P}$ is the set of total reflexive transitive binary relations on X. A typical element of $\mathcal{P}$ will be denoted R and its strict part will be denoted P. If there are n members of society, a *preference profile* is an ordered list $(R_1, \ldots, R_n)$ of preference relations, specifying the preference for each member of society.

Definition 1 *A* social welfare function φ*, or SWF, on domain* $D \subset \mathcal{P}^n$ *is a mapping* $\varphi \colon D \to \mathcal{P}$ *from a set of preference profiles to the set of preference relations. It is traditional to denote the value of* φ *at the profile* $(R_1, \ldots, R_n)$ *by* $\boldsymbol{R}$ *with no subscript.*

This definition incorporates an important assumption, namely that the social welfare relation belongs to $\mathcal{P}$. In particular, it is transitive.

Definition 2 *A SWF satisfies the* (Binary) Independence of Irrelevant Alternatives Axiom*, IIA for short, if* $(R_1, \ldots, R_n)$ *and* $(R'_1, \ldots, R'_n)$ *are profiles satisfying* $x \; R_i \; y \iff x \; R'_i \; y$ *for all* i*, then* $x \; \boldsymbol{R} \; y \iff x \; \boldsymbol{R}' \; y$.

That is, the social ranking of x and y can be determined from only the individual rankings of x and y.

Definition 3 *A SWF satisfies the* (weak) Pareto Principle *if* $x \; P_i \; y$ *for all* i *implies* $x \; \boldsymbol{P} \; y$.

Arrow's General Possibility Theorem *Assume* X *has at least three elements, and let* $\varphi \colon \mathcal{P}^n \to \mathcal{P}$ *be a social welfare function with domain* $\mathcal{P}^n$*. Assume that* φ *satisfies IIA and the Pareto Principle. Then there is some* i *such that for every preference profile, and every pair* x, y,

$$x \; P_i \; y \implies x \; \boldsymbol{P} \; y.$$

That is, some one individual dictates the social strict preference relation.

The proof of Arrow's theorem is divided into a number of small lemmas. First we shall need some definitions. A *coalition* is a nonempty subset of $N = \{1, \ldots, n\}$.

Definition 4 *A coalition S is* decisive for x over y *if for some preference profile, $x \; P_i \; y$ for all $i \in S$, $y \; P_i \; x$ for all $i \notin S$, and $x \; \boldsymbol{P} \; y$. This profile is called a profile of decisiveness for x over y via S.*

A coalition S is strictly decisive for x over y *if for every preference profile satisfying $x \; P_i \; y$ for all $i \in S$, we have $x \; \boldsymbol{P} \; y$.*

A coalition S is decisive *if it is strictly decisive for every pair of distinct alternatives.*

The definitions are a bit tricky. Note that if a coalition is decisive for x over y, then we must have $x \neq y$. On the other hand, it is vacuously true that a coalition S is strictly decisive for x over x. Obviously two decisive coalitions cannot be disjoint.

In the language of decisiveness, Arrow's theorem says that there is a decisive coalition that has only one member. A fundamental question is whether there are any decisive coalitions. The answer is yes. Indeed, the Pareto Principle may be restated as follows.

Lemma 1 *The coalition of the whole, $\{1, 2, \ldots, n\}$, is decisive.*

We now proceed to show that if a coalition is decisive for x over y, then it is decisive. In the lemmas that follow we shall use the following sort of schematic diagram for preference profiles: Columns represent coalitions. If one element in a column is higher than another, the higher one is strictly preferred. Braces are used to group elements, and within the group the ranking is unrestricted. Thus the schematic diagram

S	S^c
x	y
y	$\{x, z\}$
z	

represents any profile such that for $i \in S$, $x \; P_i \; y \; P_i \; z$, and for $i \in S^c$, $y \; P_i \; x$ and $y \; P_i \; z$.

Lemma 2 *Suppose S is decisive for x over y, and $z \notin \{x, y\}$. Then S is strictly decisive for x over z.*

Proof: IIA implies that any profile corresponding to the following schematic is a profile of decisiveness for x over y via S.

S	S^c
x	y
y	x

In particular, by adding some information about $z \notin \{x, y\}$, we do not change the social preference between x and y, so any profile corresponding to the following schematic is still a profile of decisiveness for x over y via S.

S	S^c
x	y
y	$\{x, z\}$
z	

For such profiles,

$$\begin{array}{ll} x \boldsymbol{P} y & \text{since } S \text{ is decisive for } x \text{ over } y, \\ y \boldsymbol{P} z & \text{by the Pareto Principle,} \\ x \boldsymbol{P} z & \text{by transitivity of } \boldsymbol{P}. \end{array}$$

Now erase y, and IIA implies that for any profile satisfying the schematic

S	S^c
x	$\{x, z\}$
z	

where $z \neq y$, we must have $x \boldsymbol{P} z$. ∎

Corollary 1 *If S is decisive for x over y, then for any w, S is strictly decisive for x over w.*

Proof: Lemma 2 proves this for $w \neq y$, so we need only consider the case $w = y$.

Since X has at least three elements, there is some $z \notin \{x, y\}$. Since S is decisive for x over y, Lemma 2 implies that S is strictly decisive for x over z. Since $y \notin \{x, z\}$ and S is decisive for x over z, Lemma 2 implies that S is strictly decisive for x over y. ∎

Lemma 3 *Suppose S is decisive for x over y, and $z \notin \{x, y\}$. Then S is strictly decisive for z over y.*

Proof: IIA implies that the following schematic represents a profile of decisiveness for x over y via S.

S	S^c
z	$\{y, z\}$
x	x
y	

Then

$$\begin{array}{ll} z\ \boldsymbol{P}\ x & \text{by the Pareto Principle,} \\ x\ \boldsymbol{P}\ y & \text{since } S \text{ is decisive,} \\ z\ \boldsymbol{P}\ y & \text{by transitivity of } \boldsymbol{P}. \end{array}$$

Now use IIA to erase x. ∎

The proof of the next corollary is similar to the proof of Corollary 1.

Corollary 2 *If S is decisive for x over y, then for any w, S is strictly decisive for w over y.*

Lemma 4 *Suppose that for some x and y, S is decisive for x over y. Then S is decisive.*

Proof: Let v and w be arbitrary distinct elements of X. We need to show that S is strictly decisive for v over w.

Case 1. $v = x$.

See Corollary 1.

Case 2. $w = y$.

See Corollary 2.

Case 3. $v = y$ and $w = x$.

Choose $z \notin \{x, y\}$. Then by Corollary 1, S is strictly decisive for x over z. Since $y \notin \{x, z\}$, Corollary 2 implies S is strictly decisive for y over z. Now Corollary 1 implies S is strictly decisive for y over x.

Case 4. $\{v, w\} \cap \{x, y\} = \varnothing$.

By Corollary 1, S is strictly decisive for x over w, so Corollary 2 implies S is strictly decisive for v over w.

∎

Lemma 5 *If S and T are decisive, so is $S \cap T$.*

Proof: Consider a preference profile represented by:

$S \setminus T$	$S \cap T$	$T \setminus S$	$(S \cup T)^c$
y	x	z	y
x	z	y	
z	y	x	x

Then

$$\begin{array}{ll} x \boldsymbol{P} z & \text{since } S \text{ is decisive,} \\ z \boldsymbol{P} y & \text{since } T \text{ is decisive,} \\ x \boldsymbol{P} y & \text{by transitivity of } \boldsymbol{P}. \end{array}$$

Therefore we see that $S \cap T$ is decisive for x over y, so by Lemma 4, $S \cap T$ is decisive. ∎

Lemma 6 *If S is not decisive, then S^c is decisive.*

Proof: Since S is not decisive, there is some pair x, y for which we have $x \, P_i \, y$ for all $i \in S$ and $y \, P_i \, x$ for all $i \notin S$ and $y \boldsymbol{R} x$. Since X has at least three elements, there exists some $z \notin \{x, y\}$. Consider a preference profile represented by:

S	S^c
x	y
z	x
y	z

Then

$$\begin{array}{ll} y \boldsymbol{R} x & \text{since } S \text{ is not decisive,} \\ x \boldsymbol{P} z & \text{by the Pareto Principle,} \\ y \boldsymbol{P} z & \text{by transitivity of } \boldsymbol{P}. \end{array}$$

Therefore S^c is decisive for y over z, so by Lemma 4, S^c is decisive. ∎

Lemma 7 (Arrow's Theorem) *There is a singleton decisive set.*

Proof: Clearly, if $\{i\}$ is decisive for some $i < n$, we are done. So suppose that $\{1\},\dots,\{n-1\}$ are not decisive. Then by Lemma 6, $\{1\}^c,\dots,\{n-1\}^c$ are decisive. But then by Lemma 5, $\{n\} = \bigcap_{i=1}^{n-1} \{i\}^c$ is decisive. ∎

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