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Elementary Asset Pricing Theory

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1 The simplest model

We start slowly with the simplest model. Later on, we will discuss variations on this model that make it (slightly) more descriptive. In this model, there are only two time periods, "today" ($t = 0$) and "tomorrow" ($t = 1$). There are finitely many possible **states of nature** tomorrow, and exactly one of them will be realized tomorrow. Denote the set of states by S. The state of nature tomorrow is not known today.

There are n **purely financial assets**. A purely financial asset is a contingent claim denominated in dollars (as opposed to commodities). Actually, what is important is that the payoff be unidimensional, and the payoffs of the different assets in each state of the world are perfect substitutes. That is, in state s tomorrow, a payoff of 1 from asset i and payoff of 1 from asset j are perfect substitutes for each other from everyone's point of view. We would expect that this would be the case if the payoff were denominated in dollars or in Deutchmarks or in yen. In a macroeconomic model with only one commodity, the payoffs can be expressed in units of the commodity. We would not expect this to be the case if some of the payoffs were in van Gogh paintings and others were in Malibu beach houses. Note that we are not assuming that there is perfect substitutability across states. That is, a payoff of 1 in state s need not be a perfect substitute for a payoff of 1 in state s'.

There is a **spot market** today for assets and each asset has a market price today or **spot price**. The price of asset i today is p_0^i, and it pays $p_1^i(s)$ in state s tomorrow. Throughout these notes, I will use superscripts to differentiate assets, subscripts to denote time periods, and parentheses to delimit the state of the world.

The **cash flow vector** of asset i is

$$A^i = \begin{bmatrix} -p_0^i \\ \vdots \\ p_1^i(s) \\ \vdots \end{bmatrix} \in \boldsymbol{R} \times \boldsymbol{R}^S.$$

*I am heavily indebted to Sargent [6, Chapter 7] for the applications.

The cash flow convention is that positive numbers represent cash received by the owner of the asset and negative quantities represent cash payed out by the owner. Thus the 0^{th} component of A^i is negative if p_0^i is positive, because to purchase a unit of asset i requires a cash payment if the price is positive. If p_0^i is negative, the "asset" i can be interpreted as a loan to the "owner." Thus we allow for borrowing in our framework, but whether or not the borrower defaults must be part of the specification of the payoff of the asset.

This formulation assumes that everyone agrees on what the actual cash flows will be in each state of nature. That is not the source of uncertainty. The uncertainty stems from not knowing which state of nature will occur tomorrow. This means that the description of the states of nature must be incredibly detailed. While investors do not disagree about the consequences in any state of nature, they may disagree about the likelihood of the states. Their beliefs about the states of nature affect their preferences over portfolios, which will be discussed below.

It is even possible that one of the assets may be **riskless** in that

$$p_1(s) = c \quad \text{for all } s \in S.$$

That is, the asset pays the same amount in each state of nature. (In the finance literature it is often implicitly assumed that currency is a riskless asset.) Suppose the riskless asset has spot price p_0 today. Then r defined by

$$(1 + r)p_0 = c \quad \text{or} \quad r = \frac{c}{p_0} - 1,$$

is the **riskless rate of interest**. If the riskless rate of interest is positive, then $p_0 < c$. But as long as p_0 and c are both positive we must have $r > -1$.

A portfolio is defined by the number of units of each asset held. Since there are n assets, a **portfolio** is simply a vector x in $\boldsymbol{R}^{\text{n}}$. The entry x_i indicates the number of units of asset i, which may be either positive or negative. The cash flow vector of the portfolio is just

$$\sum_{i=1}^{n} A^i x_i.$$

If $x_i < 0$, then the i^{th} asset has been sold short or issued by the portfolio holder. We will not rule this out, so a portfolio need not be a nonnegative vector.

We shall be comparing cash flows of portfolios, so we need an ordering for vectors. Here is the notation I use for vector orders in this note. Be warned that there is no universal standard notation, so be careful reading other texts.

1 Definition *For vectors* $x, y \in \boldsymbol{R}^{\text{n}}$:

$$x \geqq y \iff x_i \geqslant y_i,\ i = 1, \dots, n$$
$$x > y \iff x_i \geqslant y_i,\ i = 1, \dots, n, \text{ and } x \neq y$$
$$x \gg y \iff x_i > y_i,\ i = 1, \dots, n$$

This enables us to make the following fundamental definition.

2 Definition *An* ***arbitrage portfolio*** *is a portfolio x whose cash flow vector is semi-positive,*

$$\sum_{i=1}^{n} A^i x_i > 0.$$

If the 0^{th} component, $-\sum_{i=1}^{n} x_i p_0^i$, is strictly positive, then the portfolio can be purchased by net borrowing, and since the other components are all nonnegative, nothing need ever be paid back! If $\sum_{i=1}^{n} x_i p_0^i = 0$, then the portfolio is free, never requires a payout in any state tomorrow and returns a positive cash flow in at least one state. Arbitrage portfolios are desirable indeed.[1]

The main assumption we make about today's financial market is that the prices will adjust so as to eliminate any arbitrage portfolios. The intuition is that an arbitrage portfolio would be in universal demand so that its price would have to rise (that is, some asset prices must rise) until it is no longer an arbitrage portfolio. Implicit in this definition is the hidden assumption that everyone agrees that all states of nature have positive probability. For instance, assume that there are only two states. If I believed that state 1 would never occur and you believed that state 2 would never occur, then I could borrow from you on the condition that I only repay in state 1 and buy an asset that payed only in state 2, thus constructing an apparent arbitrage portfolio.

3 Assumption (Iron Law of Theoretical Finance) *There are no arbitrage portfolios.*

This law has the following remarkable and useful consequence:

4 Asset pricing theorem *In this model, either*

(1) *There is an arbitrage portfolio (that is, the Iron Law of Theoretical Finance fails); or else*

(2) *there are numbers $\pi(s) > 0$, $s \in S$, such that for each asset i,*

$$p_0^i = \sum_{s \in S} \pi(s) p_1^i(s).$$

Proof: In algebraic terms, alternative (1) states that there is some $x \in \boldsymbol{R}^{\text{n}}$ satisfying $Ax > 0$, where A is the $(|S|+1) \times n$ matrix whose i^{th} column is $A^i \in \boldsymbol{R} \times \boldsymbol{R}^S$. If this is

[1] You may have read about "arbitrageurs" in the business press recently. These people do *not* make money by buying arbitrage portfolios. Instead, they engage in what has been misnamed "pure risk arbitrage." That is, they buy stock in companies that are subject to merger rumors and gamble on whether the mergers will take place. They pay a slight premium for the stock but not as much as it would be worth if the merger occurs. The current owners sell at this price to avoid the risk. In the event that the merger fails, the "arbitrageurs" lose money. This is clearly a risky way to make money unless you have access to inside information. Trading on such information is illegal, but nevertheless occurs.

not true, then Stiemke's Theorem 9 states that there is $y \gg 0 \in \boldsymbol{R} \times \boldsymbol{R}^S$ such that for each i,

$$-y_0 p_0^i + \sum_{s \in S} y_s p_1^i(s) = 0.$$

Clearly the numbers

$$\pi(s) = \frac{y_s}{y_0}$$

satisfy alternative (2). It also follows from Stiemke's Theorem that alternatives (1) and (2) are inconsistent. ∎

The numbers $\pi(s)$, $s \in S$ are called **Arrow–Debreu prices**. The price $\pi(s)$ represents the current market price of a payment of \$1 in state s tomorrow. The theorem says that today's price for any asset is computed by summing the market value of its cash flow over all the future states.

Note that the theorem does *not* guarantee that these Arrow–Debreu prices are unique. They need not be. Arbitrage pricing theory by itself is unable to determine these prices. To determine the A–D prices, preferences must be taken into account to compute the supply and demand equilibrium. Nevertheless, in any supply and demand equilibrium in which investors have monotonic preferences in each state's consumption, then in equilibrium there can be no arbitrage portfolios. Thus any proposition we prove about asset prices assuming only that no arbitrage portfolios exist must also be true of a supply and demand equilibrium. However, if there are n assets and their cash flow vectors span $\boldsymbol{R}^{|S|}$, then the A–D prices are unique.

5 Exercise *Prove the claim about uniqueness.*

Risk neutral probability

Also note that if a risk free asset exists, then the risk free rate of interest r is determined by

$$r = \frac{1}{\sum_{s \in S} \pi(s)} - 1.$$

Even if there is no risk free asset, given Arrow–Debreu prices, we can still formally define a risk free rate of interest.

6 Definition *The* ***risk free rate of interest*** *(given Arrow–Debreu prices π) is defined by the equation*

$$r = \frac{1}{\sum_{s \in S} \pi(s)} - 1 \quad \text{or} \quad \sum_{s \in S} \pi(s) = \frac{1}{1+r}.$$

One problem that arises if there is not a true risk-free asset is that this risk free rate may depend on the particular Arrow–Debreu prices used to compute it. Determination of this rate is something that requires a full equilibrium analysis.

Thus the vector $(1+r)\pi$ defines a probability measure μ on S by

$$\mu(A) = (1+r)\sum_{s\in A}\pi(s).$$

The expected value $\boldsymbol{E}_\mu X$ of a random variable X under the measure μ is given by

$$\boldsymbol{E}_\mu X = (1+r)\sum_{s\in S}\pi(s)X(s),$$

so for asset i we have

$$p_0^i = \frac{1}{1+r}\,\boldsymbol{E}_\mu\, p_1^i.$$

> *That is, the price of each asset is just the present discounted value (discounted at the risk-free interest rate) of the expected value of the asset (under the probability measure μ).*

For this reason, the measure μ is called the **risk neutral probability** for the assets. If this probability is used on S, the price of each asset is simply its discounted expected value, and there are no risk premia. Note however that this formula only applies to assets in the span of the original set of assets. While we can compute this formula for nonmarketed assets, the price will depend on the particular set of Arrow–Debreu prices used.

2 Applications

Here are some simple applications of no-arbitrage theory. The choice of applications is influenced by Sargent [6, Chapter 7].

Bonds with default risk

Suppose a firm issues a bond B and promises to pay a rate of interest i, unless it can't afford to, in which case it will pay what it can afford. What can we say about i? If the firm's income is given by the random variable $p_1^X \in R^S$, then the bond's cash flow is

$$p_1^B(s) = \begin{cases} (1+i)p_0^B & \text{if } p_1^X(s) \geqslant (1+i)p_0^B \\ p_1^X(s) & \text{otherwise.} \end{cases}$$

Let $S^+ = \{s : p_1^X(s) \geqslant (1+i)p_0^B\}$ and $S^- = S \setminus S^+$, and note that the bond is truly risky only if S^- is nonempty. In that case,

$$\begin{aligned} p_0^B &= \sum_{s \in S^+} \pi(s)(1+i)p_0^B + \sum_{s \in S^-} \pi(s)p_1^X(s) \\ &< \sum_{s \in S^+} \pi(s)(1+i)p_0^B + \sum_{s \in S^-} \pi(s)(1+i)p_0^B \\ &= \sum_{s \in S} \pi(s)(1+i)p_0^B \\ &= \frac{1+i}{1+r}p_0^B. \end{aligned}$$

Thus $\frac{1+i}{1+r} > 1$ or $i > r$. Thus the rate of interest on a risky bond must be greater than the rate of interest on a riskless asset. *Note that this argument has nothing to do with risk aversion!*

Options

Given an asset X, a **call option** on X is the right to buy a unit of X at a specified exercise price on or before a given date. (American call options can be exercised any time prior to expiration; a European call option can only be exercised on the expiration data. In a two period world, the distinction does not matter.) Since the option need not be exercised, the call will only be exercised in states of nature s where $p_1^X(s)$ exceeds the exercise price. If C is a call on X with exercise price k, the time 1 cash flow from C is given by

$$p_1^C(s) = \big(p_1^X(s) - k\big)^+ = \max\{p_1^X(s) - k, 0\}.$$

A **put option** on X is the right to sell X at an exercise price k'. It will be exercised only if $p_1^X(s) < k'$. If P is a put on X, then its cash flow is

$$p_1^P(s) = \big(k' - p_1^X(s)\big)^+ = \max\{k' - p_1^X(s), 0\}.$$

See Figure 1.

7 Proposition (Put-call parity) *If there is no arbitrage and the riskless rate of interest is r, then given an asset X, a call C and a put P written on X with identical exercise price k, the spot prices at time 0 satisfy*

$$p_0^X + p_0^P - p_0^C = \frac{k}{1+r}.$$

That is, the price of the asset plus the price of the call minus the price of the put equals the present discounted value of exercise price.

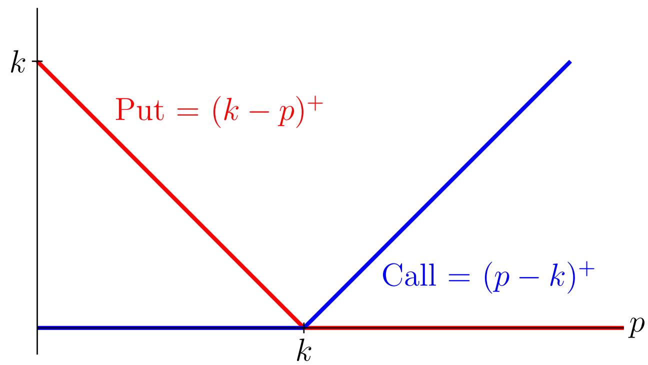

Figure 1. The payoff of a put and call option with exercise price k as a function of the underlying asset price p at the exercise date.

Mechanical proof: Let $S^+ = \{s : p_1^X(s) \geqslant k\}$ and $S^- = \{s : p_1^X(s) < k\}$. Then the prices satisfy $p_0^C = \sum_{s \in S^+} \pi(s)\big(p_1^X(s) - k\big)$ and $p_0^P = \sum_{s \in S^-} \pi(s)\big(k - p_1^X(s)\big)$. So

$$p_0^C - p_0^P = \sum_{s \in S} \pi(s)(p_1^X(s) - k) = \sum_{s \in S} \pi(s) p_1^X(s) - \sum_{s \in S} \pi(s) k = p_0^X - \frac{k}{1+r}.$$

Now rearrange the terms. ∎

Economic proof: Consider a portfolio formed by buying one unit of X, buying a put P and selling a call C. There are three cases.

1. If $p_1^X(s) < k$, the call you sold will exercised, so you receive k and give up your share of X to meet the claim. Thus you net k.

2. If $k > p_1^X(s)$, the call will not be exercised, but you can exercise your put and sell your share for k. Thus you net k.

3. If $p_1^X(s) = k$, just keep your X to get k, and neither option will be exercised. Thus you net k.

In any state of the world you receive k regardless. Thus this is a riskless portfolio so its price, $p_0^X + p_0^P - p_0^C$ is equal to $\frac{k}{1+r}$, the riskless present discounted value of k. ∎

Note that by adding options on X we are able to create riskless portfolios, even if X, P, and C are the only assets. This is why options were invented.

2.1 A Modigliani–Miller Theorem

A firm has a total income stream $p_1^X \in R^S$ and obligations in the form of stocks and bonds. Assume that the bonds promise to pay an aggregate amount B, and stockholders will receive all the firm's income after the bondholders have been paid. There is a chance that the firm may not earn enough to pay off the bondholders, but because of limited liability, the shareholders themselves need never pay the bondholders out of their own pockets. In this case, the firm declares bankruptcy, which is a complicated legal procedure, but we will make the unrealistic assumption that in the event of bankruptcy, the bondholders receive a prorated share of the entire value of the firm and the shareholders receive zero. Assume that these shares and bonds are the only obligations of the firm, and thus we may assume $p_1^X(s) \geqslant 0$ for each state s. (A value $p_1^X(s) < 0$ would imply that someone was obligated to pay $p_1^X(s)$, and neither the bond or shareholders are.)

Let p_0^B be the total market value of the bonds today. This implies a nominal interest rate i given by $(1+i)p_0^B = B$, or $i = \frac{B}{p_0^B} - 1$. The firm also has equity, which has the total value p_0^E today. Thus the total value of claims is $p_0^E + p_0^B$. In order to compute this sum we first compute the cash flow associated with equity, which is

$$\begin{cases} p_1^X(s) - B & \text{if } p_1^X(s) \geqslant B \\ 0 & \text{otherwise} \end{cases}$$

or in other words,

$$p_1^E(s) = \bigl(p_1^X(s) - B\bigr)^+. \tag{1}$$

The cash flow of the bonds is given by

$$p_1^B(s) = \begin{cases} B & \text{if } p_1^X(s) \geqslant B \\ p_1^X(s) & \text{if } p_1^X(s) < B. \end{cases}$$

Or in other words

$$p_1^B(s) = p_1^X(s) - \bigl(p_1^X(s) - B\bigr)^+ \tag{2}$$

Thus

$$\begin{aligned} p_0^E + p_0^B &= \sum_{s\in S} \pi(s)\{p_1^E(s) + p_1^B(s)\} \\ &= \sum_{s\in S} \pi(s)\{\underbrace{\bigl(p_1^X(s) - B\bigr)^+} + \underbrace{p_1^X(s) - \bigl(p_1^X(s) - B\bigr)^+}\} \\ &= \sum_{s\in S} \pi(s) p_1^X(s). \end{aligned} \tag{3}$$

Note that this is independent of the ratio of p_0^B to p_0^E. In other words, the total value of the firm is independent of how it is financed.

But increasing total indebtedness B (weakly) decreases $\bigl(p_1^X(s) - B\bigr)^+$, so by (2), an increase in B increases the total value p_0^B of the bonds, and so decreases the value of equity p_0^E.

An increase in B also raises the rate of interest on the firm's bonds. The interest rate i is determined by the discount the bonds sell at. They promise to pay B and sell for p_0^B, so $p_0^B(1+i) = B$ or

$$\begin{aligned} 1+i &= \frac{B}{p_0^B} \\ &= \frac{B}{\sum_{s\in S}\pi(s)\big\{p_1^X(s) - \big(p_1^X(s) - B\big)^+\big\}} \\ &= \frac{B}{\sum_{s\in D}\pi(s)p_1^X(s) + \sum_{s\in D^c}\pi(s)B} \\ &= \frac{1}{\sum_{s\in D}\pi(s)\frac{p_1^X(s)}{B} + \sum_{s\in D^c}\pi(s)} \end{aligned}$$

where $D = \{s \in S : p_1^X(s) < B\}$ is the set of states in which the firm defaults on its bonds. This is clearly an increasing function of B.

Note that this analysis depends on the fact that we have ignored any considerations of after-tax cash flow.

Financing investment

Now imagine the firm of the previous section contemplating a new investment. It will require an outlay of I today and result in an increment $p_1^Y(s) \geqslant 0$ in state s tomorrow. That is, the firm's new income stream will be $p_1^X(s) + p_1^Y(s)$ in state s tomorrow. What will the current equity holders want the firm to do, and does it depend on how the investment I is financed? Shareholders will want to undertake the investment if it results in a higher share price.

Let's simplify the analysis by assuming that the firm is initially debt free ($B = 0$) and its only obligations are E shares of equity with current spot price p_E. Imagine that the investment is financed through a combination of bond and equity sales with additional bonds B' and shares E'. The new bond and equity prices will be p'_B and p'_E. If the new issues just cover the cost of the investment, then

$$p'_B B' + p'_E E' = I,$$

and the new value of the firm is

$$\sum_{s\in S}\pi(s)\big(p_1^X(s) + p_1^Y(s)\big) = p'_B B' + p'_E(E + E').$$

Combining these with (3), gives

$$p_E E + \sum_{s\in S}\pi(s)p_1^Y(s) = p'_B B' + p'_E(E + E')$$

or

$$p_E E + \sum_{s \in S} \pi(s) p_1^Y(s) = I + p'_E E$$

so

$$p'_E > p_E \iff \sum_{s \in S} \pi(s) p_1^Y(s) - I > 0.$$

Another way to say this is that the shareholders will want to undertake the investment if the present discounted expected value (under the risk neutral probability) of the cash flow is greater than the current cost, and the financing method is irrelevant.

This remains true even if the investment is financed entirely through bonds. Compare this to the result above, namely that in the absence of new investment, an increase in the number of bonds decreases the price of equity. That is because the pie to be divided remains fixed. In this case, the pie has grown, and debt financing actually increases the share price.

The analysis above assumed $B = 0$. But what if that is not the case? Then we have a problem. The existence of the new income source Y can alter the payoffs of the old bonds as well as the new bonds. This can come at the expense of the current shareholders. The following example shows how.

Let there be two states a and b. Let $p_1^X(a) = 2$ and $p_1^X(b) = 0$ and suppose $\pi(a) = \pi(b) = 1/2$. (The risk free rate is zero.) Suppose initially that $E = B = 1$. The payoffs are indicated in the following table.

	State	
Asset	a	b
X	2	0
$E = 1$	1	0
$B = 1$	1	0

Then the value of the firm is $\pi(a)p_1^X(a) + \pi(b)p_1^X(b) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = 1$, the price of equity is $p_E = \pi(a)1 + \pi(b)0 = \frac{1}{2}$, and the price of a bond is $p_B = \pi(a)1 + \pi(b)0 = \frac{1}{2}$.

Consider now Y where $p_1^Y(a) = 0$, $p_1^Y(b) = 2$, so $\pi(a)p_1^Y(a) + \pi(b)p_1^Y(b) = 1$, and the current investment $I = 3/4$ (so Y acts as an insurance policy for the bondholders.) Then $\pi(a)p_1^Y(a) + \pi(b)p_1^Y(b) > I$. Suppose that the firm finances this investment entirely by issuing new bonds $0 < B' < 1$. The payoffs are indicated in the following table.

	State	
Asset	a	b
X	2	0
Y	0	2
$X+Y$	2	2
$E = 1$	$1 - B'$	$1 - B'$
$B + B' = 1 + B'$	$1 + B'$	$1 + B'$

Then the new value of the firm is $\pi(a)(X+Y)(a) + \pi(b)(X+Y)(b) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$, the new price of equity is $p'_E = \pi(a)(1 - B') + \pi(b)(1 - B') = 1 - B'$, and the new price of a bond is $p'_B = \pi(a)1 + \pi(b)1 = 1$. The condition that $p^{B'}B' = I$ gives $B' = 3/4$, so $1/4 = p'_E < p_E = 1/2$. What has happened is that by issuing new bonds to finance Y, the old bonds become valuable in states where before they were worthless. This cuts into the returns on equity.

This contradicts the analysis in Sargent [6, pp. 157–158], where it is claimed that the initial shareholders will benefit whenever $\sum_S \pi(s)p_1^Y(s) > I$. He implicitly assumes that the initial shareholders can capture all the increase in value. One way we could imagine this happening is that the initial shareholders first buy all the outstanding bonds, then issue new bonds, and then make the investment, which increases the value of the bonds now held by the initial shareholders. Another way to do this would be to pull an Enron. Create a new entity “off the books” of the parent, with income p_1^Y financed by bonds that only pay off if $p_1^Y > 0$. The old bonds would only pay off if $p_1^X > 0$. This avoids the problem of the new investment redefining the payoffs of the old bonds.

3 A more dynamic model

In this model there are three periods: “today” ($t = 0$), “tomorrow” ($t = 1$), and “later” ($t = 2$). The set S of states has the structure $S = U \times V$, where u is revealed tomorrow and v is revealed later. We assume that each asset i pays nothing tomorrow and $p_2^i(u, v)$ later. The **spot price** of asset i today is p_0^i. Its spot price tomorrow in state u will be $p_1^i(u)$.

A **dynamic portfolio** is a vector

$$x = \left((x_0^i)_{i=1,\dots,n}, \left(x_1^i(u)\right)_{i=1,\dots,n,\ u \in U} \right) \in \boldsymbol{R}^{\mathrm{n}} \times \boldsymbol{R}^{\mathrm{n} \times |U|}.$$

The dynamic portfolio x is **self-financing** if

$$\sum_{i=1}^{n} p_0^i x_0^i \leqslant 0,$$

and for each $u \in U$,

$$\sum_{i=1}^{n} p_1^i(u)\big(x_1^i(u) - x_0^i\big) \leqslant 0.$$

The cash flow of a dynamic portfolio x is

$$\begin{array}{c} \\ \hline \vdots \\ u \\ \vdots \\ \hline \vdots \\ (u,v) \\ \vdots \end{array} \left[\begin{array}{c} -\sum_{i=1}^{n} p_0^i x_0^i \\ \hline \vdots \\ -\sum_{i=1}^{n} p_1^i(u)\big(x_1^i(u) - x_0^i\big) \\ \vdots \\ \hline \vdots \\ \sum_{i=1}^{n} p_2^i(u,v) x_1^i(u) \\ \vdots \end{array}\right]$$

A **dynamic arbitrage portfolio** is a portfolio that has a semi-positive cash flow. Note that this implies that the portfolio is self-financing.

8 Dynamic pricing theorem *If (and only if) there are no dynamic arbitrage portfolios, then there are probability measures $\hat{\mu}$ and μ on $S = U \times V$, a "one-period risk-free interest rate" $r_{0,1}$ between periods 0 and 1, a "two-period risk-free interest rate" $r_{0,2}$ between periods 0 and 2, and for each partial state u, there is a "one-period risk-free interest rate" $r_{1,2}(u)$ between period 1 in state u and period 2, such that the following properties are satisfied.*

1. *For each asset i, today's spot price is the expected present discounted value of future prices. Specifically,*

$$p_0^i = \frac{1}{1 + r_{0,1}} \boldsymbol{E}_{\hat{\mu}}\, p_1^i = \frac{1}{1 + r_{0,2}} \boldsymbol{E}_{\mu}\, p_2^i.$$

2. *The measures $\hat{\mu}$ and μ have the same conditional probabilities. That is, for every (u, v),*

$$\hat{\mu}(v|u) = \mu(v|u).$$

3. *For each partial state u, for each asset i, tomorrow's spot price $p_1^i(u)$ in state u is the conditional expected present discounted value of the payoffs later. That is,*

$$p_1^i(u) = \frac{1}{1 + r_{1,2}(u)} \boldsymbol{E}_{\hat{\mu}}(p_2^i \mid u) = \frac{1}{1 + r_{1,2}(u)} \boldsymbol{E}_{\mu}(p_2^i \mid u).$$

4. The ***term structure*** *of interest rates and discount factors satisfies*

$$1 + r_{0,2} = (1 + r_{0,1})\, \boldsymbol{E}_\mu (1 + r_{1,2}), \qquad \frac{1}{1 + r_{0,2}} = \frac{1}{1 + r_{0,1}}\, \boldsymbol{E}_{\hat{\mu}} \frac{1}{1 + r_{1,2}}.$$

Proof: A dynamic portfolio x is a dynamic arbitrage portfolio if it satisfies

$$
\begin{array}{r}
\\ 0\\ \\ u' \\ \\ \\ (u'',v) \\ \\
\end{array}
\left[\begin{array}{c|ccc}
j & & (i,u) & \\
-p_0^j & 0 & \cdots & 0 \\ \hline
\vdots & & & \\
p_1^j(u') & & -p_1^i(u)\delta_{u,u'} & \\
\vdots & & & \\ \hline
\vdots & & & \\
0 & & p_2^i(u,v)\delta_{u,u''} & \\
\vdots & & &
\end{array}\right]
\left[\begin{array}{c} x_0^j \\ \hline \vdots \\ x_1^i(u) \\ \vdots \end{array}\right] > 0.
$$

(Figure 2 illustrates this matrix inequality for $n = 2$, $U = \{1, 2, 3\}$, and $V = \{1, 2\}$.)

The (Stiemke) alternative is that there is some

$$\pi = \Big(\pi_0, \big(\pi_1(u)\big)_{u \in U}, \big(\pi_2(u,v)\big)_{(u,v) \in U \times V}\Big) \gg 0$$

such that for each $j = 1, \dots, n$

$$-p_0^j \pi_0 + \sum_{u \in U} p_1^j(u)\pi_1(u) = 0,$$

and also for each (i, u), $i = 1, \dots, n$, $u \in U$,

$$-p_1^i(u)\pi_1(u) + \sum_{v \in V} p_2^i(u,v)\pi_2(u,v) = 0.$$

This is homogeneous in π, so without loss of generality $\pi_0 = 1$, so we have

$$p_0^i = \sum_{u \in U} p_1^i(u)\pi_1(u). \tag{$\star$}$$

and

$$p_1^i(u) = \sum_{v \in V} p_2^i(u,v) \frac{\pi_2(u,v)}{\pi_1(u)} \tag{$\star\star$}$$

so that

$$p_0^i = \sum_{(u,v) \in U \times V} p_2^i(u,v)\pi_2(u,v). \tag{$\star\star\star$}$$

$$
\begin{array}{c}
\\
0 \\ \hline
u'{=}1 \\
u'{=}2 \\
u'{=}3 \\ \hline
(u'',v){=}(1,1) \\
(u'',v){=}(1,2) \\
(u'',v){=}(2,1) \\
(u'',v){=}(2,2) \\
(u'',v){=}(3,1) \\
(u'',v){=}(3.2)
\end{array}
\begin{array}{cc|cccccc}
j{=}1 & j{=}2 & \substack{(i,u)=\\(1,1)} & \substack{(i,u)=\\(2,1)} & \substack{(i,u)=\\(1,2)} & \substack{(i,u)=\\(2,2)} & \substack{(i,u)=\\(1,3)} & \substack{(i,u)=\\(2,3)} \\
-p_0^1 & -p_0^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
p_1^1(1) & p_1^2(1) & -p_1^1(1) & -p_1^2(1) & 0 & 0 & 0 & 0 \\
p_1^1(2) & p_1^2(2) & 0 & 0 & -p_1^1(2) & -p_1^2(2) & 0 & 0 \\
p_1^1(3) & p_1^2(3) & 0 & 0 & 0 & 0 & -p_1^1(3) & -p_1^2(3) \\ \hline
0 & 0 & p_2^1(1,1) & p_2^2(1,1) & 0 & 0 & 0 & 0 \\
0 & 0 & p_2^1(1,2) & p_2^2(1,2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_2^1(2,1) & p_2^2(2,1) & 0 & 0 \\
0 & 0 & 0 & 0 & p_2^1(2,2) & p_2^2(2,2) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_2^1(3,1) & p_2^2(3,1) \\
0 & 0 & 0 & 0 & 0 & 0 & p_2^1(3,2) & p_2^2(3,2)
\end{array}
\begin{bmatrix} x_0^1 \\ x_0^2 \\ \hline x_1^1(1) \\ x_1^2(1) \\ x_1^1(2) \\ x_1^2(2) \\ x_1^1(3) \\ x_1^2(3) \end{bmatrix}
>
\begin{bmatrix} 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Figure 2. An arbitrage portfolio for $n = 2$, $U = \{1, 2, 3\}$, and $V = \{1, 2\}$.

Thus, we may interpret the $\pi_1(u)$ and $\pi_2(u,v)$ as today's prices for a dollar at the various dates and states of the world. As before we can normalize these prices to define an interest rate and a probability measure.

Equation ($\star\star\star$) suggests we define $r_{0,2}$ by

$$(1 + r_{0,2}) \sum_{(u,v)\in U\times V} \pi_2(u,v) = 1. \tag{4}$$

It is the riskless rate of interest between periods 0 and 2. The corresponding probability measure μ on $U \times V$ is defined by

$$\mu(u,v) = (1 + r_{0,2})\pi_2(u,v). \tag{5}$$

Then ($\star\star\star$) becomes

$$\boxed{p_0^i = \frac{1}{1 + r_{0,2}} \boldsymbol{E}_\mu \, p_2^i.} \tag{6}$$

Similarly, equation ($\star$) suggests defining $r_{0,1}$ by

$$(1 + r_{0,1}) \sum_{u\in U} \pi_1(u) = 1. \tag{7}$$

It is the risk free one period rate between periods today and tomorrow. It determines a probability $\hat{\mu}_\bullet$ on U by

$$\hat{\mu}_\bullet(u) = (1 + r_{0,1})\pi_1(u). \tag{8}$$

Then ($\star$) can be rewritten as

$$p_0^i = \frac{1}{1 + r_{0,1}} \boldsymbol{E}_{\hat{\mu}_\bullet} \, p_1^i. \tag{9}$$

Equation ($\star\star$) suggests that for each $u \in U$, we define $r_{1,2}(u)$ by

$$\big(1 + r_{1,2}(u)\big) \sum_{v\in V} \frac{\pi_2(u,v)}{\pi_1(u)} = 1. \tag{10}$$

It is the riskless rate of interest at time 1 in state u. (From the point of view of period 0, the rate $r_{1,2}$ is a random variable.) We also have a probability measure $\hat{\mu}(\cdot \mid u)$ on V defined by

$$\hat{\mu}(v \mid u) = \big(1 + r_{1,2}(u)\big) \frac{\pi_2(u,v)}{\pi_1(u)}. \tag{11}$$

Therefore

$$p_1^i(u) = \frac{1}{1 + r_{1,2}(u)} \boldsymbol{E}_{\hat{\mu}|u} \, p_2^i. \tag{12}$$

Now define the measure $\hat{\mu}$ on $U \times V$ by

$$\hat{\mu}(u,v) = \hat{\mu}(v \mid u)\hat{\mu}_\bullet(u). \tag{13}$$

Then $\hat{\mu}_\bullet$ is the marginal of $\hat{\mu}$ on U and $\hat{\mu}(\cdot \mid u)$ is the conditional probability on V given u. So (9) becomes

$$\boxed{p_0^i = \frac{1}{1+r_{0,1}} \boldsymbol{E}_{\hat{\mu}}\, p_1^i.}$$

and (12) becomes

$$\boxed{p_1^i(u) = \frac{1}{1+r_{1,2}(u)} \boldsymbol{E}_{\hat{\mu}}(p_2^i \mid u).} \tag{14}$$

Also observe that

$$\begin{aligned} \hat{\mu}(u,v) &= \hat{\mu}(v \mid u)\hat{\mu}_\bullet(u) && (13) \\ &= (1+r_{0,1})\pi_1(u)\big(1+r_{1,2}(u)\big)\frac{\pi_2(u,v)}{\pi_1(u)} && \text{equations (8) and (11)} \\ &= (1+r_{0,1})\big(1+r_{1,2}(u)\big)\pi_2(u,v). \end{aligned} \tag{15}$$

What is the relationship between $\hat{\mu}$ and μ? From (15) and (5) we have

$$\mu(u,v) = \frac{1+r_{0,2}}{(1+r_{0,1})\big(1+r_{1,2}(u)\big)}\hat{\mu}(u,v) \tag{16}$$

Conditioning on u then gives

$$\begin{aligned} \mu(v \mid u) = \frac{\mu(u,v)}{\sum_{v'} \mu(u,v')} &= \frac{\dfrac{1+r_{0,2}}{(1+r_{0,1})\big(1+r_{1,2}(u)\big)}\hat{\mu}(u,v)}{\sum_{v'} \dfrac{1+r_{0,2}}{(1+r_{0,1})\big(1+r_{1,2}(u)\big)}\hat{\mu}(u,v')} \\ &= \frac{\hat{\mu}(u,v)}{\sum_{v'} \hat{\mu}(u,v')} = \hat{\mu}(v \mid u). \end{aligned}$$

Another way to see this is to note that (5) implies

$$\mu(v \mid u) = \pi_2(u,v) / \sum_{v'} \pi_2(u,v')$$

and equations (10) and (11) imply

$$\hat{\mu}(v \mid u) = \pi_2(u,v) / \sum_{v'} \pi_2(u,v').$$

Either way

$$\boxed{\mu(v \mid u) = \hat{\mu}(v \mid u).}$$

Thus (14) can also be written as

$$\boxed{p_1^i(u) = \frac{1}{1+r_{1,2}(u)} \boldsymbol{E}_\mu(p_2^i \mid u).}$$

Summing both sides of (16) over $U \times V$ gives

$$\boldsymbol{E}_{\hat\mu} \frac{1+r_{0,2}}{(1+r_{0,1})(1+r_{1,2})} = 1.$$

In other words, the term structure satisfies

$$\boxed{\frac{1}{1+r_{0,2}} = \frac{1}{1+r_{0,1}} \boldsymbol{E}_{\hat\mu} \frac{1}{1+r_{1,2}}.}$$

On the other hand, rewriting (16) as

$$(1+r_{0,1})\big(1+r_{1,2}(u)\big)\mu(u,v) = (1+r_{0,2})\hat\mu(u,v)$$

and summing, we see that

$$\boxed{1+r_{0,2} = (1+r_{0,1})\, \boldsymbol{E}_\mu(1+r_{1,2})}$$

■

3.1 American vs. European options

Recall that a call option on an asset Z is the right to buy a unit of Z at a specified exercise price on or before a given date. American call options can be exercised any time prior to expiration; a European call option can only be exercised on the expiration data. In a two period world, the distinction does not matter, but in our three period world there may be a difference.

Let Z be an asset that pays $p_2^Z(u,v)$ in state (u,v) in period 2, with no other payouts. Let E be a European call on Z with strike price k. That is, E entitles you to buy one share of Z in period 2 at the price k. The timing is such that you will then receive the payment $p_2^Z(u,v)$. Let A be an American call on Z with the same strike price k. This option entitles you to purchase a share of Z for the price k at either time $t=1$ or $t=2$. Intuitively, since an American option can duplicate the performance of a European option, it must have a price at least as great. Can it ever have a strictly greater price? Let us examine this in more detail.

First observe that if $p_1^A(u) < p_1^E(u)$, then the portfolio $x^A(u) = 1$, $x^E(u) = -1$, $x_t^i(u') = 0$ everywhere else, is an arbitrage portfolio, so we must have

$$p_1^A(u) \geqslant p_1^E(u) \quad \text{for all } u. \tag{17}$$

We proceed by backward induction. In period 2, state (u,v) we have

$$p_2^E(u,v) = p_2^A(u,v) = (p_2^Z(u,v) - k)^+.$$

Thus by our asset pricing theorem, in period 1, state u, we have

$$\begin{aligned} p_1^Z(u) &= \frac{1}{1+r_{1,2}(u)} \boldsymbol{E}_\mu(p_2^Z \mid u) \\ p_1^E(u) &= \frac{1}{1+r_{1,2}(u)} \boldsymbol{E}_\mu\big((p_2^Z - k)^+ \mid u\big). \end{aligned}$$

The price $p_1^A(u)$ of the American call is a little more subtle. It is not hard to see that

$$p_1^A(u) = \begin{cases} p_1^Z(u) - k & \text{if it is exercised in state } u \\ p_1^E(u) & \text{if it is not exercised in state } u. \end{cases}$$

But now observe the mathematical fact that

$$(p-k)^+ + k \geqslant p$$

for all p. (The economic interpretation of this fact is that it is better to have k dollars and the option than it is to have only the underlying asset.) Therefore if the American option is exercised in state u, we have

$$\begin{aligned} p_1^A(u) + k &= p_1^Z(u) \\ &= \frac{1}{1+r_{1,2(u)}} \boldsymbol{E}_\mu(p_2^Z \mid u) \\ &\leqslant \frac{1}{1+r_{1,2(u)}} \boldsymbol{E}_\mu\Big(\big(p_2^Z(u,v) - k\big)^+ + k \mid u\Big) \\ &= p_1^E(u) + \frac{1}{1+r_{1,2(u)}} k \\ &\leqslant p_1^A(u) + \frac{1}{1+r_{1,2(u)}} k, \end{aligned}$$

where the last inequality is just inequality (17). As a consequence we see that the option can only be exercised if $-1 < r_{1,2}(u) \leqslant 0$. Note that if $r_{1,2}(u) = 0$, then we have equality everywhere, so $(p_2^Z(u,v) - k)^+ = p_2^Z(u,v) - k$ for all v. That is, $p_2^Z(u,v) \geqslant k$ for all v so any European option is sure to be exercised in period 2, from which it follows that $p_1^E(u) = p_1^Z(u) - \frac{1}{1+r_{1,2(u)}} k = p_1^Z(u) - k$, and consequently $p_1^E(u) = p_1^A(u)$.

In other words

$$r_{1,2}(u) \geqslant 0 \quad \Longrightarrow \quad p_1^E(u) = p_1^A(u).$$

If $r_{1,2}(u) < 0$, which is unlikely but not theoretically impossible, the story is different. Consider the following: $U = \{1,2\}$, $V = \{1,2\}$, and we have Arrow–Debreu prices $\pi_1 = .5$, $\pi_2 = .4$, and $\pi_{1,1} = \pi_{1,2} = .3$, $\pi_{2,1} = \pi_{2,2} = .2$. The corresponding measures $\hat{\mu}$ and μ are shown in Figure 3.

State		$\pi_2(u,v)$	$\hat{\mu}$	μ	$p_2^Z(u,v)$
$\pi_1(1)$	$(1,1)$	.3	$\frac{25}{90}$	$\frac{27}{90}$	10
$= .5$	$(1,2)$	.3	$\frac{25}{90}$	$\frac{27}{90}$	6
$\pi_1(2)$	$(2,1)$	.2	$\frac{20}{90}$	$\frac{18}{90}$	5
$= .4$	$(2,2)$	.2	$\frac{20}{90}$	$\frac{18}{90}$	5

Figure 3. Example with negative risk-free interest rate.

Then

$$r_{0,2} = 0, \quad r_{0,1} = \frac{1}{9}, \quad r_{1,2}(1) = -\frac{1}{6} \quad r_{1,2}(2) = 0.$$

Let's just double check our term rate structure to make sure we haven't made any algebraic mistakes. We should have

$$1 = \frac{1}{1+r_{0,2}} = \frac{1}{1+r_{0,1}} \boldsymbol{E}_{\hat{\mu}} \frac{1}{1+r_{1,2}} = \frac{9}{10}\left(\frac{5}{9}\cdot\frac{6}{5} + \frac{4}{9}\cdot\frac{1}{1}\right) = 1$$

and

$$1 = 1 + r_{0,2} = (1+r_{0,1})\, \boldsymbol{E}_{\mu}(1+r_{1,2}) = \frac{10}{9}\left(\frac{6}{10}\cdot\frac{5}{6} + \frac{4}{10}\cdot\frac{1}{1}\right) = 1.$$

But you already knew that, right?

Let Z have the payoffs indicated in Figure 3. Then

$$p_0^Z = 6.8, \quad p_1^Z(1) = 9.6, \quad p_1^Z(2) = 5.$$

Now consider American and European call options with strike price $k = 5$. Then our asset pricing formula gives

$$p_0^E = 1.8, \quad p_1^E(1) = 3.6, \quad p_1^E(2) = 0,$$

while

$$p_0^A = 2.3, \quad p_1^A(1) = 4.6 \text{ (the option is exercised early)}, \quad p_1^A(2) = 0.$$

4 The cash conundrum

The last example may have you screaming, how can the interest rate be negative? What if cash, C, is one of the assets? Well that depends on what you mean by cash. Remember, in our simple model no asset pays off until period 2. Suppose by cash you mean an asset that has a payoff $p_2^C(u,v) = 1$ for all (u,v), where the value 1 is in some unit of account, say dollars. Then there is no reason that the prices $p_1^C(u)$ or p_0^C should equal one in our unit of account. But suppose that it is true that $p_0^C = p_1^C(u) = p_2^C(u,v)$ for all (u,v). Then we must have $\sum_u \pi_1(u) = 1$ and $\sum_{(u,v)} \pi_2(u,v) = 1$, which imply $r_{0,2} = r_{0,1} = r_{1,2}(u) = 0$ for all u, and also that $\hat{\mu} = \mu$. So the existence of cash in this strong sense implies that all risk-free interest rates are zero. This is not as unrealistic as it may sound. Given that assets only pay off in period 2 and period 1 is just for updating and portfolio rebalancing, since cash is "consumed" only in period 2, its interest rate perhaps should be zero. But read on.

4.1 Change of unit of account

Suppose a set of prices $p^1,\dots,p^n$ in $\boldsymbol{R} \times \boldsymbol{R}^U \times \boldsymbol{R}^{U\times V}$ is arbitrage free, and let λ in $\boldsymbol{R} \times \boldsymbol{R}^U \times \boldsymbol{R}^{U\times V}$ satisfy $\lambda \gg 0$. Define new prices $\hat{p}^1,\dots,\hat{p}^n$ by

$$\begin{aligned}
\hat{p}_0^i &= \lambda_0 p_0^i \\
\hat{p}_1^i(u) &= \lambda_1(u) p_1^i(u) \\
\hat{p}_2^i(u,v) &= \lambda_2(u,v) p_2^i(u,v).
\end{aligned}$$

The new prices are arbitrage free and have as Arrow–Debreu prices the vector $\hat{\pi}$ defined by

$$\begin{aligned}
\hat{\pi}_1(u) &= \frac{\lambda_0 \pi_1(u)}{\lambda_1(u)} \\
\hat{\pi}_2(u,v) &= \frac{\lambda_0 \pi_2(u,v)}{\lambda_2(u,v)}.
\end{aligned}$$

To see this, just rewrite equations $(\star)$–$(\star\star\star)$ as

$$\begin{aligned}
\hat{p}_0^i = \lambda_0 p_0^i &= \sum_u \lambda_1(u) p_1^i(u) \frac{\lambda_0 \pi_1(u)}{\lambda_1(u)} = \sum_u \hat{p}_1^i(u) \hat{\pi}_1(u) \\
\hat{p}_1^i(u) = \lambda_1(u) p_1^i(u) &= \sum_v \lambda_2(u,v) p_2^i(u,v) \frac{\lambda_0 \pi_2(u,v)}{\lambda_2(u,v)} \frac{\lambda_1(u)}{\lambda_0 \pi_1(u)} = \sum_v \hat{p}_2^i(u,v) \frac{\hat{\pi}_2(u,v)}{\hat{\pi}_1(u)} \\
\hat{p}_0^i = \lambda_0 p_0^i &= \sum_{(u,v)} \lambda_2(u,v) p_2^i(u,v) \frac{\lambda_0 \pi_2(u,v)}{\lambda_2(u,v)} = \sum_{(u,v)} \hat{p}_2^i(u,v) \hat{\pi}_2(u,v).
\end{aligned}$$

We can think of this as changing the unit of account in each state of the world, say by using different currencies. This is not deep. But now suppose I take as my vector λ the vector of inverse prices of some asset. That is, let $*$ be an asset with $p^* \gg 0$, and express the price of every asset in terms of $*$. No arbitrage opportunity is created this way, but now $p_t^*(s) = 1$ for every time and state. In other words by changing the unit of account, we have made a riskless asset out of a risky asset. Cash, if there was cash to start with, is now a risky asset. In our purely financial world, there is no reason to prefer any one asset over another. (Well we do need $p^* \gg 0$, and you may argue that cash is the only asset in the real world with this property, but I doubt that even cash has that property.)

A Inequalities and a Theorem of the Alternative

The following result is an example of what is called a theorem of the alternative. This version may be found in Gale [2, Corollary 2, p. 49], who gives an elementary (but not necessarily easy) algebraic proof, or in Nikaidô [4, Theorem 3.7, p. 36], who attributes it to Stiemke [7]. Beware, the alternatives may be transposed in different references.

9 Stiemke's Theorem *Let A be an $n \times m$ matrix. Either*

(1) *the system of inequalities*

$$Ax > 0$$

has a solution $x \in \boldsymbol{R}^{\mathrm{m}}$,

Or else

(2) *the system of equations*

$$pA = 0$$

has a strictly positive solution $p \gg 0$ in $\boldsymbol{R}^{\mathrm{n}}$.

(But not both.)

Proof: Clearly both (1) and (2) cannot be true, for then we must have both $pAx = 0$ (as $pA = 0$) and $pAx > 0$ (as $p \gg 0$ and $Ax > 0$). So it suffices to show that if (1) fails, then (2) must hold.

Let $\Delta = \{z \in \boldsymbol{R}^{\mathrm{n}} : z \geqq 0 \text{ and } \sum_{j=1}^{n} z_j = 1\}$ be the the unit simplex in $\boldsymbol{R}^{\mathrm{n}}$. In geometric terms, (1) asserts that the span M of the columns $\{A^1, \ldots, A^n\}$ intersects the nonnegative orthant $\boldsymbol{R}^{\mathrm{m}}_{+}$ at a nonzero point, namely Ax. Since M is a linear subspace, if M intersects the nonnegative orthant at a nonzero point z, then $\frac{1}{\sum_i z_i} z$ belongs to $M \cap \Delta$. Thus the negation of (1) is equivalent to the disjointness of M and Δ.

So assume that condition (1) fails. Then since Δ is compact and convex and M is closed and convex, there is a hyperplane strongly separating Δ and M, see, e.g., [1, Theorem 2.9, p. 11]. That is, there is some nonzero $p \in \boldsymbol{R}^{\mathrm{n}}$ and some $\varepsilon > 0$ satisfying

$$p \cdot y + \varepsilon < p \cdot z \quad \text{for all } y \in M,\ z \in \Delta.$$

Since M is a linear subspace, we must have $p \cdot y = 0$ for all $y \in M$.[2] Consequently $p \cdot z > \varepsilon > 0$ for all $z \in \Delta$. Since the j^{th} unit coordinate vector e^j belongs to Δ, we see that $p_j = p \cdot e^j > 0$. That is, $p \gg 0$.

Since each $A^i \in M$, we have that $p \cdot A^i = 0$, i.e.,

$$pA = 0.$$

This completes the proof. ∎

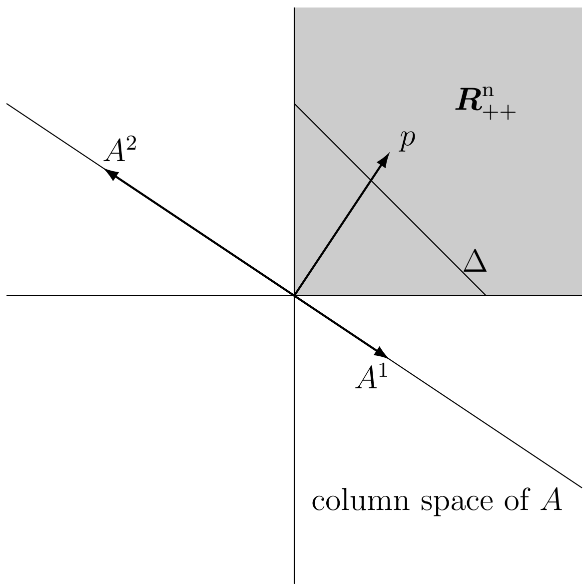

Figure 4. Geometry of the Stiemke Alternative

[2]To see this, suppose $\bar{y} \in M$ and $p \cdot \bar{y} \neq 0$. Then for any real number α, the vector $y_\alpha = \frac{\alpha}{p \cdot \bar{y}} \bar{y}$ belongs to M and $p \cdot y_\alpha = \alpha$. This contradicts the fact that $p \cdot y$ is bounded above on M by $p \cdot z$ for any $z \in \Delta$.

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