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Exercise on Saddlepoints and the Second Welfare Theorem

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Consider an n-person pure exchange economy with aggregate endowment $\omega \in \boldsymbol{R}^{\mathrm{m}}$. Assume $\omega > 0$. Let $u_i \colon \boldsymbol{R}^{\mathrm{m}}_{+} \to \boldsymbol{R}$ denote person i's utility function. (This implicitly assumes that preferences are selfish.) Recall that an *allocation* is a vector $x = (x^1, \dots, x^n) \in (\boldsymbol{R}^{\mathrm{m}}_{+})^{\mathrm{n}}$ satisfying $\sum_{i=1}^{n} x^i = \omega$. An allocation $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ is *Pareto efficient* if there is no allocation $x = (x^1, \dots, x^n)$ satisfying

$$u_i(x^i) \geqslant u_i(\bar{x}^i) \text{ for all } i = 1, \dots, n \quad \text{and} \quad u_i(x^i) > u_i(\bar{x}^i) \text{ for some } i.$$

An allocation $\tilde{x} = (\tilde{x}^1, \dots, \tilde{x}^n)$ is a *competitive market allocation* with respect to the allocation $(\omega^1, \dots, \omega^n)$ if there exists a nonzero price vector $p \in \boldsymbol{R}^{\mathrm{m}}$ such that for every $i = 1, \dots, n$,

$$u^i(\tilde{x}^i) \geqslant u^i(z) \text{ for all } z \in \boldsymbol{R}^{\mathrm{m}}_{+} \text{ satisfying } p \cdot z \leqslant p \cdot \omega^i.$$

That is, everyone is maximizing their utility subject to their budget constraint.

Assume now that each utility function is concave and strictly monotonic. Use the Saddlepoint Theorem to show that every strictly positive Pareto efficient allocation is a competitive market allocation with respect to itself.

Sample Answer

Let $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ be a strictly positive Pareto efficient allocation. Set $v_i = u^i(\bar{x}^i)$. Then $\bar{x}$ solves the following constrained maximization problem.

$$\underset{(x^1,\dots,x^n)\in(\boldsymbol{R}^{\mathrm{m}}_{+})^{\mathrm{n}}}{\text{maximize}} \; u^1(x^1) \text{ subject to } u^i(x^i) \geqslant v_i, \; \imath = 2, \dots, n, \text{ and } \sum_{i=1}^{n} x^i = \omega.$$

Since each u^i is monotonic, we may replace the resource constraints with the inequality constraints $\omega_j - \sum_{i=1}^{n} x^i_j \geqslant 0$, for $j = 1, \dots, m$. Since each $\bar{x}^i > 0$ and each u^i is monotonic, we see that $\tilde{x}$, defined by $\tilde{x}^1 = 0$ and $\tilde{x}^i = \bar{x}^i + \frac{1}{n}\bar{x}^1$ for $i = 2, \dots, n$, satisfies $u^i(\tilde{x}^i) - v_i > 0$ for $i = 2, \dots, n$ and $\omega_j - \sum_{i=1}^{n} \tilde{x}^i_j = \frac{1}{n}\bar{x}^1_j > 0$, so Slater's Condition is satisfied. Now observe that all the constraints are defined by concave functions.

*This is based on my solution (2–24–1976) to a problem in Ec 8–161 taught by Leonid Hurwicz. That solution was for the 2-person case.

Thus by the Saddlepoint Theorem there exist nonnegative multipliers $\bar\mu_2, \ldots, \bar\mu_n$ and $\bar\pi_1, \ldots, \bar\pi_m$ such that $(\bar x; \bar\mu, \bar\pi)$ is a saddlepoint of the Lagrangean

$$L(x; \mu, \pi) = u^1(x^1) + \sum_{i=2}^n \mu_i\big(u^i(x^i) - v_i\big) + \sum_{j=1}^m \pi_j \big[\omega_j - \sum_{i=1}^n x^i_j\big]$$

over $(\boldsymbol{R}^{\mathrm{m}}_+)^{\mathrm{n}} \times \big[\boldsymbol{R}^{\mathrm{n}}_+ \times \boldsymbol{R}^{\mathrm{m}}_+\big]$. To make things more symmetric, set $\bar\mu^1 = 1$. Then the saddlepoint conditions become

$$\begin{aligned} &\sum_{i=1}^n \bar\mu_i\big(u^i(x^i) - v_i\big) + \sum_{j=1}^m \bar\pi_j \big[\omega_j - \sum_{i=1}^n x^i_j\big] \\ \leqslant \quad & + \sum_{i=1}^n \bar\mu_i\big(u^i(\bar x^i) - v_i\big) + \sum_{j=1}^m \bar\pi_j \big[\omega_j - \sum_{i=1}^n \bar x^i_j\big] \qquad (1) \\ \leqslant \quad & \sum_{i=1}^n \mu_i\big(u^i(\bar x^i) - v_i\big) + \sum_{j=1}^m \pi_j \big[\omega_j - \sum_{i=1}^n \bar x^i_j\big] \qquad (2) \end{aligned}$$

for all $x \in (\boldsymbol{R}^{\mathrm{m}}_+)^{\mathrm{n}}$ and all $(\mu, \pi) \in \boldsymbol{R}^{\mathrm{n}}_+ \times \boldsymbol{R}^{\mathrm{m}}_+$. Furthermore, the complementary slackness conditions

$$\bar\mu_i\big(u^i(\bar x^i) - v_i\big) = 0 \quad i = 1, \ldots, n$$

and

$$\bar\pi_j \big[\omega_j - \sum_{i=1}^n \bar x^i_j\big] \quad j = 1, \ldots, m$$

are satisfied.

We now show that no $\bar\pi_j$ is zero. For suppose $\bar\pi_k = 0$. Let e^k denote the kth unit coordinate vector in $\boldsymbol{R}^{\mathrm{m}}$. Let us now evaluate (1) for x given by $x^1 = \bar x^1 + e^k$ and $x^i = \bar x^i$ for $i = 2, \ldots, n$. This yields

$$\begin{aligned} & u^1(\bar x^1 + e^k) - v_1 + \sum_{i=2}^n \bar\mu_i\big(\bar u^i(\bar x^i) - v_i\big) + \sum_{j=1} \bar\pi_j \big[\omega_j - \sum_{i=1}^n \bar x^i_j\big] - \bar\pi_k \\ \leqslant \quad & u^1(\bar x^1) - v_1 + \sum_{i=2}^n \bar\mu_i\big(u^i(\bar x^i) - v_i\big) + \sum_{j=1}^m \bar\pi_j \big[\omega_j - \sum_{i=1}^n \bar x^i_j\big], \end{aligned}$$

which in light of the assumption that $\bar\pi_k = 0$ boils down to

$$u^1(\bar x^1 + e^k) \leqslant u^1(\bar x^1),$$

which contradicts the strict monotonicity of u^1.

Moreover, no $\bar\mu_i = 0$ either. For suppose $\bar\mu_k = 0$ for some $k > 1$. Consider equation (1) for x given by $x^1 = \bar x^1 + \bar x^k$, $x^k = 0$, and $x^i = \bar x^i$ for $i \neq 1, k$. Then we get

$$u^1(\bar x^1 + \bar x^k) \leqslant u^1(\bar x^1),$$

which again contradicts the strict monotonicity of u^1.

We now show that for each $i = 1, \dots, n$, the point $(\bar{x}^i; \frac{1}{\bar{\mu}_i})$ is a saddlepoint of the function

$$L_i(z;\nu) = u^i(z) + \nu\big(\sum_{j=1}^{m} \bar{\pi}_j(\bar{x}^i_j - z_j)\big) \tag{3}$$

over $\boldsymbol{R}^{\mathrm{m}}_+ \times \boldsymbol{R}_+$. That is, we need to show that

$$\begin{aligned} u^i(z) + \frac{1}{\bar{\mu}_i}\big(\sum_{j=1}^{m} \bar{\pi}_j(\bar{x}^i_j - z_j)\big) \quad &\leqslant \quad u^i(\bar{x}^i) + \frac{1}{\bar{\mu}_i}\big(\sum_{j=1}^{m} \bar{\pi}_j(\bar{x}^i_j - \bar{x}^i_j)\big) && (4) \\ &\leqslant \quad u^i(\bar{x}^i) + \nu\big(\sum_{j=1}^{m} \bar{\pi}_j(\bar{x}^i_j - \bar{x}^i_j)\big) && (5) \end{aligned}$$

for all $z \in \boldsymbol{R}^{\mathrm{m}}_+$ and all $\nu \in \boldsymbol{R}_+$. Clearly (5) is true. Suppose by way of contradiction that for some k and some $z^k \in \boldsymbol{R}^{\mathrm{m}}$, inequality (4) is violated. That is,

$$u^k(z^k) + \frac{1}{\bar{\mu}_k}\big(\sum_{j=1}^{m} \bar{\pi}_j(\bar{x}^k_j - z^k_j)\big) > u^k(\bar{x}^k). \tag{6}$$

Then subtracting v_k from each side and multiplying by the positive scalar $\bar{\mu}_k$ yields

$$\bar{\mu}_k\big(u^k(z^k) - v_k\big) + \sum_{j=1}^{m} \bar{\pi}_j(\bar{x}^k_j - z^k_j) > \bar{\mu}_k\big(u^k(\bar{x}^k) - v_k\big).$$

Evaluating (1) evaluated at $x \in (\boldsymbol{R}^{\mathrm{m}}_+)^{\mathrm{n}}$ defined by $x^i = \bar{x}^i$ for $i \neq k$ and $x^k = z^k$, we get

$$\begin{aligned} \sum_{i \neq k} \bar{\mu}_i\big(u^i(\bar{x}^i) - v_i\big) + \sum_{j=1}^{m} \bar{\pi}_j\big[\omega_j - \sum_{i \neq k} \bar{x}^i_j\big] + \bar{\mu}_k\big(u^k(z^k) - v_k\big) - \sum_{j=1}^{m} \bar{\pi}_j z^k_j \\ \leqslant \quad \sum_{i=1}^{n} \bar{\mu}_i\big(u^i(\bar{x}^i) - v_i\big) + \sum_{j=1}^{m} \bar{\pi}_j\big[\omega_j - \sum_{i=1}^{n} \bar{x}^i_j\big], \end{aligned}$$

which implies

$$u^k(z^k) + \frac{1}{\bar{\mu}_k}\big(\sum_{j=1}^{m} \bar{\pi}_j(\bar{x}^k_j - z^k_j)\big) \leqslant u^k(\bar{x}^k),$$

which in turn contradicts (6). This contradiction shows that $(\bar{x}^k; \frac{1}{\bar{\mu}_k})$ is a saddlepoint of (3). But now by the easy half of the Saddlepoint Theorem, we see that $\bar{x}^k$ maximizes $u^k(z)$ over $\boldsymbol{R}^{\mathrm{m}}_+$ subject to $\bar{\pi} \cdot z \leqslant \bar{\pi}\bar{x}^k$ for each k. That is, $\bar{x}$ is a competitive market allocation with respect to itself at the prices $\bar{\pi}$.