Chapter 11

Options

Road Map

Part A Introduction to finance.

Part B Valuation of assets, given discount rates.

Part C Determination of risk-adjusted discount rate.

Part D Introduction to derivatives.

- Forwards and futures.
- Options.
- Real options.

Main Issues

- Introduction to Options
- Use of Options
- Properties of Option Prices
- Valuation Models of Options

1 Introduction to Options

1.1 Definitions

Option types:

Call: Gives owner the right to purchase an asset (the underlying asset) for a given price (exercise price) on or before a given date (expiration date).

Put: Gives owner the right to sell an asset for a given price on or before the expiration date.

Exercise styles:

European: Gives owner the right to exercise the option only on the expiration date.

American: Gives owner the right to exercise the option on or before the expiration date.

Key elements in defining an option:

- Underlying asset and its price S
- Exercise price (strike price) K
- Expiration date (maturity date) T (today is 0)
- European or American.

1.2 Option Payoff

The payoff of an option on the expiration date is determined by the price of the underlying asset.

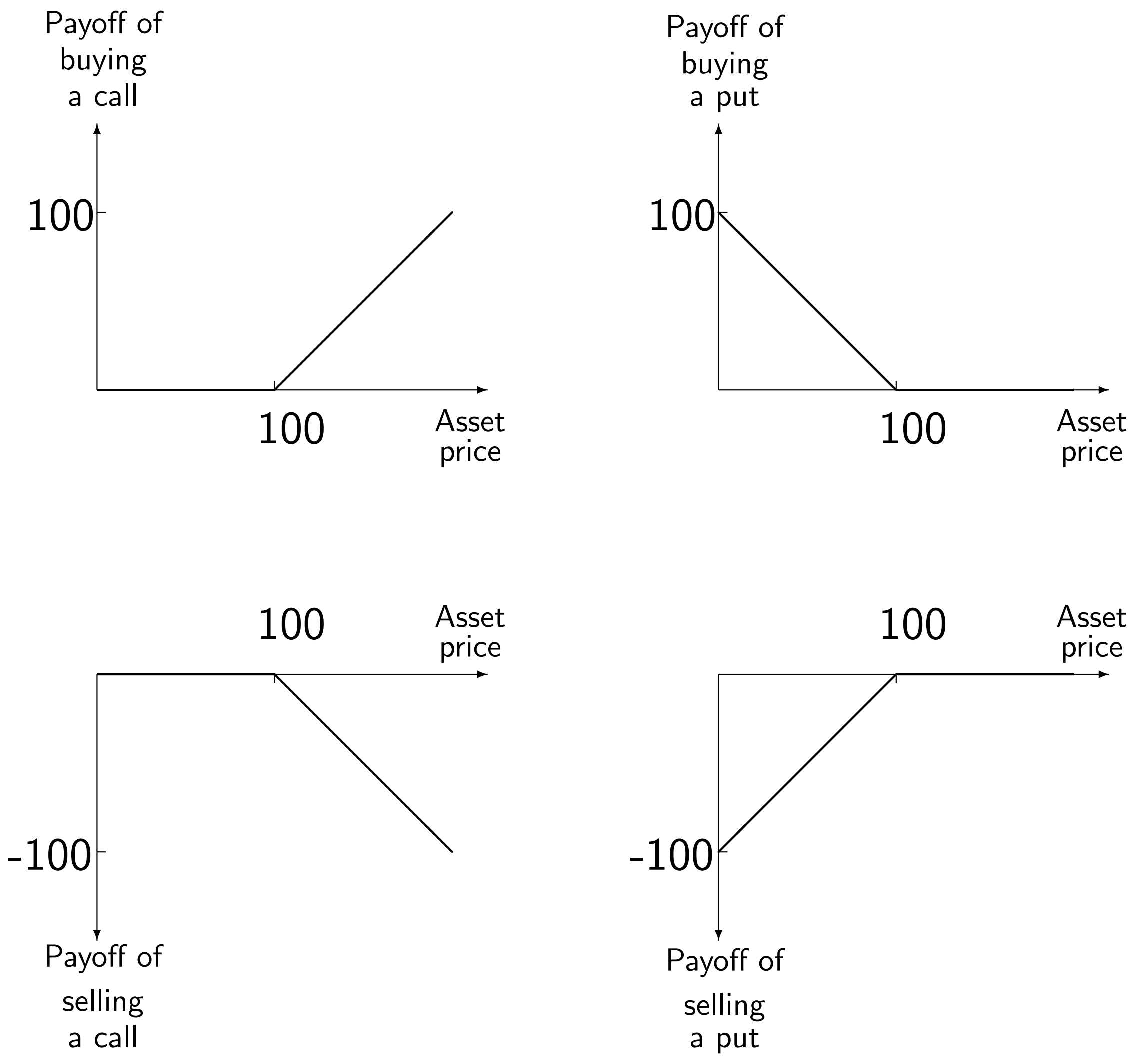
Example. Consider a European call option on IBM with exercise price \$100. This gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at \$100 on the expiration date. Depending on the share price of IBM on the expiration date, the option owner's payoff looks as follows:

IBM Price	Action	Payoff
$\vdots$	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
$\vdots$	Exercise	$S_T - 100$

Note:

- The payoff of an option is never negative.
- Sometimes, it is positive.
- Actual payoff depends on the price of the underlying asset.

► Payoffs of calls and puts can be described by plotting their payoffs at expiration as function of the price of the underlying asset:

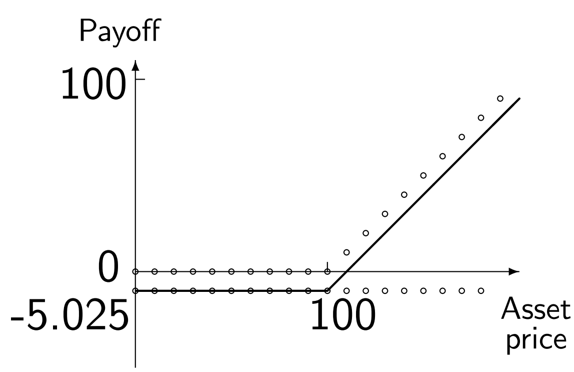


The net payoff from an option must includes its cost.

Example. A European call on IBM shares with an exercise price of \$100 and maturity of three months is trading at \$5. The 3-month interest rate, not annualized, is 0.5%. What is the price of IBM that makes the call break-even?

At maturity, the call's net payoff is as follows:

IBM Price	Action	Payoff	Net payoff
⋮	Not Exercise	0	- 5.025
80	Not Exercise	0	- 5.025
90	Not Exercise	0	- 5.025
100	Not Exercise	0	- 5.025
110	Exercise	10	4.975
120	Exercise	20	14.975
130	Exercise	30	24.975
⋮	Exercise	$S_T - 100$	$S_T - 100 - 5.25$



The break even point is given by:

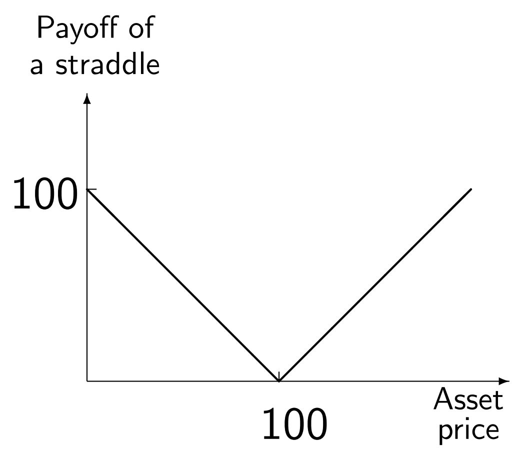
$$\text{Net payoff} = S_T - 100 - (5)(1 + 0.005) = 0$$

or

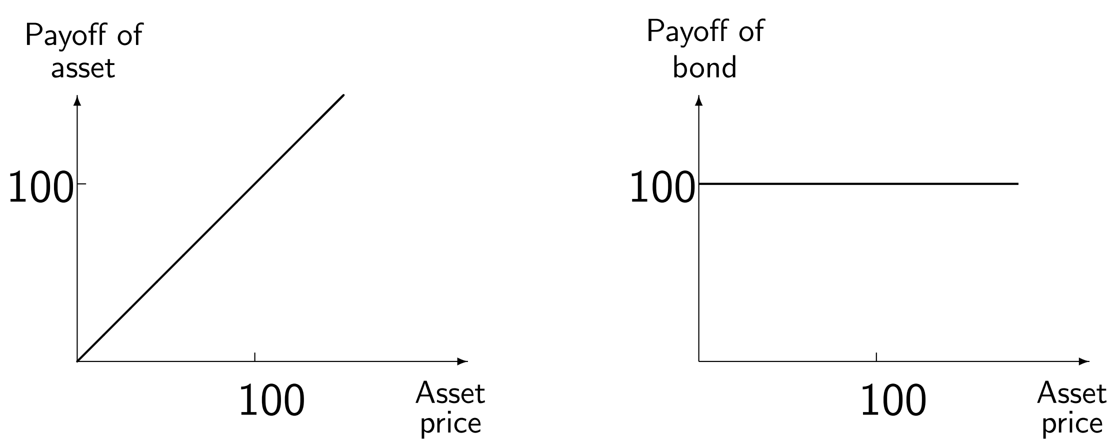
$$S_T = \$105.025.$$

Using the payoff diagrams, we can also examine the payoff of a portfolio consisting of options as well as other assets.

Example. Consider the following portfolio (a straddle): buy one call and one put (with the same exercise price). Its payoff is:



Example. The underlying asset and the bond (with face value \$100) have the following payoff diagram:



1.3 Corporate Securities as Options

Example. Consider two firms, A and B, with identical assets but different capital structures (in market value terms).

Balance sheet of A

Asset	\$30	\$0	Bond
		30	Equity
	---	---	
	\$30	\$30	

Balance sheet of B

Asset	\$30	\$25	Bond
		5	Equity
	---	---	
	\$30	\$30	

- Firm B's bond has a face value of \$50. Thus default is likely.
- Consider the value of stock A, stock B, and a call on the underlying asset of firm B with an exercise price \$50:

Asset Value	Value of Stock A	Value of Stock B	Value of Call
\$20	20	0	0
40	40	0	0
50	50	0	0
60	60	10	10
80	80	30	30
100	100	50	50

- Stock B gives exactly the same payoff as a call option written on its asset.
- Thus B's common stocks really are call options.

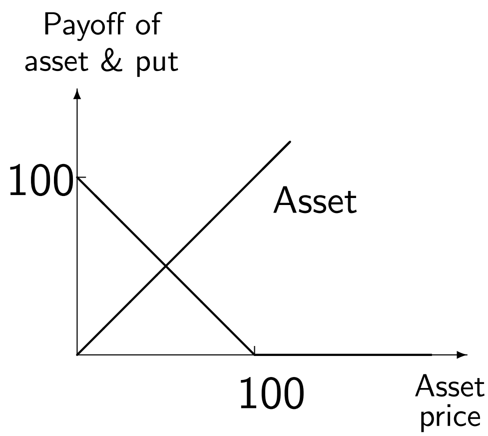
Indeed, many corporate securities can be viewed as options:

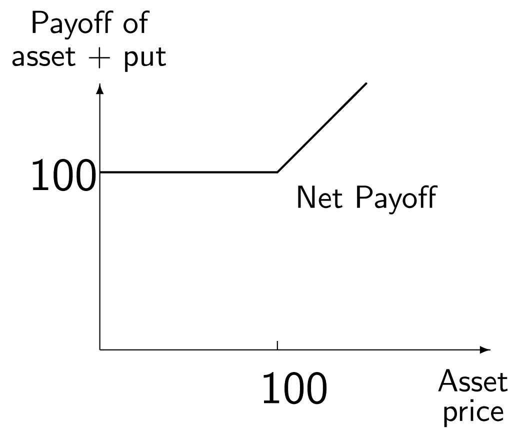
Common Stock:	A call option on the assets of the firm with the exercise price being its bond's redemption value.
Bond:	A portfolio combining the firm's assets and a short position in the call with exercise price equal bond redemption value.
Warrant:	Call options on the stock issued by the firm.
Convertible bond:	A portfolio combining straight bonds and a call option on the firm's stock with the exercise price related to the conversion ratio.
Callable bond:	A portfolio combining straight bonds and a call written on the bonds.

2 Use of Options

Hedging Downside while Keeping Upside.

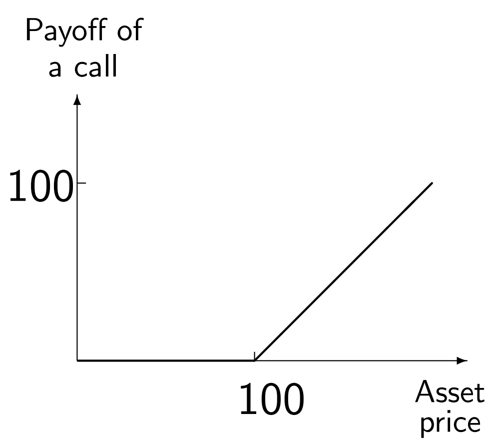
The put option allows one to hedge the downside risk of an asset.

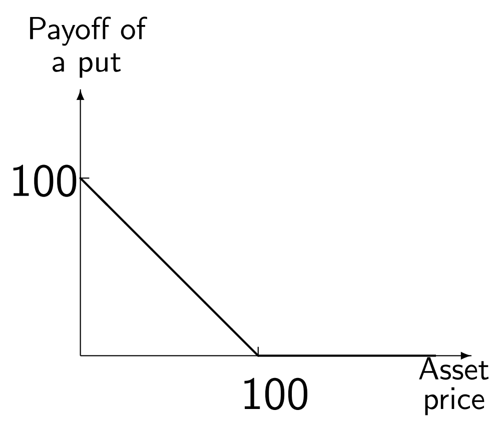




Speculating on Changes in Prices

Buying puts (calls) is a convenient way of speculating on decreases (increases) in the price of the underlying asset. Options require only a small initial investment.





3 Properties of Options

For convenience, we refer to the underlying asset as stock. It could also be a bond, foreign currency or some other asset.

Notation:

S: Price of stock now

S_T: Price of stock at T

B: Price of discount bond with face value \$1 and maturity T (clearly, $B \leq 1$)

C: Price of a European call with strike price K and maturity T

P: Price of a European put with strike price K and maturity T

c: Price of an American call with strike price K and maturity T

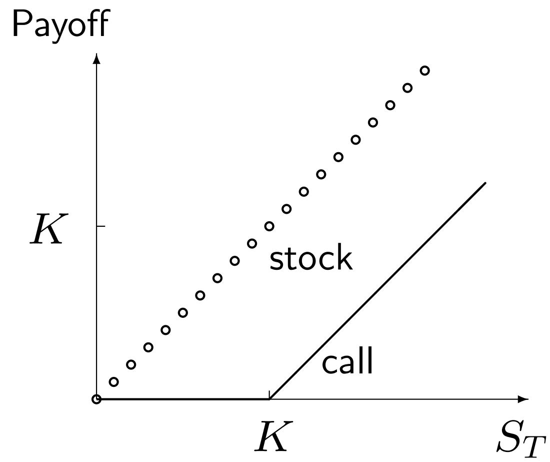
p: Price of an American put with strike price K and maturity T.

Price Bounds

First consider European options on a non-dividend paying stock.

1. $C \geq 0$.

2. $C \leq S$ — The payoff of stock dominates that of call:

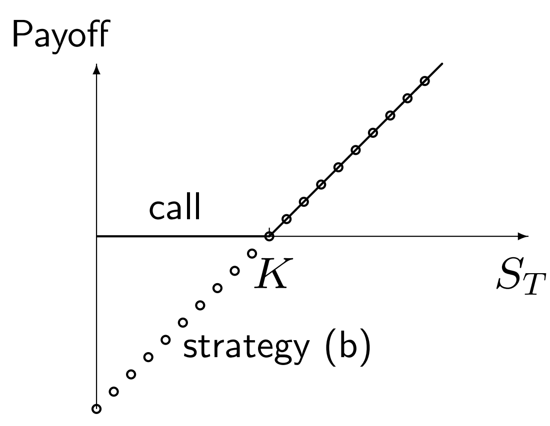


3. $C \geq S - KB$ (assuming no dividends).

   Strategy (a): Buy a call

   Strategy (b): Buy a share of stock by borrowing KB.

   The payoff of strategy (a) dominates that of strategy (b):

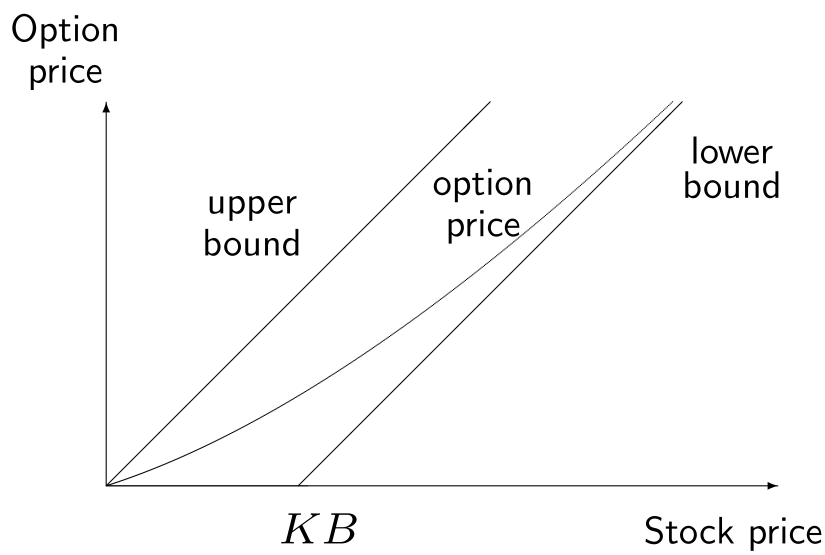


Since $C \geq 0$, we have

$$C \geq \max[S - KB,\ 0].$$

4. Combining the above, we have

$$\max[S - KB,\ 0] \leq C \leq S.$$

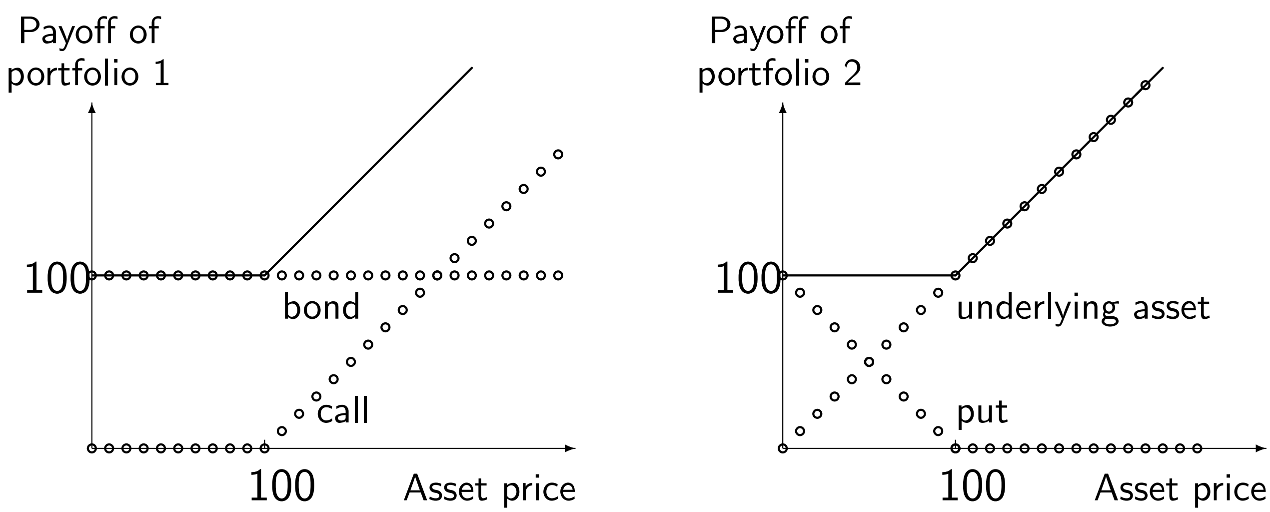


Put-Call Parity

Consider the following two portfolios:

1. A portfolio of a call with exercise price \$100 and a bond with face value \$100.
2. A portfolio of a put with exercise price \$100 and a share of the underlying asset.

Their payoffs are



Their payoffs are identical, so must be their prices:

$$C + K/(1+r)^T = P + S.$$

This is called the put-call parity.

American Options and Early Exercise

1. American options are worth more than their European counterparts.

2. Without dividends, never exercise an American call early.

   - Exercising prematurely requires paying the exercise price early, hence loses the time value of money.
   - Exercising prematurely foregoes the option value

   $$c(S, K, T) = C(S, K, T).$$

3. Without dividends, it can be optimal to exercise an American put early.

   Example. A put with strike \$10 on a stock with price zero.

   - Exercise now gives \$10 today
   - Exercise later gives \$10 later.

Effect of Dividends

1. With dividends,

   $$\max[S - KB - PV(D),\ 0] \leq C \leq S.$$

2. Dividends make early exercise more likely for American calls and less likely for American puts.

Option Value and Asset Volatility

Option value increases with the volatility of underlying asset.

Example. Two firms, A and B, with the same current price of \$100. B has higher volatility of future prices. Consider call options written on A and B, respectively, with the same exercise price \$100.

	Good state	bad state
Probability	p	$1-p$
Stock A	120	80
Stock B	150	50
Call on A	20	0
Call on B	50	0

Clearly, call on stock B should be more valuable.

4 Binomial Option Pricing Model

Determinants of Option Value

Key factors in determining option value:

1. price of underlying asset S
2. strike price K
3. time to maturity T
4. interest rate r
5. dividends D
6. volatility of underlying asset σ.

Additional factors that can sometimes influence option value:

7. expected return on the underlying asset
8. additional properties of stock price movements
9. investors' attitude toward risk, ...

Price Process of Underlying Asset

In order to have a complete option pricing model, we need to make additional assumptions about

1. Price process of the underlying asset (stock)
2. Other factors.

We will assume, in particular, that:

- Prices do not allow arbitrage.
- Prices are "reasonable".
- A benchmark model — Price follows a binomial process.

One-period Bimomial Model

Example. Valuation of a European call on a stock.

- Current stock price is \$50.
- There is one period to go.
- Stock price will either go up to \$75 or go down to \$25.
- There are no cash dividends.
- The strike price is \$50.
- one period borrowing and lending rate is 10%.

The stock and bond present two investment opportunities:

$$50 \begin{cases} 75 \\ 25 \end{cases} \qquad\qquad 1 \begin{cases} 1.1 \\ 1.1 \end{cases}$$

The option's payoff at expiration is:

$$C_0 \begin{cases} 25 \\ 0 \end{cases}$$

Question: What is C_0, the value of the option today?

Claim: We can form a portfolio of stock and bond that gives identical payoffs as the call.

Consider a portfolio $(a,\, b)$ where

- a is the number of shares of the stock held
- b is the dollar amount invested in the riskless bond.

We want to find $(a,\, b)$ so that

$$\begin{aligned} 75a + 1.1b &= 25 \\ 25a + 1.1b &= 0. \end{aligned}$$

There is a unique solution

$$a = 0.5 \quad \text{and} \quad b = -11.36.$$

That is

- buy half a share of stock and sell \$11.36 worth of bond
- payoff of this portfolio is identical to that of the call
- present value of the call must equal the current cost of this "replicating portfolio" which is

  $$(50)(0.5) - 11.36 = 13.64.$$

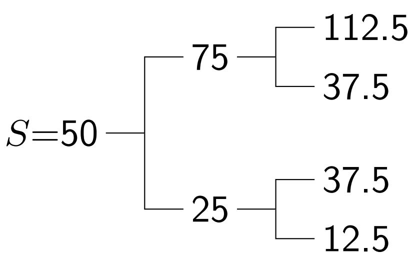
Definition: Number of shares needed to replicate one call option is called *hedge ratio* or *option delta*.

In the above problem, the option delta is a:

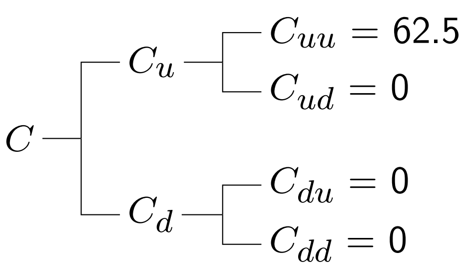
$$\text{Option delta} = 1/2.$$

Two-period Binomial Model

Now we generalize the above example when there are two periods to go: period 1 and period 2. The stock price process is:



The call price follows the following process:



where

- the terminal value of the call is known, and
- C_u and C_d denote the option value next period when the stock price goes *up* and goes *down*, respectively.

We derive current value of the call by working backwards: first compute its value next period, and then its current value.

Step 1. Start with Period 1:

1. Suppose the stock price goes up to \$75 in period 1:
   - Construct the replicating portfolio at node ($t = 1$, up):

     $$\begin{aligned} 112.5a + 1.1b &= 62.5 \\ 37.5a + 1.1b &= 0. \end{aligned}$$

   - The unique solution is

     $$a = 0.833 \quad \text{and} \quad b = -28.4.$$

   - The cost of this portfolio is

     $$(0.833)(75) - 28.4 = 34.075.$$

   - The exercise value of the option is

     $$75 - 50 = 25 < 34.075.$$

   - Thus, $C_u = 34.075$.

2. Suppose the stock price goes down to \$25 in period 1. Repeat the above for node ($t = 1$, down):
   - The replicating portfolio is

     $$a = 0 \quad \text{and} \quad b = 0.$$

   - The call value at the lower node next period is $C_d = 0$.

Step 2. Now go back one period, to Period 0:

- The option's value next period is either 34.075 or 0 depending upon whether the stock price goes up or down:

$$C_0 \begin{cases} C_u = 34.075 \\ C_d = 0 \end{cases}$$

- If we can construct a portfolio of the stock and bond to "replicate" the value of the option next period, then the cost of this "replicating portfolio" must equal the option's present value.

- Find a and b so that

$$\begin{aligned} 75a + 1.1b &= 34.075 \\ 25a + 1.1b &= 0. \end{aligned}$$

- The unique solution is

$$a = 0.6815 \quad \text{and} \quad b = -15.48.$$

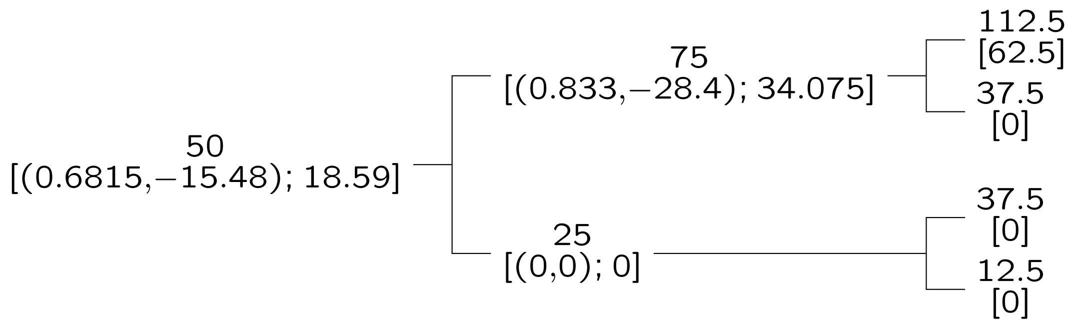
- The cost of this portfolio is

$$(0.6815)(50) - 15.48 = 18.59.$$

- The present value of the option must be $C_0 = 18.59$ (which is greater than the exercise value 0).

We have also confirmed that the option will not be exercised before maturity.

Summary of the replicating strategy:



"Play Forward" —

1. In period 0: spend \$18.59 and borrow \$15.48 at 10% interest rate to buy 0.6815 shares of the stock.
2. In period 1:
   (a) When the stock price goes up, the portfolio value becomes 34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing 28.4 at 10%.
      - One period hence, the payoff of this portfolio exactly matches that of the call.

   (b) When the stock price goes down, the portfolio becomes worthless. Close out the position.
      - The portfolio payoff one period hence is zero.

Thus

- No early exercise.
- Replicating strategy gives payoffs identical to those of the call.
- Initial cost of the replicating strategy must equal the call price.

Lessons from the Binomial Model

What we have used to calculate option's value

- current stock price
- magnitude of possible future changes of stock price – volatility
- interest rate
- strike price
- time to maturity.

What we have *not* used

- probabilities of upward and downward movements
- investor's attitude towards risk.

Questions on the Binomial Model

- What is the length of a period?
- Price can take more than two possible values.
- Trading takes place continuously.

Response: The length of a period can be arbitrarily small.

5 "Risk-Neutral" Pricing: A Shortcut

Consider the "up" and "down" *digital options*:

$$-\!\!\left[\begin{array}{l}1\\ 0\end{array}\right. \quad \text{and} \quad -\!\!\left[\begin{array}{l}0\\ 1\end{array}\right.$$

Let their prices be d_u and d_d, respectively.

Now, consider a security with the following payoff:

$$-\!\!\left[\begin{array}{l}CF_u\\ CF_d\end{array}\right. = CF_u \times -\!\!\left[\begin{array}{l}1\\ 0\end{array}\right. + CF_d \times -\!\!\left[\begin{array}{l}0\\ 1\end{array}\right.$$

No free-lunch requires:

$$\text{PV}(CF) = d_u CF_u + d_d CF_d.$$

In particular,

- Stock:

$$d_u(75) + d_d(25) = 50.$$

- Bond:

$$d_u(1.1) + d_d(1.1) = 1.$$

Thus, we have

$$d_u = \frac{0.6}{1.1} \quad \text{and} \quad d_d = \frac{0.4}{1.1}.$$

Consequently,

- Call with strike of $50:

$$\frac{0.6}{1.1}(25) + \frac{0.4}{1.1}(0) = \frac{15}{1.1} = 13.64.$$

- Put with strike of $36:

$$\frac{0.6}{1.1}(0) + \frac{0.4}{1.1}(11) = 4.$$

Let

$$q_u = \frac{d_u}{d_u + d_d} = 0.6, \quad q_d = \frac{d_d}{d_u + d_d} = 0.4 \quad \left(d_u + d_d = \frac{1}{1+r} \right)$$

we can write:

$$\text{PV}(CF) = \frac{q_u\, CF_u + q_d\, CF_d}{1+r}$$

We call q_u and q_d risk-adjusted probabilities.

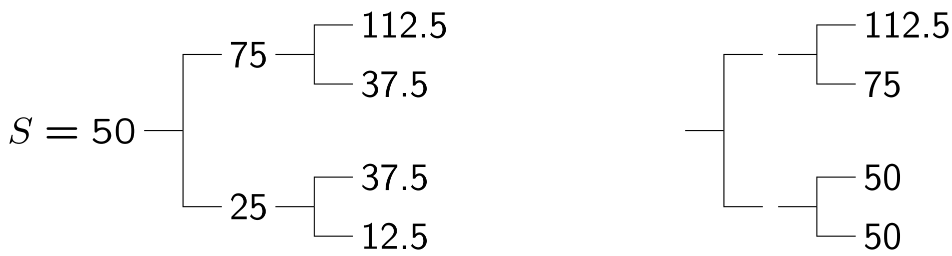
Thus, the price of a security equals the expected payoff using the risk-adjusted probabilities, discounted at the risk-free rate:

$$\text{PV}(CF) = \frac{q_u\, CF_u + q_d\, CF_d}{1+r} = \frac{\mathrm{E}_q[CF]}{1+r}.$$

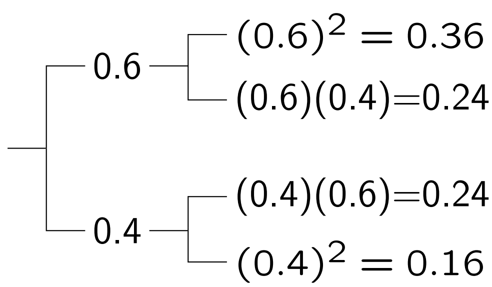
This is called the *risk-neutral* pricing formula.

- Risk-adjusted probabilities are (normalized) prices.
- They are different from the true probabilities.
- The market in general is *not* risk-neutral.

Example. We want to price an exotic financial contract that pays in period-2 the maximum the stock price has achieved between now and then:



We can easily find the risk-neutral probabilities for two periods:



The price of the contract is

$$
\begin{aligned}
PV_0 &= \frac{(0.36)(112.5)+(0.24)(75)+(0.24)(50)+(0.16)(50)}{(1.1)^2} \\
&= 78.5/1.21 = 64.88
\end{aligned}
$$

6 Black-Scholes Formula

If we let the period-length get smaller and smaller, we obtain the Black-Scholes option pricing formula:

$$C(S, K, T) = SN\big(x\big) - KR^{-T}N\big(x - \sigma\sqrt{T}\big)$$

where

- x is defined by
$$x = \frac{\ln\big(S/KR^{-T}\big)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$
- T is in units of a year
- R is one plus the annual riskless interest rate
- σ is the volatility of annual returns on the underlying asset
- $N(\cdot)$ is the normal cumulative density function.

An interpretation of the Black-Scholes formula:

- The call is equivalent to a levered long position in the stock.
- The replicating strategy:
  - $SN(x)$ is the amount invested in the stock
  - $KR^{-T}N\big(x - \sigma\sqrt{T}\big)$ is the dollar amount borrowed
  - The option delta is $N(x) = C_S$.

Example. Consider a European call option on a stock with the following data:

1. $S = 50$, $K = 50$, $T = 30$ days
2. The volatility σ is 30% per year
3. The current annual interest rate is 5.895%.

Then

$$x = \frac{\ln\left(50/50(1.05895)^{-\frac{30}{365}}\right)}{(0.3)\sqrt{\frac{30}{365}}} + \frac{1}{2}(0.3)\sqrt{\frac{30}{365}} = 0.0977$$

$$\begin{aligned} C &= 50N(0.0977) - 50(1.05895)^{-\frac{30}{365}} N\left(0.0977 - 0.3\sqrt{\frac{30}{365}}\right) \\ &= 50(0.53890) - 50(0.99530)(0.50468) \\ &= 1.83. \end{aligned}$$

7 Homework

Readings:

- BMA Chapters 20.
- BKM Chapters 20, 21.

Assignment:

- Problem Set 7.