Chapter 8

CAPM and APT

Road Map

Part A Introduction to finance.

Part B Valuation of assets, given discount rates.

Part C Determination of risk-adjusted discount rates.

- Introduction to return and risk.
- Portfolio theory.
- CAPM and APT.

Part D Introduction to derivative securities.

Main Issues

- Derivation of CAPM
- Risk and Return in CAPM
- Applications of CAPM
- Empirical Evidence
- APT

1 Introduction

Portfolio theory analyzes investors' asset demand given asset returns.

1. Diversify to eliminate non-systematic risk.
2. Hold only the risk-free asset and the tangent portfolio.

This chapter studies how investors' asset demand determines the relation between assets' risk and return in a market equilibrium.

► A model to price risky assets.

$$\mathrm{E}[\tilde{r}_i] = ?$$

2 The Market Portfolio

Definition: The market portfolio is the portfolio of all risky assets traded in the market.

The market capitalization of an asset is its total market value.

Suppose there are a total of $i = 1, \ldots, n$ risky assets. Asset i's market capitalization is

$$\mathsf{MCAP}_i = (\text{price per share})_i \times (\# \text{ of shares outstanding})_i.$$

The total market capitalization of all risky assets is

$$\mathsf{MCAP}_M = \sum_{i=1}^{n} \mathsf{MCAP}_i.$$

The market portfolio is the portfolio with weights in each risky asset i being

$$w_i = \frac{\mathsf{MCAP}_i}{\sum_{j=1}^{n} \mathsf{MCAP}_j} = \frac{\mathsf{MCAP}_i}{\mathsf{MCAP}_M}.$$

We denote the market portfolio by $\mathbf{w}_M$.

3 Derivation of CAPM

Assumptions for this chapter:

1. Investors agree on the distribution of asset returns.
2. Investors hold efficient frontier portfolios.
3. There is a risk-free asset:
    - paying interest rate r_F
    - in zero net supply.
4. Demand of assets equals supply in equilibrium.

Implications:

1. Every investor puts their money into two pots:
    - the riskless asset
    - a *single* portfolio of risky assets – the tangent portfolio.
2. All investors hold the *risky* assets in same proportions
    - they hold the same risky portfolio, the tangent portfolio.
3. The tangent portfolio is the market portfolio.

CAPM requires that in equilibrium total asset holdings of all investors must equal the total supply of assets.

We show this through the example below.

There are only three risky assets, A, B and C. Suppose that the tangent portfolio is

$$\mathbf{w}_T = (w_{\mathrm{A}}, w_{\mathrm{B}}, w_{\mathrm{C}}) = (0.25, 0.50, 0.25).$$

There are only three investors in the economy, 1, 2 and 3, with total wealth of 500, 1000, 1500 billion dollars, respectively. Their asset holdings (in billion dollars) are:

Investor	Riskless	A	B	C
1	100	100	200	100
2	200	200	400	200
3	-300	450	900	450
Total	0	750	1500	750

Claim:

The market portfolio is the tangent portfolio:

$$\mathbf{w}_M = \mathbf{w}_T.$$

In equilibrium, the total dollar holding of each asset must equal its market value:

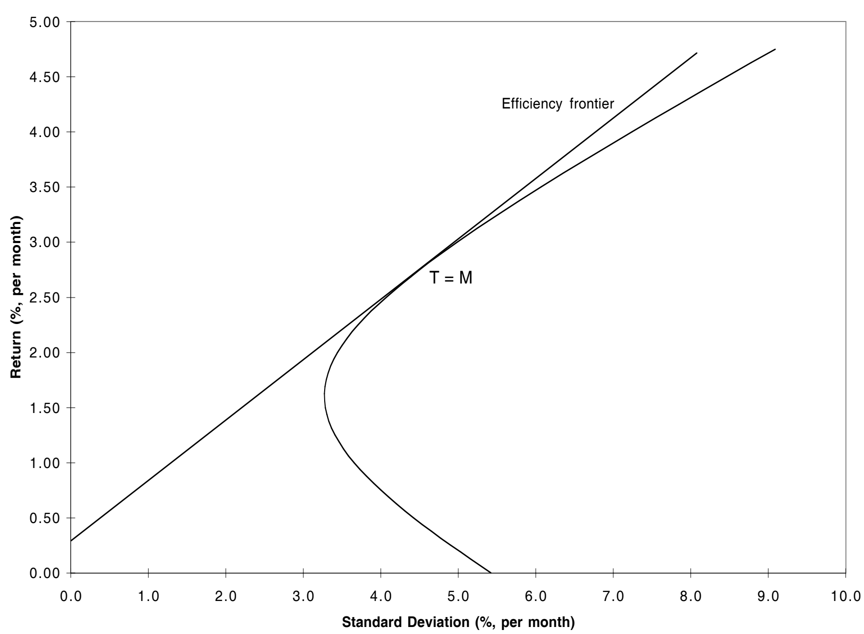
$$\begin{aligned}\text{Market capitalization of A} &= \$750 \text{ billion}\\ \text{Market capitalization of B} &= \$1500 \text{ billion}\\ \text{Market capitalization of C} &= \$750 \text{ billion.}\end{aligned}$$

The total market capitalization is

$$750 + 1500 + 750 = \$3,000 \text{ billion.}$$

The market portfolio is the tangent portfolio:

$$\mathbf{w}_M = \left(\frac{750}{3000}, \frac{1500}{3000}, \frac{750}{3000}\right) = (0.25, 0.50, 0.25) = \mathbf{w}_T.$$



4 The CAPM

4.1 Contribution of An Asset to A Portfolio

In the presence of a risk-free asset, a portfolio's return is

$$\tilde{r}_p = \left(1 - \sum_{i=1}^{n} w_i\right) r_F + \sum_{i=1}^{n} w_i \tilde{r}_i = r_F + \sum_{i=1}^{n} w_i \left(\tilde{r}_i - r_F\right).$$

Expected return.

The expected portfolio return is

$$\bar{r}_p = r_F + \sum_{i=1}^{n} w_i \left(\bar{r}_i - r_F\right).$$

The marginal contribution of risky asset i to the expected portfolio return is its risk premium:

$$\frac{\Delta \bar{r}_p}{\Delta w_i} = \bar{r}_i - r_F.$$

("Marginal contribution of x to A" means the incremental changes of A when x changes by a small amount.)

Risk.

The variance of portfolio return is the sum of all entries of the table:

	$w_1 r_1$	$w_2 r_2$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$w_1 w_2 \sigma_{12}$	$\cdots$	$w_1 w_n \sigma_{1n}$
$w_2 r_2$	$w_2 w_1 \sigma_{21}$	$w_2^2 \sigma_2^2$	$\cdots$	$w_2 w_n \sigma_{2n}$
$\cdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_n w_1 \sigma_{n1}$	$w_n w_2 \sigma_{n2}$	$\cdots$	$w_n^2 \sigma_n^2$

The sum of the entries of the i-th-row and the i-th column is the *total* contribution of asset i to portfolio variance:

$$w_i^2 \sigma_i^2 + 2 \sum_{j \neq i}^{n} w_i w_j \sigma_{ij}.$$

The marginal contribution of asset i to portfolio variance is:

$$\begin{aligned}
\frac{\Delta \sigma_p^2}{\Delta w_i} &= 2 w_i \sigma_i^2 + 2 \sum_{j \neq i}^{n} w_j \sigma_{ij} = 2 \sum_{j=1}^{n} w_j \sigma_{ij} \\
&= 2\,\mathrm{Cov}[\tilde{r}_i, \tilde{r}_p].
\end{aligned}$$

The marginal contribution of asset i to portfolio volatility is

$$\frac{\Delta \sigma_p}{\Delta w_i} = \frac{1}{2\sigma_p} \frac{\Delta \sigma_p^2}{\Delta w_i} = \mathrm{Cov}[\tilde{r}_i, \tilde{r}_p]/\sigma_p = \sigma_{ip}/\sigma_p.$$

4.2 Individual Asset and the Market Portfolio

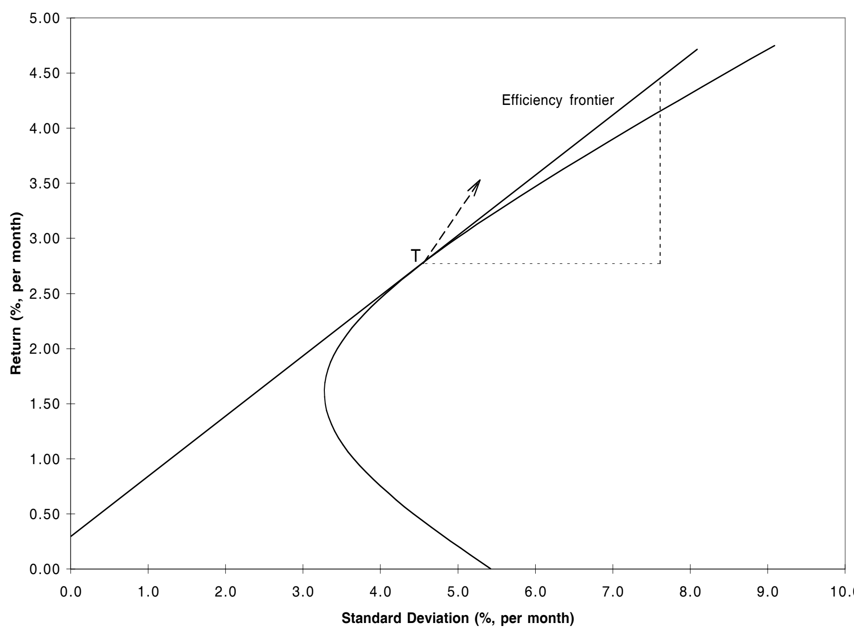
Definition: The (marginal) return-to-risk ratio (RRR) of risky asset i in a portfolio p is:

$$\text{RRR}_i = \frac{\text{marginal return}}{\text{marginal risk}} = \frac{(\Delta\bar{r}_p/\Delta w_i)}{(\Delta\sigma_p/\Delta w_i)} = \frac{\bar{r}_i - r_F}{(\sigma_{ip}/\sigma_p)}.$$

Claim: For the market (tangent) portfolio, the return-to-risk ratio of all risky assets must be the same:

$$\text{RRR}_i = \frac{\bar{r}_i - r_F}{(\sigma_{iM}/\sigma_M)} = \text{RRR}_T = \frac{\bar{r}_M - r_F}{\sigma_M}.$$

Intuition: The RRR of a frontier portfolio cannot be improved.



Re-writing

$$\frac{\bar{r}_i - r_F}{(\sigma_{iM}/\sigma_M)} = \frac{\bar{r}_M - r_F}{\sigma_M}$$

we have the following important result:

$$\boxed{\bar{r}_i - r_F = \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_F) = \beta_{iM}(\bar{r}_M - r_F)}$$

where

$$\beta_{iM} = \sigma_{iM}/\sigma_M^2$$

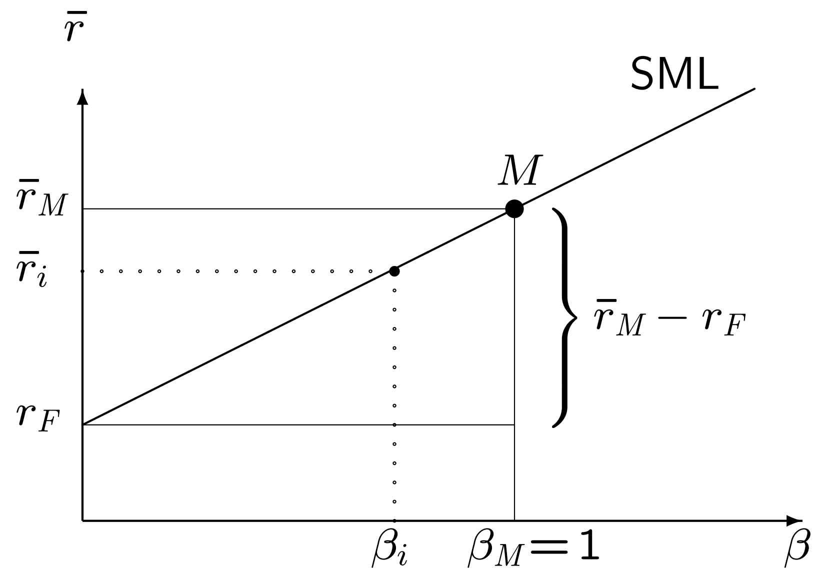
is the beta of asset i with respect to the market portfolio.

This is the CAPM:

- β_{iM} gives a measure of asset i's systematic risk
- $\bar{r}_M - r_F$ gives the premium per unit of systematic risk
- The risk premium on asset i equals the amount of its systematic risk times the premium per unit of the risk.

The relation between an asset's premium and its market beta is called the "Security Market Line" (SML).

Security Market Line (SML)



Given an asset's beta, we can determine its expected return.

Example. Suppose that CAPM holds. The expected market return is 14% and T-bill rate is 5%.

1. What should be the expected return on a stock with $\beta = 0$?

   Answer: Same as the risk-free rate, 5%. Note:

   - The stock may have significant uncertainty in its return.
   - This uncertainty is uncorrelated with the market return.

2. What should be the expected return on a stock with $\beta = 1$?

   Answer: The same as the market return, 14%.

3. What should be the expected return on a portfolio made up of 50% T-bills and 50% market portfolio?

   Answer: the expected return should be

   $$\bar{r} = (0.5)(0.05) + (0.5)(0.14) = 9.5\%.$$

4. What should be expected return on stock with $\beta = -0.6$?

   Answer: The expected return should be

   $$\bar{r} = 0.05 + (-0.6)(0.14 - 0.05) = -0.4\%.$$

   How can this be?

5 Understanding Risk/Return in CAPM

We can decompose an asset's return into three pieces:

$$\tilde{r}_i - r_F = \alpha_i + \beta_{iM}(\tilde{r}_M - r_F) + \tilde{\varepsilon}_i$$

where

- $\mathrm{E}[\tilde{\varepsilon}_i] = 0$
- Cov$[\tilde{r}_M,\ \tilde{\varepsilon}_i] = 0$.

Three characteristics of an asset:

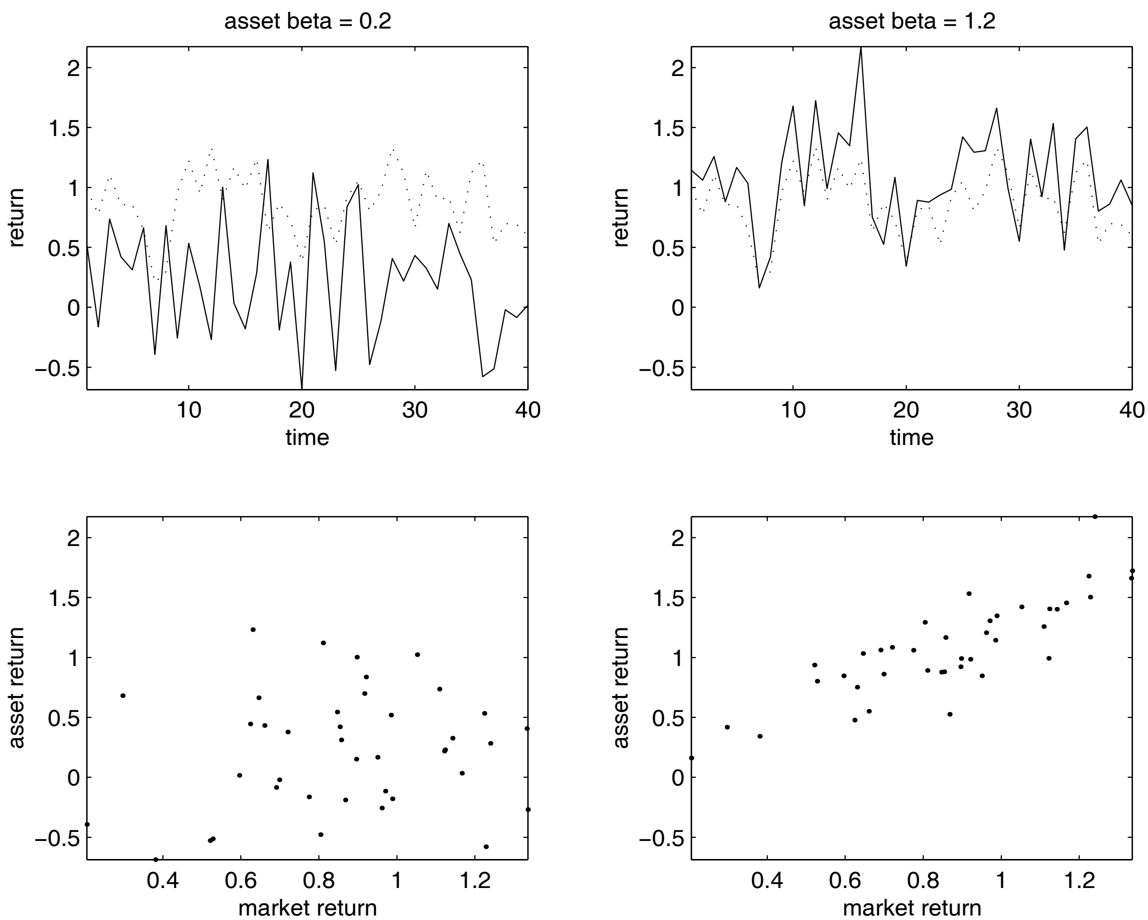
- Beta.
- Sigma = StD $(\tilde{\varepsilon}_i)$.
- Alpha.

Beta

$$\tilde{r}_i - r_F = \alpha_i + \boxed{\beta_{iM}} (\tilde{r}_M - r_F) + \tilde{\varepsilon}_i$$

- Beta measures an asset's systematic risk.
- Assets with higher betas are more sensitive to the market.

Two assets with same total volatility but different betas



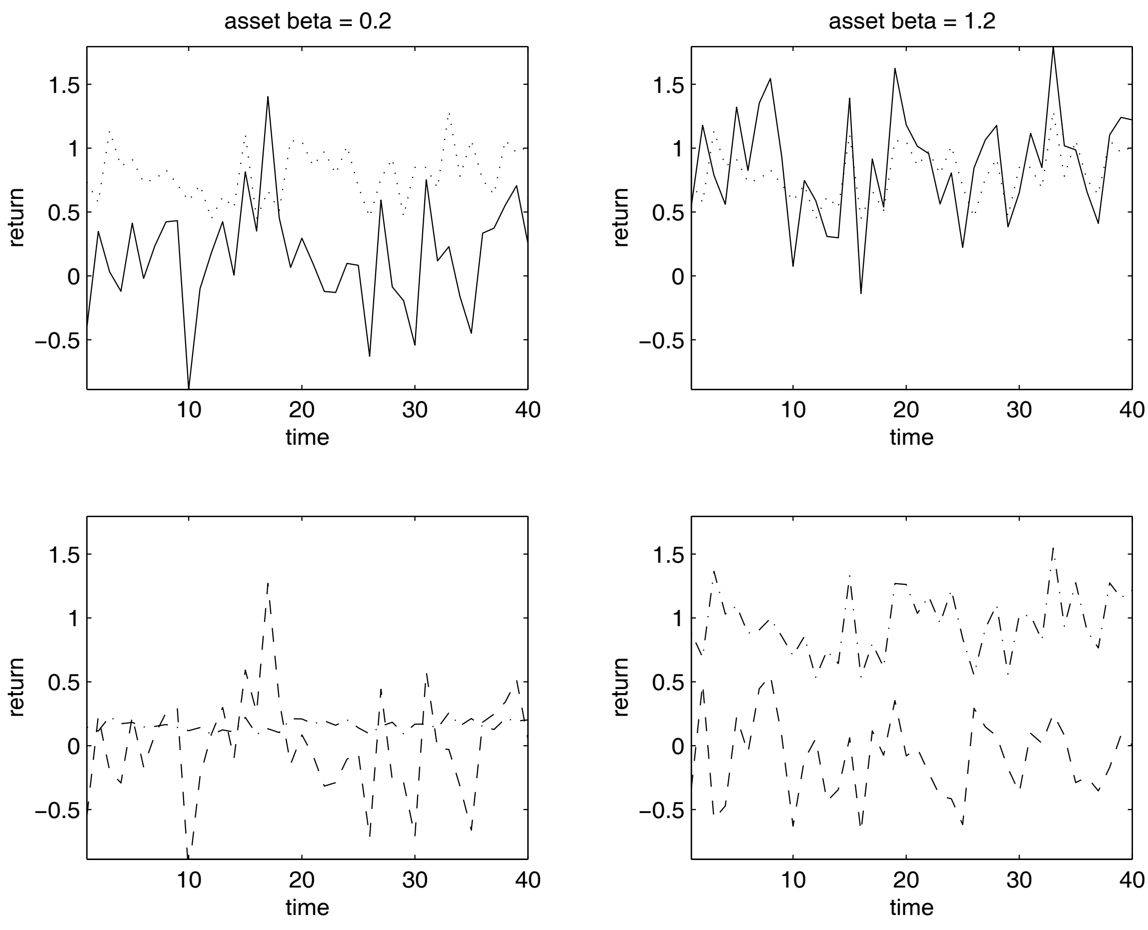
(Market premium = 8%, market volatility = 25%, asset volatility = 40%.)
Solid lines – asset returns. Dotted lines – market returns.

Sigma

$$\tilde{r}_i - r_F = \alpha_i + \beta_{iM}(\tilde{r}_M - r_F) + \boxed{\tilde{\varepsilon}_i}$$

- An asset's sigma measures its non-systematic risk.
- Non-systematic risk is uncorrelated with systematic risk.

Two assets with same total volatility but different betas



(Market premium = 8%, market volatility = 25%, asset volatility = 40%.)
Solid lines – asset returns. Dotted lines – market returns.
Dashdot lines – market component. Dashed lines – idiosyncratic component.

We can decompose return and risk as follows:

$$\tilde{r}_i - r_F = \overbrace{\beta_{iM}(\tilde{r}_M - r_F)}^{\substack{\text{systematic}\\\text{component}}} + \overbrace{\tilde{\varepsilon}_i}^{\substack{\text{non-systematic}\\\text{component}}} .$$

$$\overbrace{\mathsf{Var}[\tilde{r}_i]}^{\substack{\text{total}\\\text{risk}}} = \overbrace{\beta_{iM}^2\mathsf{Var}[\tilde{r}_M]}^{\substack{\text{systematic}\\\text{risk}}} + \overbrace{\mathsf{Var}[\tilde{\varepsilon}_i]}^{\substack{\text{non-systematic}\\\text{risk}}} .$$

Example. Systematic risk is only a part of return volatility. Consider an asset with

- annual volatility (σ) of 40%
- market beta of 1.2.

Suppose that the annual volatility of the market is 25%. What percentage of the total variance of the asset is attributable to non-systematic risk?

$$\begin{aligned}(0.4)^2 &= (1.2)^2(0.25)^2 + \mathsf{Var}[\tilde{\varepsilon}].\\ \mathsf{Var}[\tilde{\varepsilon}] &= 0.0700.\\ \sigma_\varepsilon &= 0.2646.\end{aligned}$$

$$\frac{\text{Non-systematic risk}}{\text{Total risk}} = \frac{0.07}{0.16} = 43.75\%.$$

Example. Two assets with the same *total* risk can have very different systematic risks.

Suppose that $\sigma_M = 20\%$.

Stock	Business	Market beta	Residual variance
1	Steel	1.5	0.10
2	Software	0.5	0.18

What is the total variance of each return?

$$\begin{aligned}\sigma_1^2 &= \beta_{1M}^2\sigma_M^2 + \sigma_{1\varepsilon}^2\\ &= (1.5)^2(0.2)^2 + 0.10\\ &= 0.19\end{aligned}$$

$$\begin{aligned}\sigma_2^2 &= \beta_{2M}^2\sigma_M^2 + \sigma_{2\varepsilon}^2\\ &= (0.5)^2(0.2)^2 + 0.18\\ &= 0.19.\end{aligned}$$

However

$$R_1^2 = \frac{(1.5)^2(0.2)^2}{0.19} = 47\%$$

$$R_2^2 = \frac{(0.5)^2(0.2)^2}{0.19} = 5\%.$$

Alpha

$$\tilde{r}_i - r_F = \boxed{\alpha_i} + \beta_{iM}(\tilde{r}_M - r_F) + \tilde{\varepsilon}_i$$

- According to CAPM, α should be zero for all assets.
- α measures an asset's return in excess of its risk-adjusted award according to CAPM.

What to do with an asset with a positive α?

- Check estimation error.
- Past value of α may not predict its future value.
- Positive α may be compensating for other risks.
- ...

6 Applications of CAPM

Example. Required rates of return on IBM and Dell.

1. Use the value-weighted stock portfolio as a proxy for the market portfolio.
2. Regress historic returns of IBM and Dell on the returns on the value-weighted portfolio. Suppose the beta estimates are

   $\beta_{\text{IBM,VW}} = 0.73$ and $\beta_{\text{Dell,VW}} = 1.63.$

3. Use historic excess returns on the value weighted portfolio to estimated average market premium:

   $\pi = \bar{r}_{\text{VW}} - r_F = 8.6\%.$

4. Obtain the current riskless rate. Suppose it is

   $r_F = 4\%.$

5. Applying CAPM:

$$\begin{aligned} \bar{r}_{\text{IBM}} &= r_F + \beta_{\text{IBM,VW}}(\bar{r}_{\text{VW}} - r_F) \\ &= 0.04 + (0.73)(0.086) = 0.1028. \end{aligned}$$

The expected rate of return on IBM (under CAPM) is 10.28%.

Similarly, the expected rate of return on Dell is 18.02%.

Reading the Beta Book

MLPF&S's Beta Book

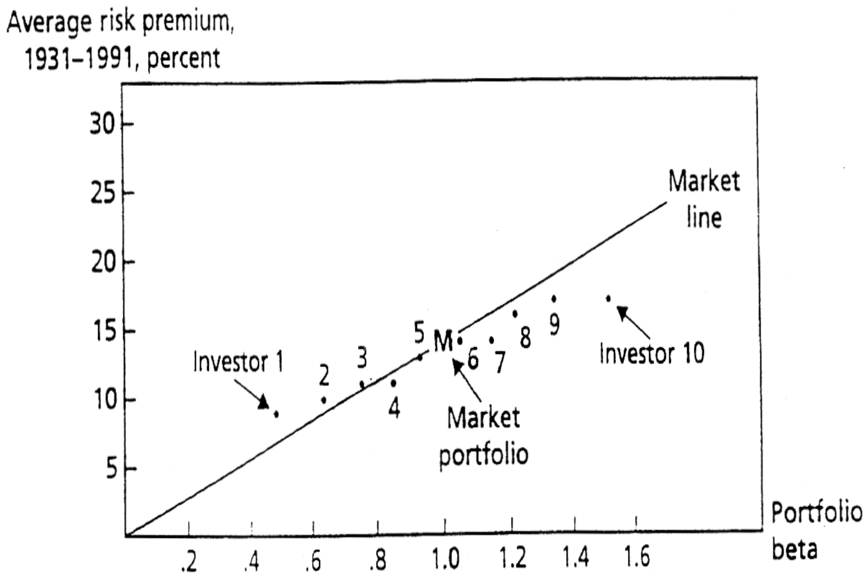
Ticker Symbol		00/03 Close Price	Beta	Alpha	R-Sqr.	Resid Std Dev-N.	–Std. Err.– of Beta	of Alpha	Adj. Beta	Num. of Obs.
AALA	AmerAlia	2.250	-2.25	10.05	0.03	42.72	1.30	6.06	-1.15	60
AOL	America Online	67.438	2.40	4.12	0.25	17.17	0.52	2.44	1.93	60
GNLK	GeneLink	0.290	-7.74	16.15	0.44	37.30	2.10	9.92	-4.79	17
GM	General Mtrs	82.813	1.01	-0.57	0.27	6.98	0.21	0.99	1.01	60
TSN	Tyson Foods	11.125	1.13	-2.31	0.23	8.55	0.26	1.21	1.09	60

Note

(a) S&P 500 is used as a proxy for the market.
(b) Betas are estimated with raw returns, not risk risk premiums. The alpha, according to CAPM, is $r_F(1-\beta)$.
(c) Adjusted beta is obtained using other information.

7 Empirical Evaluation of CAPM

1. Long-run average returns are significantly related to beta:

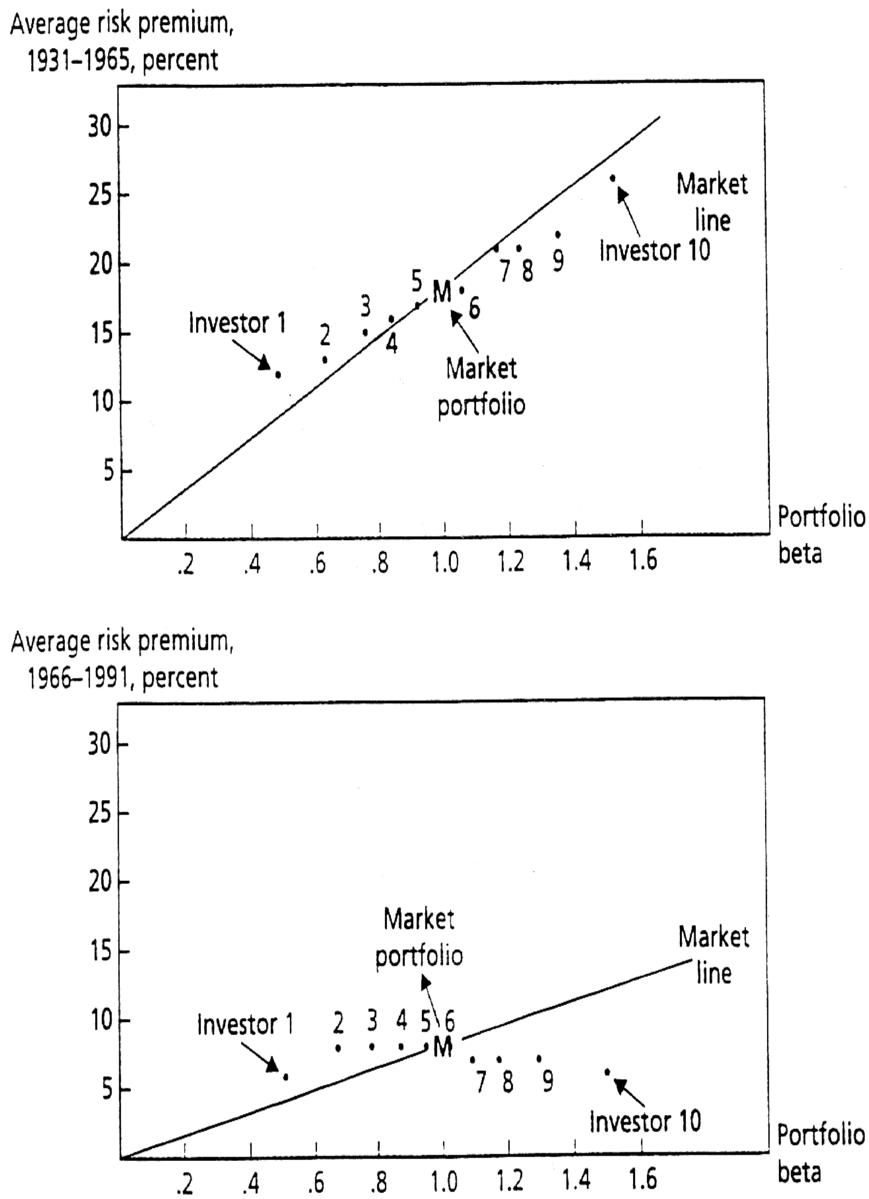


(Source: Fisher Black, "Beta and return.")

The dots show the actual average risk premiums from portfolios with different betas.

- high beta portfolios generated higher average returns
- high beta portfolios fall below SML
- low beta portfolios land above SML
- a line fitted to the 10 portfolios would be flatter than SML.

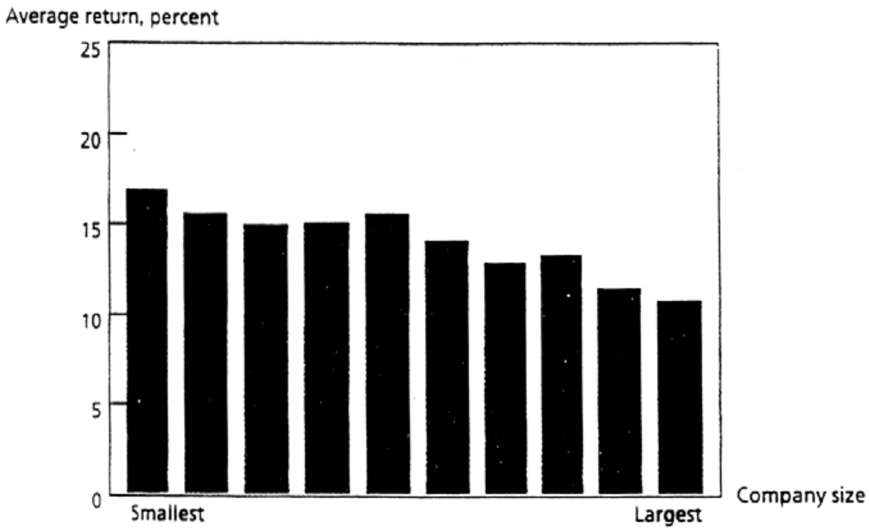
2. CAPM does not seem to work well over the last 30 years:

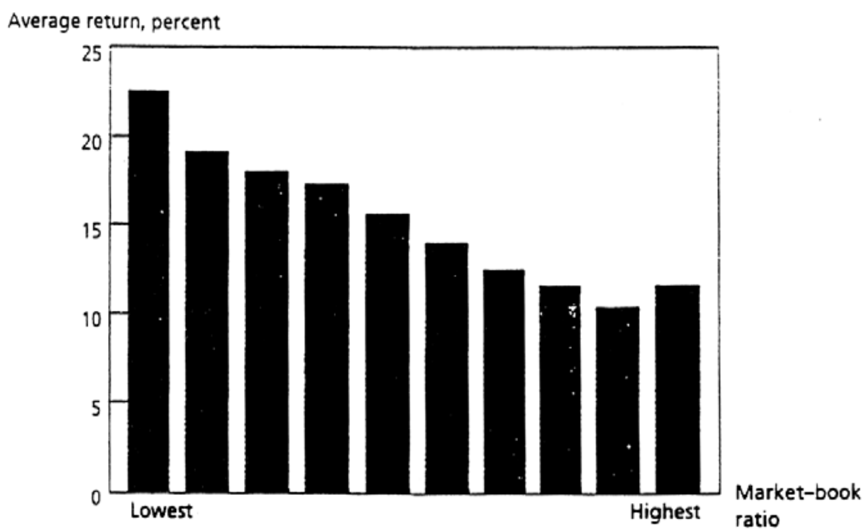


Source: Fischer Black, "Beta and return."

The dots show the actual average risk premiums from portfolios with different betas over different periods. The relation between beta and actual average return has been much weaker since the mid-1960s.

3. Factors other than beta seem important in pricing assets:





Source: G. Fama and K. French, "The Cross-Section of Expected Stock Returns".

Since mid-1960s:

- Small stocks have outperformed large stocks
- Stocks with low ratios of market-to-book value have outperformed stocks with high ratios.

8 Summary of CAPM

CAPM determines risk-return trade-off:

- Invest only in the risk-free asset and the market portfolio.
- Beta measures systematic risk.
- Required rate of return is proportional to beta.

CAPM is simple and sensible:

- It is built on modern portfolio theory.
- It distinguishes systematic risk and non-systematic risk.
- It provides a simple pricing model.

CAPM is controversial:

- It is difficult to test (to identify the market portfolio).
- Empirical evidence is mixed.
- Alternative pricing models might do better.
  - Multiple risks.

9 Arbitrage Pricing Theory (APT)

We can extend the market-risk model to include multiple risks:

$$\tilde{r}_i - r_F = \alpha_i + b_{iM}\,(\tilde{r}_M - r_F) + \cdots + b_{iN}\,(\tilde{r}_N - r_F) + \tilde{\varepsilon}_i$$

where

- $\tilde{r}_M$ and $\tilde{r}_N$ represent common risk factors
- b_{iM} and b_{iN} define asset i's exposure to risk factors
- $\tilde{\varepsilon}_i$ is part of asset i's risk unrelated to risk factors.

The Arbitrage Pricing Theory states that:

$$\boxed{\bar{r}_i - r_F = b_{iM}(\bar{r}_M - r_F) + \cdots + b_{iN}(\bar{r}_N - r_F)}$$

where

- $\bar{r}_k - r_F$ is the premium on factor k
- b_{ik} is asset i's loading of factor k

(i.e., $\alpha_i = 0$).

Example. Suppose that there are two factors represented by:

(1) return on the market portfolio $\tilde{r}_M$

(2) return on Treasury bond portfolio $\tilde{r}_N$:

$$\tilde{r}_i - r_F = b_{iM}(\tilde{r}_M - r_F) + b_{iN}(\tilde{r}_N - r_F) + \tilde{\varepsilon}_i.$$

Suppose that

r_F	$\bar{r}_M - r_F$	$\bar{r}_N - r_F$
5%	8%	2%

APT implies that an asset's risk premium is given by

$$\bar{r}_i - r_F = b_{iM}\,(8\%) + b_{iN}\,(2\%).$$

Suppose for assets A, B and C, we have

Asset	b_M	b_N
A	1.0	1.0
B	1.5	0.2
C	1.0	0.6

Then, APT implies that individual assets have to offer returns consistent with their factor exposures and factor premiums.

$$\begin{aligned}
\bar{r}_A &= 0.05 + (1.0)(0.08) + (1.0)(0.02) = 15.0\% \\
\bar{r}_B &= 0.05 + (1.5)(0.08) + (0.2)(0.02) = 17.4\% \\
\bar{r}_C &= 0.05 + (1.0)(0.08) + (0.6)(0.02) = 14.2\%.
\end{aligned}$$

Suppose that $\bar{r}_A$ was instead 10% (and it has only factor risks).

We would then have an arbitrage:

(a) buy \$100 of market portfolio

(b) buy \$100 of bond portfolio

(c) sell \$100 of asset A

(d) sell \$100 of risk-free asset.

This portfolio has the following characteristics:

- requires zero initial investment (an arbitrage portfolio)
- bears no factor risk (and no idiosyncratic risk)
- pays $(13 + 3 - 10 - 5) = \$1$ surely.

Thus, in absence of arbitrage, APT holds.

10 Implementation of APT

The implementation of APT involves three steps:

1. Identify the factors
2. Estimate factor loadings of assets
3. Estimate factor premia.

Strength and Weaknesses of APT

1. The model gives a reasonable description of return and risk.
2. Model itself does not say what the right factors are.

Differences between APT and CAPM

- APT is based on the factor model of returns and "arbitrage."
- CAPM is based on investors' portfolio demand and equilibrium.

11 Homework

Readings:

- BKM Chapters 9.
- BMA Chapters 8, 9.1, 9.3.

Assignment:

- Problem Set 5.