BEM 103: Introduction to Finance.
Homework 1: time and money

Solutions

October 9, 2013

1. **Present value**

   Consider a claim that pays you \$100 each October 7 starting October 7 2014 and ending October 7 2023.

   (a) *If the annual interest rate is 3% what is the value of the claim?* \$853

   $$V = \frac{\$100}{1.03} + \frac{\$100}{1.03^2} + \ldots + \frac{\$100}{1.03^{10}} = \frac{\$100}{1.03} \cdot \frac{1 - \frac{1}{1.03^{10}}}{1 - \frac{1}{1.03}} \approx \$853$$

   (b) *If individuals were perfectly patient what would be the value of the claim?*
   $\$100 \cdot 10$ years $= \$1000$.

   (c) *What is the implied interest rate that validates a price of \$400 for the claim?* 21.4%

   $$\begin{aligned} \$400 &= \frac{\$100}{1+r} + \frac{\$100}{(1+r)^2} + \ldots + \frac{\$100}{(1+r)^{10}} = \$100 \cdot \frac{1 - \frac{1}{(1+r)^{10}}}{r} \\ 1 - \frac{1}{(1+r)^{10}} &= 4r \\ r &\approx 0.214 \end{aligned}$$

2. **Present value and prices**

   Suppose there are two bonds in the market. Both are 5 year maturity. Both have face value 100. Bond 1 has a coupon of 7%, Bond 2 has a coupon of 4%.

   (a) *If the price of Bond 1 is 102, what is the price of Bond 2?* 90.

   Denote r is the interest rate. Then

   $$102 = \sum_{i=1}^{5} \frac{100 \cdot 0.07}{(1+r)^i} + \frac{100}{(1+r)^5} = 7 \cdot \frac{1 - \frac{1}{(1+r)^5}}{r} + \frac{100}{(1+r)^5}$$

$$\Rightarrow \quad r \approx 0.065$$

The price of the second bond is

$$\sum_{i=1}^{5} \frac{100 \cdot 0.04}{(1+r)^i} + \frac{100}{(1+r)^5} = 4 \cdot \frac{1 - \frac{1}{(1+r)^5}}{r} + \frac{100}{(1+r)^5} \approx 90$$

(b) *Suppose the current interest rate is 5%. What are the price of Bond 1 and Bond 2?* 109 and 96.

The price of bond 1:

$$7 \cdot \frac{1 - \frac{1}{(1+r)^5}}{r} + \frac{100}{(1+r)^5} = 7 \cdot \frac{1 - \frac{1}{(1.05)^5}}{0.05} + \frac{100}{(1.05)^5} \approx 109$$

The price of bond 2:

$$4 \cdot \frac{1 - \frac{1}{(1.05)^5}}{0.05} + \frac{100}{(1.05)^5} \approx 96$$

(c) *Now suppose the bond has a 1 year maturity and the interest rate is 5%. What are the price of Bond 1 and Bond 2?* 102 and 99.

The price of bond 1 is $\frac{107}{1.05} \approx 102$.

The price of bond 2 is $\frac{104}{1.05} \approx 99$.

3. **Present value and project selection**

A company that is cash constrained has 1 million dollars and can invest in one of two projects.

Project 1 involves an expansion of the core business and is expected to yield a \$100,000 in additional profits a year and expires at T after which the \$100,000 annual profits will disappear.

Project 2 involves the development of a new technology that it expected to sell for 1.5 Million dollars.

(a) *Is this enough information such that you can decide at what time t^* the new technology (project 2) will mature such that project 1 is preferred if $t > t^*$ and project 2 is preferred if $t < t^*$? If not, what is missing? (In other words, what additional variables should be specified?)* T and d.

The value of project 1 is

$$V_1(T, d) = \$100,000 \sum_{i=1}^{T} d^i.$$

The value of project 2 that will mature at t is

$$V_2(t, d) = \$1,500,000d^t.$$

So,

$$V_1(T, d) = V_2(t^*, d) \quad \Leftrightarrow \quad \$100,000 \sum_{i=1}^{T} d^i = \$1,500,000 d^{t^*}.$$

To solve this equation for t^*, we need to know T and d.

(b) *Now suppose project 1 has a life time of 20 years ($T = 20$) and project 2 matures in 5 years, what discount rate d^* is such that if and only if $d > d^*$ project 1 is preferred to project 2?* 0.94

$$V_1(T = 20, d^*) = V_2(t = 5, d^*) \quad \Leftrightarrow \quad \$100,000 \sum_{i=1}^{20} (d^*)^i = \$1,500,000(d^*)^5$$

$$\frac{d^*(1 - (d^*)^{20})}{1 - d^*} = 15(d^*)^5$$

$$d^* \approx 0.94$$

Note: there is another solution to this equation which is $d^* \approx 0.67$. However, discount rate 0.67 looks unrealistically low.

(c) *Suppose $T = 20$ and project 2 matures after 5 years, the firm can borrow an additional million dollars at an interest rate of 10% (and it could also lend out its money at 10% a year). What project(s) does it carry out if any?* None.

$$\begin{aligned} V_1(T = 20, r = 0.1) &= \sum_{i=1}^{20} \frac{\$100,000}{1.1^i} = \$100,000 \cdot \frac{1 - \frac{1}{1.1^{20}}}{0.1} \\ &\approx \$851,356 < \$1,000,000 \\ V_2(t = 5, r = 0.1) &= \frac{\$1,500,000}{1.1^5} \approx \$931,382 < \$1,000,000 \end{aligned}$$

(d) *Suppose $T = 20$ and project 2 matures after 5 years, the firm can borrow an additional million dollars at an interest rate of 5% (and it could also lend out its money at 5% a year). What project(s) does it carry out if any?* Project 1 and 2.

$$\begin{aligned} V_1(T = 20, r = 0.05) &= \sum_{i=1}^{20} \frac{\$100,000}{1.05^i} = \$100,000 \cdot \frac{1 - \frac{1}{1.05^{20}}}{0.05} \\ &\approx \$1,246,221 > \$1,000,000 \\ V_2(t = 5, r = 0.05) &= \frac{\$1,500,000}{1.05^5} \approx \$1,175,289 > \$1,000,000 \end{aligned}$$

4. **Going to business school**

   Jane, a freshman in college, needs \$55,000 in 4 years to start studying for an MBA. Her investments earn 5% interest per year.

   (a) *How much must she invest today to have that amount at graduation from college?* \$45,249.

   $$I \cdot 1.05^4 = \$55,000 \quad \Rightarrow \quad I \approx \$45,249.$$

   (b) *If she invested once a year for four years beginning today until the end of the 4 years how much must she invest?* \$12,153.

   Assuming the same investment I each time,

   $$I \cdot (1.05^4 + 1.05^3 + 1.05^2 + 1.05) = \$55,000 \quad \Rightarrow \quad I \approx \$12,153.$$

5. **Machines**

   A company is considering its options for a machine that it needs. At a cost of 47 the company can make small repairs to their machine which will last two years. At a cost of 90 they can make repairs that will last four years. A new machine costs 300 and will last 8 years. The company faces an interest rate of 10%.

   (a) *What should it do?* Small repairs.

   The price of two small repairs:

   $$P_{ss} = 47\left(1 + \frac{1}{1.1^2}\right) \approx 86.$$

   The price of 1 large repair is 90, so two small repairs are better than 1 large repair.

   The price of the small repairs (repeated 4 times) is

   $$P_s = 47\left(1 + \frac{1}{1.1^2} + \frac{1}{1.1^4} + \frac{1}{1.1^6}\right) \approx 144.$$

   The price of buying a new machine is 300.

   So, the cheapest way to keep the machine working for 8 years is doing the small repairs 4 times.

(b) *Suppose 47 allows you to produce 1 million units and 90 allows you to produce 2 million units while the new machine is good for 4 million units. The economy is in a slump so output will be half a million units, for the next two years, before returning to 1 million units a year. What do you do (note if you repair now you can always buy the new machine later)?* Invest 47 now and continue investing 47 each year starting from the second year.

On the trend:

- two 47 investments are better than one 90 investment:

$$47\left(1+\frac{1}{1.1}\right) \approx 89.7 < 90$$

- four 47 investments are better than one 300 investment:

$$47\left(1+\frac{1}{1.1}+\frac{1}{1.1^2}+\frac{1}{1.1^3}\right) \approx 163.9 < 300$$

So, the investment of 47 each year is the best strategy on the trend.

If the company invest 47 today, then the best it can do later is to invest 47 each year starting from the second year from now. The total value of all investments for 5 years today is

$$V_s = 47\left(1+\frac{1}{1.1^2}+\frac{1}{1.1^3}+\frac{1}{1.1^4}\right) \approx 153$$

If the company invest 90 today, then the best it can do later is to invest 47 each year starting from the third year from now. The total value of all investments for 5 years today is

$$V_l = 90 + 47\left(\frac{1}{1.1^3}+\frac{1}{1.1^4}\right) \approx 157$$

If the company invest 300 today, then the best it can do later is to invest 47 each year starting from the fifth year from now. The total value of all investments for 5 years today is 300.

(c) *Suppose the economy is in a boom (out 2 million units for the next two years before reverting to trend a 1 million units) what should be done?* Invest 90 in first and in second years, then invest 47 each year.

From the previous point, the investment of 47 each year is the best strategy on the trend.

During the boom:

- one 90 investment is better than two 47 investments:

$$47\left(1+\frac{1}{1.05}\right)\approx 92 > 90$$

- two 90 investments are better than one 300 investment:

$$90\left(1+\frac{1}{1.1}\right)\approx 172 < 300$$

6. **Another machine**

   The C&C *company recently installed a new bottling machine. The machine's initial cost is 2000, and can be depreciated on a straight-line basis to a zero salvage in 5 years. The machine's per year fixed cost is 1500, and its variable cost is 0.50 per unit. The selling price per unit is 1.50.* C&C*'s tax rate is 34%, and it uses a 16% discount rate. Calculate the machine's accounting break-even point on the new machine (i.e., the production rate such that the accounting profits are zero), as well as the present value break-even point (i.e., the production rate such that the NPV is zero).* 1900 and 2220.

   Denote x the production rate.

   The accounting profit for 1 year is

$$\left(1.5x - 0.5x - 1500 - \frac{2000}{5}\right)\cdot(1-0.34) = 0.66(x-1900).$$

   So, if $x = 1900$ the accounting profits are zero.

   Assuming the machine works only for 5 years, the NPV is

$$\left(1.5x - 0.5x - 1500 - 0.34\left(1.5x - 0.5x - 1500 - \frac{2000}{5}\right)\right)\sum_{i=1}^{5}\frac{1}{1.16^i} - 2000$$
$$\approx 2.16x - 4796$$

   Thus, if $x = 2220$ the NPV is zero.

7. **Pharma**

   After extensive medical and marketing research, PillAdvent Inc, believes it can penetrate the pain reliever market. It can follow one of two strategies. The first is to manufacture a medication aimed at relieving headache pain. The second strategy is to make a pill designed to relieve headache and arthritis pain. Both products would be introduced at a price of 4 per package in real terms. The *broader remedy would probably sell 10 million packages a year. This is twice the sales rate for the headache-only medication.*

Cash costs of production in the first year are expected to be 1.50 per package in real terms for the headache-only brand. Production costs are expected to be 1.70 in real terms for the more general pill. All prices and costs are expected to rise at the general inflation rate of 5%.

Either strategy would require further investment in plant.

The headache-only pill could be produced using equipment that would cost 10.2 million, last three years, and have no resale value.

The machinery required to produce the broader remedy would cost 12 million and last three years. At this time the firm would be able to sell it for 1 million in real terms. The production machinery would need to be replaced every three years at constant real costs. For both projects the firm will dace a straight line depreciation, it faces tax rates 34% (on profits and capital gains) and it believes the appropriate discount rate is 13%.

(a) *What product should the firm produce?* The broader remedy.

The first strategy:

- Investment: 10,200,000 now.
- First year after-tax profit:

$$(4-1.5)\cdot 5,000,000-0.34\left((4-1.5)\cdot 5,000,000-\frac{10,200,000}{3}\right)=9,406,000$$

- Second year after-tax profit:

$$1.05\cdot(4-1.5)\cdot 5,000,000-0.34\left(1.05\cdot(4-1.5)\cdot 5,000,000-\frac{10,200,000}{3}\right)$$
$$=9,818,500$$

- Third year after-tax profit:

$$1.05^2\cdot(4-1.5)\cdot 5,000,000-0.34\left(1.05^2\cdot(4-1.5)\cdot 5,000,000-\frac{10,200,000}{3}\right)$$
$$=10,251,625$$

- So, the total three-years net profit is

$$\frac{9,406,000}{1.13}+\frac{9,818,500}{1.13^2}+\frac{10,251,625}{1.13^3}-10,200,000\approx 12,918,110$$

The second strategy:

- Investment: 12,000,000 now.
- Reselling: $1,000,000 \cdot 1.05^2 = 1,102,500$ in three years.
- First year after-tax profit:

$$(4-1.7)\cdot 10,000,000-0.34\left((4-1.7)\cdot 10,000,000-\frac{12,000,000-1,102,500}{3}\right)$$
$$= 16,415,050$$

- Second year after-tax profit:

$$\begin{aligned}&1.05\cdot(4-1.7)\cdot 10,000,000\\&-0.34\left(1.05\cdot(4-1.7)\cdot 10,000,000-\frac{12,000,000-1,102,500}{3}\right)\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 17,174,050\end{aligned}$$

- Third year after-tax profit:

$$\begin{aligned}&1.05^2\cdot(4-1.7)\cdot 10,000,000\\&-0.34\left(1.05^2\cdot(4-1.7)\cdot 10,000,000-\frac{12,000,000-1,102,500}{3}\right)\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 17,971,000\end{aligned}$$

- So, the total three-years net profit is

$$\frac{16,415,050}{1.13}+\frac{17,174,050}{1.13^2}+\frac{17,971,000}{1.13^3}-12,000,000+\frac{1,102,500}{1.13^3}\approx 29,195,285$$

(b) *Suppose Congress abolishes all corporate taxes what product should the firm produce?* The broader remedy.

The first strategy:

- Investment: 10,200,000 now.
- First year after-tax profit:

$$(4-1.5)\cdot 5,000,000 = 12,500,000$$

- Second year after-tax profit:

$$1.05\cdot(4-1.5)\cdot 5,000,000 = 13,125,000$$

- Third year after-tax profit:

$$1.05^2 \cdot (4 - 1.5) \cdot 5,000,000 = 13,781,250$$

- So, the total three-years net profit is

$$\frac{12,500,000}{1.13} + \frac{13,125,000}{1.13^2} + \frac{13,781,250}{1.13^3} - 10,200,000 \approx 20,691,845$$

The second strategy:

- Investment: 12,000,000 now.
- Reselling: $1,000,000 \cdot 1.05^2 = 1,102,500$ in three years.
- First year after-tax profit:

$$(4 - 1.7) \cdot 10,000,000 = 23,000,000$$

- Second year after-tax profit:

$$1.05 \cdot (4 - 1.7) \cdot 10,000,000 = 24,150,000$$

- Third year after-tax profit:

$$1.05^2 \cdot (4 - 1.7) \cdot 10,000,000 = 25,357,500$$

- So, the total three-years net profit is

$$\frac{23,000,000}{1.13} + \frac{24,150,000}{1.13^2} + \frac{25,357,500}{1.13^3} - 12,000,000 + \frac{1,102,500}{1.13^3} \approx 45,605,082$$