Basic definitions: normal-form games

Let $(N, \{S_i : i \in N\}, \{u_i : i \in N\})$ be a normal-form game, with $N = \{1, \ldots, n\}$.

- A strategy $\bar{\sigma}_i \in \Delta(S_i)$ is a *dominant strategy* if
$$u_i(\bar{\sigma}_i, \sigma_{-i}) > u_i(\tilde{\sigma}_i, \sigma_{-i}),$$
for all $\tilde{\sigma}_i \in \Delta(S_i)$ and all $\sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$.
- A strategy $\sigma_i \in \Delta(S_i)$ is a *dominated strategy* if there exists a strategy $\bar{\sigma}_i \in \Delta(S_i)$ such that
$$u_i(\bar{\sigma}_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}),$$
for all $\sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$. In this case we say that $\bar{\sigma}_i$ *dominates* σ_i.
- A strategy $\sigma_i \in \Delta(S_i)$ is a *weakly dominated strategy* if there exists a strategy $\bar{\sigma}_i \in \Delta(S_i)$ such that $u_i(\bar{\sigma}_i, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$, for all σ_{-i}, and such that there is some σ_{-i} for which $u_i(\bar{\sigma}_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$.

In the problem set, you are asked to prove that $\bar{\sigma}_i$ dominates σ_i iff $u_i(\bar{\sigma}_i, s_{-i}) > u_i(\sigma_i, s_{-i})$, for all pure strategy profiles $s_{-i} \in S_{-i}$.

- A *belief* for a player i is a probability distribution $\mu \in \Delta(S_{-i})$.
- A strategy $\sigma_i \in \Delta(S_i)$ is a *best response* to beliefs μ if it solves the problem of maximizing $\mathbb{E}_\mu u(\tilde{\sigma}_i, s_{-i})$ over $\tilde{\sigma}_i \in \Delta(S_i)$.[1]
- Denote the set of best responses to beliefs μ by $\beta_i(\mu)$. Note that $\beta_i(\mu)$ may not be a singleton.
- A mixed strategy σ_i is in $\beta_i(\mu)$ iff every pure strategy in the support of σ_i is in $\beta_i(\mu)$. In particular, every strategy in the support of σ_i yields the same payoff to i.

A *Nash equilibrium* is a strategy profile σ such that, for every player i, σ_i is a best response to beliefs μ, and μ concide with σ_{-i}. Succinctly, a Nash equilibrium is a profile σ where $\sigma_i \in \beta_i(\sigma_{-i})$.

Notice: if σ is a Nash equilibrium profile, then σ_i cannot dominated. In fact σ_i is rationalizable.

A normal-form game is *finite* if S_i is a finite set for all i.

Theorem. *Any finite normal-form game has a Nash equilibrium.*

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[1] Remember that $\mathbb{E}_\mu u(\tilde{\sigma}_i, s_{-i}) = \sum_{s_{-i} \in S_{-i}} \mu(s_{-i}) u(\tilde{\sigma}_i, s_{-i})$.