

Dynamics of Contract Design with Screening

Jakša Cvitanić[†], Xuhu Wan[‡] and Huali Yang[§]

This draft: October 24, 2011

ABSTRACT

We analyze a novel principal-agent problem of moral hazard and adverse selection in continuous time. The constant private shock revealed at time zero when the agent selects the contract has a long-term impact on the optimal contract. The latter is based not only on the continuation value of the agent who truthfully reports, but also contingent upon the continuation value of the agent who misreports, called temptation value. The good agent is retired when the temptation value of the bad agent becomes large, because then it is expensive to motivate the good agent. The bad agent is retired when the temptation value of the good agent becomes small, because then the future payment does not provide sufficient incentives. We also compare the efficiency of the shutdown contract and the screening contract, and explore the optimal contracts when the agents are patient.

Key words: Adverse selection, constant private shock, principal-agent model, continuous-time, continuation value, temptation value, dynamic moral hazard.

*The research of J. Cvitanić was supported in part by NSF grant 10-08219. The research of X. Wan and H. Yang was supported by Hong Kong government general research fund grant GRF 620909.

[†]Caltech, Humanities and Social Sciences, M/C 228-77, 1200 E. California Blvd. Pasadena, CA 91125. Ph: (626) 395-1784. E-mail: cvitanic@hss.caltech.edu.

[‡]Department of Information and Systems Management, School of Business and Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Ph: +852 2358 7731. E-mail: xuhu.wan@gmail.com.

[§]Department of Information and Systems Management, School of Business and Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. E-mail: yanghualizsu@163.com.

1. Introduction

We investigate a novel problem of optimal dynamic contracting under moral hazard and adverse selection, in which the agent's preference is subject to a constant private shock. Adverse selection is important in practice, as agents (managers) may vary in terms of productivity and preferences, privately known only to themselves. Similarly, investors may lack information on the quality and the future profitability of a project of an entrepreneur who seeks financing.

Despite the important role of adverse selection in managerial compensation and contract design, dynamic contracting literature on this topic is very scarce.¹ To the best of our knowledge, there are no discrete-time models with infinite periods investigating constant private shock and dynamic moral hazard. Sung (2005) was the first to investigate continuous-time contracting with both adverse selection and moral hazard with constant private type, in a model in which a risk-neutral principal hires agents with CARA preferences, and all the decisions are taken at time zero; then, the optimal contract is linear, and effort is constant. Continuous-time papers dealing with pure adverse selection problem subject to repeated persistent shock include Zhang's (2009) Markov chain model in which the agent's utility is affected by a persistent random shock observed only by the agent, and Williams (2009) who considers a persistent private random shock with a continuum of states. Fong (2009) considers a dynamic mixed model with instantaneous payment in analyzing the dynamic environment of health care provision; however, different from our model, that paper imposes the assumption that the good agent has no conflict of interest with the principal. In Sannikov (2007a) a principal employs an agent to manage a project whose drift and outcome are observed only by the agent; the paper assumes that the agent consumes only at a finite horizon and uses a non-standard methodology. Giat, Hackman, and Subramanian (2010) consider the model in which the project value is observed, but its "risk premium" (drift term) is not observed, and the principal and the agent may have different prior beliefs about it. Similarly, Prat and Jovanovic (2010) extend Sannikov (2008) model to the case of unobserved drift. Different from those papers, our main methodological contribution is that we extend the continuation value based approach to models that combine dynamic moral hazard and adverse selection.

¹Dynamic models without adverse selection include the seminal paper by Holmstrom and Milgrom (1987), the first to explore continuous-time moral hazard models. Their work was generalized and extended by many authors, including Schättler and Sung (1993, 1997), Sung (1995, 1997). Cvitanic, Wan and Zhang (2008) generalized Holmstrom-Milgrom model to allow for general utility functions. Sannikov (2008) was the first to consider a dynamic moral hazard model with continuous payment, in a model in which the agent's continuation value process is the unique state variable. Williams (2006) investigates a general version of the same problem. Demarzo and Sannikov (2006) analyze dynamic capital security design with hidden savings. Biais, Mariotti, Plantin, and Rochet (2007) consider a model in which the arrival rate of investment opportunities is controlled by the agent.

In our model the private shock is constant and it models agent's skill, type or preferences, that remain unchanged throughout the agent's lifetime. In addition, our model involves dynamic moral hazard, which, in combination with a constant private shock, makes the problem difficult. The reason for the difficulty is that the contract payment transferred to the agent not only has to provide instantaneous incentives for the agent to work, but also to provide aggregate incentives for the agent to report the true shock value at time zero. Dealing with both simultaneously is quite challenging.

The model is a generalization of Sannikov (2008) to the case of adverse selection. He develops a continuation value based approach to explore the dynamic moral hazard problem that is a continuous-time analogue of the model in Spear and Srivastava (1987). The agent's continuation value is the total future expected utility conditional on the past history. Sannikov (2008) manages to reduce the agent's incentive problem to instantaneous conditions involving the volatility of the continuation value process. In our model, with private information at time zero, it is not enough to consider the volatility of each agent's continuation value, because of the additional concern that the agent may not be truthful about her type. Hence, continuation value is not the only state variable. Rather, we need to consider also the continuation value if the agent untruthfully reports her type, that we call the "temptation value process". The temptation value process, implicitly determined by the payment stream offered to the honest agent, provides incentives to the dishonest agent to exert effort. Then, by restricting the initial value of the temptation value process, the principal can induce the agent to report truthfully. Hence, when the principal designs the contract, he not only needs to consider how to provide incentives for exerting effort from an honest agent, but also how to control the temptation value process of her dishonest counterpart. The continuation value and temptation value processes then both affect the optimal payment stream. That is, the optimal contract is based on two state variables.

The main difficulty relative to the pure moral hazard model, is that, with the continuation value and the temptation value processes being coupled, it is not straightforward to identify appropriate boundary conditions and the domain of the relevant value functions. This domain, called the "credible set", is the set of pair values that can be implemented as expected utility values by admissible payment streams. In other words, if a pair consisting of the initial values of the continuation value and the temptation value processes lies outside the credible set, then there exists no payment stream that implements the honest and dishonest agents' expected payoffs at time zero.

Motivated by Abreu, Pearce and Stacchetti (1990) and Sannikov (2007b), we construct a method for computing the credible set. It has two boundaries, that we call "stationary boundary" and "extreme boundary". When the state variable processes reach the stationary boundary of the credible set, the contract is terminated. In our version of the pure moral hazard problem, the contract is terminated (the agent retires) simply when the continuation value reaches the minimum or the maximum possible value. However, when the moral hazard is mixed with adverse selection, it is possible that the contract is terminated at any level of the continuation value process, depending on the temptation value process. The contract of the good agent is terminated when the temptation value process of the bad agent becomes large, as the good agent becomes too expensive to motivate. The contract of the bad agent is terminated when the temptation value process of the good agent becomes small, because the contract offers too few incentives. When the state variable processes reach the extreme boundary of the credible set, they continue moving along the tangential direction of the set. While the stationary boundary is a line, the extreme boundary is more complex, it is a solution to a ordinary differential equations (ODE).

We also consider shutdown contracts, that is, the contracts that the bad agent would not accept. We compare the efficiency of the optimal screening contract and the optimal shutdown contract for different utility reservation values of the bad agent. We find that, when the reservation value is high, it is more profitable for the principal to offer the shutdown contract. When the reservation value is low, it is better for the principal to offer the screening contract (that agents of both types will accept). In static models (see Laffont and Martimort 2002), a significant inefficiency is a feature of the shutdown contract, because the bad agent will not be producing. In our model, however, it may or may not be more efficient to offer the screening contract. Expanding on Sannikov (2008), who identifies the "income effect" inefficiency in dynamic moral hazard problems, we find that the good agent's incentives are affected not only by her own income effect, but also by the dishonest agent's income effect. When the reservation value is low, the shutdown contract leads to a low rent for the good agent and the expected utility of the dishonest agent has to be lower than the reservation value. However, low expected payoff for the dishonest agent may reduce incentives to the honest agent. By comparison, under the screening contract and with low reservation value, both the good and bad agents' expected payoffs are not binding at reservation utility, thus providing better incentives to the good agent. When the reservation utility is high and the income effect of the dishonest agent becomes less relevant, then the shutdown contract is better because it brings down the good agent's rent. Hence, we conclude that the screening contract may be better not only because the bad agent does not

produce if not offered a contract, but also because screening provides strong incentives to the good agent if the reservation utility is low.

Finally, we analyze the contract design when the agents are patient, that is, when the discount rate converges to zero. Different from the dynamic moral hazard model of Sannikov (2008), in which the second-best contract attains the first-best efficiency in the limit, in our model efficiency is not attained in the limit. This is because the cost of incentive provisions induced by the private benefit of the dishonest agent does not vanish as the discount rate becomes smaller.

The remainder of this paper is organized as follows. In Section 2 we discuss the model and setup. In Section 3 we find the optimal contract with pure moral hazard. In Section 4 we investigate the optimal shutdown contract with both moral hazard and adverse selection. The optimal screening contract is presented in Section 5. In Section 6 we discuss the credible set and optimal contract when agents are patient. We conclude the paper in Section 7. The proofs are presented in the appendix.

2. Model

Time is continuous. A standard Brownian motion $Z = \{Z(t), \mathcal{F}(t)\}_{t \geq 0}$ on (Ω, \mathcal{F}, P) drives the output process. The total output $Y(t)$ produced up to time t evolves according to

$$dY(t) = a(t)dt + \sigma dZ(t),$$

where $a(t)$ is the manager's choice of effort level and σ is a constant. The manager's effort is a stochastic process $a = \{a(t)\}_{t \geq 0}$ that is progressively measurable with respect to $\mathcal{F}(t)$, where the set of feasible effort levels \mathcal{A} is binary: $\mathcal{A} = \{0, a_M\}$. The effort is costly with instantaneous cost $g(a)$ such that $g(a_M) > g(0) = 0$, and is measured in the same units as the utility of the agent's consumption.

A firm owner (the principal, he) signs a contract with a manager (the agent, she) at time 0. As determined by the contract, the principal makes an instantaneous payment $c = \{c(t)\}_{t \geq 0}$ to the manager, and the manager's utility of payment is $\theta u(c(t))$, where $u(\cdot)$ is increasing and concave. We normalize $u(0) = 0$ and denote the inverse function of $u(\cdot)$ by $v(\cdot)$. The instantaneous payment $c(t)$ can take values only in a compact set $C = [0, c_M]$. Parameter θ stands for the manager's type. We assume that the managers in the labor market have only two types taking values in the set $\Theta = \{\theta_g, \theta_b\}$ with $\theta_b < \theta_g$. We call the manager

good (bad) if her type is θ_g (θ_b). Moreover, it is common knowledge that the proportion of the managers of type $\theta_i, i = g, b$, is p_i .

The output process Y is publicly observable by both the firm owner and the manager. $\mathcal{F}^Y(t)$ is the information flow generated by $\{Y(s)\}_{s \leq t}$. The firm owner cannot observe the manager's effort a or her type θ , known only by the manager. Hence, we have a contracting problem with both adverse selection and moral hazard.

The firm owner offers a menu of contracts $\Psi_i = \{c_i, a_i\}, i = g, b$, that specifies a bounded flow of payments $c_i = \{c_i(t)\}_{t \geq 0}$ and desired effort $a_i = \{a_i(t)\}_{t \geq 0}$ based on his observations of output and the agent's reported type. The desirable level of effort is the level that the firm owner recommends to the manager.

Assume that both the firm owner and the manager discount the flow of profits and utility at a common rate r . If the manager's type is θ_i , with payment c_i and chosen effort level a_i , then her expected utility is given by

$$V(\theta_i, c_i, a_i) = rE \left\{ \int_0^\infty e^{-rs} [\theta_i u(c_i(s)) - g(a_i(s))] ds \right\},$$

and the firm owner is risk neutral with expected profit

$$r \sum_{i=g,b} p_i E \left\{ \int_0^\infty e^{-rs} [dX(s) - c_i(s) ds] \right\} = r \sum_{i=g,b} p_i E \left\{ \int_0^\infty e^{-rs} [a_i(s) - c_i(s)] ds \right\}. \quad (1)$$

Factor r in front of the integrals normalizes the cumulative payoffs to the same scale as the flow payoffs.

2.1. Formulation of firm owner's problem

Assume that the reservation utility for managers of both types is R . The owner's problem is then to offer a contract menu $\{\Psi_i\}_{i=g,b}$ that maximizes his profit (1) subject to delivering to the agent a required initial utility value of at least R . We write these "individual reservation" (IR) constraints as

$$V(\theta_i, c_i, a_i) \geq R_i, \quad i = g, b. \quad (2)$$

There are also two incentive compatibility conditions:

$$V(\theta_i, c_i, a_i) = \max_{\hat{a}} V(\theta_i, c_i, \hat{a}), \quad (3)$$

$$V(\theta_i, c_i, a_i) \geq \max_{\hat{a}} V(\theta_i, c_j, \hat{a}), \quad (4)$$

where $i \neq j$, $i, j = g, b$. Such a contract is called a “screening contract.” Condition (3) states that, given c_i , the principal’s effort recommendation is the agent’s best response when she truthfully reports her type. Condition (4) means that, if the agent adversely selects a contract, then her expected utility cannot be better than what it would be if she truthfully reported her type at time 0.

We first derive the optimal contract under pure moral hazard, without adverse selection, as the main benchmark. Next, we derive the “optimal shutdown contract”, that is, the contract which deliberately excludes the bad type. Finally, we find the optimal screening contract.

3. Optimal Contract with Pure Moral Hazard

In this section, we assume that the manager’s type is publicly known and discuss optimal contracting under pure moral hazard. This contract can be found by familiar methods that summarize the agent’s incentives using her continuation value, i.e., her future expected payoff when she chooses the principal’s desired effort,² that is

$$W_i(t) = rE_t \left\{ \int_t^\infty e^{-r(s-t)} [\theta_i u(c_i(s)) - g(a_i(s))] ds \right\}, \quad i = g, b.$$

The optimal contract can then be derived using the dynamic programming approach. Denote by $F^i(W_i)$, $i = g, b$, the principal’s expected profit if the agent’s type is θ_i .

Note that the agents’ continuation values are bounded. The following definition introduces the domain of their feasible payoffs.

Definition 3.1 *Set*

$$\mathcal{V} = [0, \theta_b u(c_M)] \times [0, \theta_g u(c_M)] \subset \mathcal{R}^2$$

²Continuation value based methods were developed by Green (1987), Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990), and in continuous time by Sannikov (2008).

is called the “feasible set” for the expected payoff pairs $[W_b, W_g]$ of bad and good managers.³

It follows from the results below that the unique way of delivering $W_i(t) = 0$ is to offer zero payment after time t , in which case the agent’s effort is zero. The corresponding principal’s expected profit is $F^i(0) = 0$. Moreover, the unique way to deliver $W_i(t) = \theta_i u(c_M)$ is to make a constant payment $c_i(s) = c_M$ after time t , in which case the agent’s optimal effort is $a_i(s) = 0$, for $s \geq t$, and so $F^i(\theta_i u(c_M)) = -c_M$. These are the boundary conditions needed to find the optimal contract in this setting. The method we apply is that of Sannikov (2008), and the proofs of Lemma 1 and Proposition 1 below, are the same as in that paper, although our pure moral hazard setting differs from Sannikov’s (2008) in the boundary conditions.

The following result gives the instantaneous incentive compatibility conditions for the managers.

Lemma 1 *Given a payment process c_i and effort process a_i , there exists an adapted process β^i such that the agent’s continuation value evolves according to*

$$dW_i(t) = r[W_i(t) + g(a_i(t)) - \theta_i u(c_i(t))] dt + r\beta_i(t)(dY(t) - a_i(t)dt), \quad i = g, b. \quad (5)$$

Moreover, the agent with type θ_i will optimally exert the recommended effort a_i if and only if the following incentive compatibility condition holds:

$$-g(a_i) + \beta_i a_i \geq -g(\hat{a}) + \beta_i \hat{a}, \quad \text{for all } \hat{a} \in \mathcal{A}. \quad (6)$$

For the sake of smoother terminology, we introduce the following definition.

Definition 3.2 *If (a_i, β_i) satisfies instantaneous incentive compatibility condition (6), then we say that β_i enforces effort level a_i .*

The reason behind the incentive compatibility condition is that the drift of the agent’s continuation value in (5) depends on $\beta_i a_i - g(a_i)$, so the best response for an agent of type θ_i is to maximize $-g(a) + \beta_i a$.

Given the desired effort a_i , the principal will choose process β_i that enforces a_i and has the smallest absolute value among such process. The choice of the smallest absolute value process is due to the concavity of the principal’s value function, as we will see below. We use $\gamma(a_i)$ to denote such β_i .

³This domain is determined by the bounds on the instantaneous benefit $\theta_g u(c) - g(a)$.

Proposition 1 *The optimal (incentive compatible) contract $\{c_i(W_i(t)), a_i(W_i(t))\}$ is determined by the maximization in the optimality (HJB) equation for the principal's value function*

$$F^i(W) = \max_{a_i, c_i} \left\{ a_i - c_i + [W + g(a_i) - \theta_i u(c_i)] F_W^i(W) + \frac{r\sigma^2\gamma^2(a_i)}{2} F_{WW}^i(W) \right\},$$

satisfying boundary conditions

$$F^i(0) = 0, \quad F^i(\theta_i u(c_M)) = -c_M. \quad (7)$$

Here, $W_i(t)$ is the continuation value process of the agent with type θ_i following the dynamics (5), and its initial value is any value such that $W_i(0) \in \arg \max_{\hat{w} \geq R, \in [0, \theta_i u(c_M)]} F^i(\hat{w})$.

An important finding in Sannikov (2008) is that the agent's initial expected payoff at time 0 may be strictly larger than the reservation utility R , if the reservation utility is low enough. A typical form of the value function $F^i(W)$, together with $c_i(W)$ and $a_i(W)$, is shown in Figure 1. Numerical results show that the optimal contract motivates the good manager to work throughout the contract period. However, it may be too costly for the principal to compensate the bad manager for her effort, and so the desired and optimal effort is 0 when her continuation value is sufficiently large. Moreover, consistent with the findings in Sannikov (2008), if the manager's continuation value is low enough, then even without being payed she still may have an incentive to work in order to move $W_i(t)$ away from the low retirement point (equal to zero).

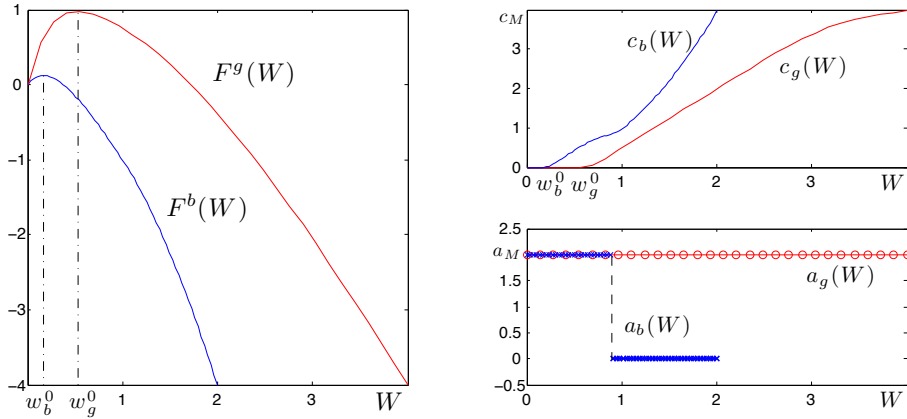


Figure 1

Function $F^i(W)$ for $u(c) = \sqrt{c}$, $\theta_g = 2$, $\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$. Points w_b^0 and w_g^0 are the maximum of $F^b(W)$ and $F^g(W)$ respectively.

4. Optimal Shutdown Contract under Adverse Selection and Moral Hazard

Before discussing the optimal screening contract, it is helpful to investigate the shutdown contract first, in which the principal deliberately excludes the bad manager from hiring. Assume that the firm owner only wants to hire the good manager. He offers a contract $\Psi_g = \{c_g(s), a_g(s)\}_{s \geq 0}$, which only the good manager accepts, whereas the bad manager prefers to take outside opportunity R . The principal's problem is to choose Ψ_g to maximize

$$rE \left\{ \int_0^\infty e^{-rs} [a_g(s) - c_g(s)] ds \right\},$$

such that conditions (2) and (3) for the good manager hold, and

$$R \geq \max_{\hat{a} \in \mathcal{A}} V(\theta_b, c_g, \hat{a}). \quad (8)$$

Here, the right-hand side is the maximum expected utility that the bad manager can obtain if she takes the shutdown contract. Under constraint (8), she would not do it, rather, she would take the outside opportunity.

4.1. Credible set

Definition of credible set

At time 0, the principal offers payment stream $c_g = \{c_g(t)\}_{t \geq 0}$, which is progressively measurable with respect to $\mathcal{F}^Y = \{\mathcal{F}_t^Y\}_{t \geq 0}$. Both the good and bad managers may choose to take it. When their best efforts are exerted by the managers, their continuation value processes in general will be different. We use $W_b^c = \{W_b^c(t)\}_{t \geq 0}$ to denote the bad manager's continuation value, if she takes the contract and behaves optimally. To distinguish it from the continuation value of the good manager, we call it the bad manager's "temptation value process." We use a superscript "c" in order to distinguish it from W_b in the screening contract, which is the bad manager's continuation if she truthfully reports her type obtains payment stream $c_b = \{c_b(t)\}_{t \geq 0}$ that is designed for the bad manager. It is crucial to distinguish between $W_b(t)$ and $W_b^c(t)$ in solving for the optimal contracts. Denote the bad manager's best effort choice by $a_b^c = \{a_b^c(t)\}_{t \geq 0}$. By Proposition 1, we have

$$dW_g(t) = r[W_g(t) + g(a_g(t)) - \theta_g u(c_g(t))] dt + r\beta_g(t)[dY(t) - a_g(t)dt], \quad (9)$$

$$dW_b^c(t) = r[W_b^c(t) + g(a_b^c(t)) - \theta_b u(c_g(t))]dt + r\beta_b^c(t)[dY(t) - a_b^c(t)dt], \quad (10)$$

where the conditions of incentive compatibility are

$$a_g \in \arg \max_{a \in \mathcal{A}} -g(a(t)) + \beta_g(t)a(t), \quad (11)$$

$$a_b^c \in \arg \max_{a \in \mathcal{A}} -g(a(t)) + \beta_b^c(t)a(t). \quad (12)$$

If the good manager takes the contract, $W_g(t)$ is the continuation value process of the good manager and $\frac{1}{\sigma}[dY(t) - a_g(t)dt]$ is the increment of the Brownian motion process. If the bad manager takes the contract, $W_b^c(t)$ is the continuation value process, or “temptation value process,” of the bad manager and $\frac{1}{\sigma}[dY(t) - a_b^c(t)dt]$ is the increment of the Brownian motion process.

When the principal designs the contract, he not only needs to consider the good manager’s incentive, as he would in the pure moral hazard setting, but he also needs to identify the possible outcomes if the bad manager takes the contract, and process W_b^c is the one which summarizes the bad manager’s incentives. Hence, the optimal contract design should be based on two state variables, W_g and W_b^c , which are fully coupled through c_g and Y . The principal has to satisfy the following constraints: condition (3) for the good manager, equivalent to (11), condition (8), equivalent to (12), and the exclusion condition

$$W_b^c(0) \leq R. \quad (13)$$

Condition (13) states that if the bad manager pretends to be a good manager, then her expected utility at time 0 cannot be better than her reservation R . Hence, only the good manager will take the contract, assuming that

$$W_g(0) \geq R. \quad (14)$$

Thus, by utilizing the continuation value processes, we transform the global conditions into instantaneous conditions and initial value conditions, thereby greatly simplifying the contracting problem. It is to be noted that, although asymmetric information exists only at time 0, it has a long-term effect on contract design and the dynamics of optimal contracts.

In order to solve the problem, we will need to identify the right boundary conditions. From Section 4, we know that $\{W_b^c(t), W_g(t)\}$ cannot move outside the feasible set \mathcal{V} . However, not every value pair in

the feasible set can be implemented by incentive-compatible contracts. Motivated by Abreu, Pearce and Stacchetti (1990) and Sannikov (2007b), we define the “credible set” as follows.

Definition 4.3 Consider the set \mathcal{E} of initial value pairs (w_b^c, w_g) in \mathcal{V} for which there exists a tuple $[c_g, a_g, a_b^c, \beta_g, \beta_b^c]$ such that the corresponding payoff processes pair $\{W_b^c(t), W_g(t)\}$, with dynamics (9) and (10), satisfies (11) and (12), and take values in \mathcal{V} for all t , almost surely. Set \mathcal{E} is called the “credible set”.

In other words, given an initial value pair (w_b^c, w_g) outside the credible set, there exists no payment stream $c = \{c(t)\}_{t \geq 0}$ taking values in \mathcal{C} , such that, if the bad (good) manager takes the contract, then her optimal expected utility at time 0 is w_b^c (w_g). That is, given such initial value pair (w_b^c, w_g) , with any payment stream, the corresponding pair $(W_b^c, W_g(t))$ will move out of the feasible set with positive probability.

Characterization of the credible set

Sannikov (2007b) developed a curvature-based approach for characterizing credible sets. Motivated by that work, we introduce a method useful in finding credible sets when the optimal contract is based on several coupled state processes.

We want to know, given $W_b^c(t)$, what the largest or the smallest expected utility is that the good manager can achieve at time t . We denote the largest utility by $U(W_b^c(t))$ and the smallest by $L(W_b^c(t))$. The credible set is then given by

$$\mathcal{E} = \{(w_b^c, w_g) \in \mathcal{V}, \text{ s.t. } w_b^c \in [0, \theta_b u(c_M)] \text{ and } L(w_b^c) \leq w_g \leq U(w_b^c)\}.$$

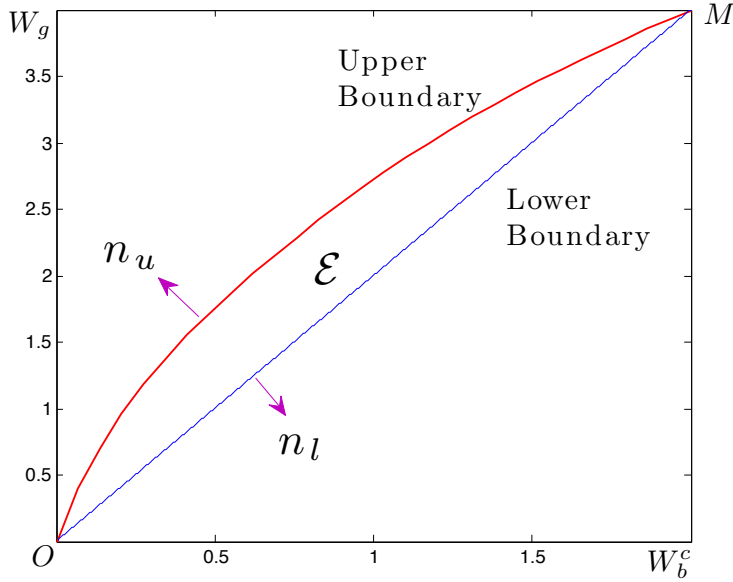
We call $U(w_b^c)$ ($L(w_b^c)$) the upper (lower) boundary of the credible set. Figure 2 presents an example of the credible set.

We next provide an informal discussion of the results stated below. Taking the bad manager’s temptation value process as a state variable, $U(w_b^c)$ is determined as follows.

$$e^{-rv}U(w_b^c) = \max_{c_g, a_g, a_b^c, \beta_g} rE \left\{ \int_v^\infty e^{-rs} [\theta_g u(c_g(s)) - g(a_g(s))] ds | W_b^c(v) = w_b^c \right\}, \quad (15)$$

Figure 2

The credible set. The lower boundary is the line segment connecting $O = (0,0)$ and $M = (\theta_b u(c_M), \theta_g u(c_M))$. Vector $n_u = (1, -U_W(W_b^c))$ ($n_l = (\frac{\theta_g}{\theta_b}, -1)$) is the normal vector at the upper (lower) boundary (pointing outward). The parameters are $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1, c_M = 4, a_M = 2, g(a_M) = 1, r = 2, \sigma = 1$.



subject to conditions (10), (12) and (3) for the good manager. Hence, the increment of Brownian motion is $dZ(t) = \frac{1}{\sigma}[dY(t) - a_g(t)dt]$, and the bad manager's temptation process from the good manager's perspective is

$$dW_b^c(t) = r \left(W_b^c - \theta_b u(c_g(t)) + g(a_b^c(t)) + \underbrace{\beta_b^c(t)(a_g(t) - a_b^c(t))}_{\text{Private benefit of the bad manager}} \right) dt + r\sigma\beta_b^c(t) dZ(t), \quad (16)$$

with (β_b^c, a_b^c) satisfying (12). The term $\beta_b^c(t)(a_g(t) - a_b^c(t))$ is the bad manager's instantaneous benefit from the good manager's perspective. Applying Ito's Lemma to $U(W_b^c(t))$, we can derive the optimality equation for $U(\cdot)$, taking into account the enforceability of $a_g(t)$. Value function (15) does not explicitly depend on $\beta_g(t)$. However, the diffusion term of $W_g(t) = U(W_b^c(t))$ is

$$\sigma\beta_b^c(t) U_W(W_b^c(t)) dZ(t) = \beta_b^c(t) U_W(W_b^c(t)) [dY(t) - a_g(t)dt].$$

From Proposition 1, we see that $\beta_g(t) = \beta_b^c(t) U_W(W_b^c(t))$ has to enforce $a_g(t)$:

$$-g(a_g(t)) + \beta_b^c(t) U_W(W_b^c(t)) a_g(t) \geq -g(\hat{a}) + \beta_b^c(t) U_W(W_b^c(t)) \hat{a}.$$

Moreover, the optimality, HJB equation for $U(W_b^c)$ is:

$$U(W) = \max_{c_g, a_g, a_b^c, \beta_b^c} \left\{ \theta_g u(c_g) - g(a_g) + U_W(W) [W - \theta_b u(c_g) + g(a_b^c) + \beta_b^c(a_g - a_b^c)] + \frac{r\sigma^2}{2} (\beta_b^c)^2 U_{WW}(W) \right\}, \quad (17)$$

such that β_b^c enforces a_b^c , $\beta_b^c U_W(W_b^c)$ enforces a_g , $U(0) = 0$ and $U(\theta_b u(c_M)) = \theta_g u(c_M)$. The last two conditions are boundary conditions. Obviously, if $W_b^c(t) = 0$ ($W_b^c(t) = \theta_b u(c_M)$), then the bad manager's expected utility at time t if she takes the contract is 0 ($\theta_b u(c_M)$). The payment after time t would be $\{c_g(s) = 0\}_{s \geq t}$ ($\{c_g(s) = c_M\}_{s \geq t}$). Then, the good manager's expected utility at time t would be 0 ($\theta_g u(c_M)$).

Similarly, we can find the lower boundary of the credible set as a function $W_g = L(W_b^c)$. We summarize our main findings for the credible set as follows.

Proposition 2 *Upper boundary $U(W)$ of the credible set is the unique solution of optimality equation (17), that is strictly increasing and strictly concave. The lower boundary is given by equation $W_g = L(W_b^c) = \frac{\theta_g}{\theta_b} W_b^c$ on $[0, \theta_b u(c_M)]$. Moreover, if the continuation value pair $(W_b^c(t), W_g(t))$ reaches the upper boundary, then it will move along the boundary following the strategy determined by equation (17) until it is absorbed by $(0, 0)$ or $(\theta_b u(c_M), \theta_g u(c_M))$. If $(W_b^c(t), W_g(t))$ reaches the lower boundary at $P^* = (\theta_b w^*, \theta_g w^*)$ for any $w^* \in [0, u(c_M)]$ at time t , then it will stay at P^* forever and the payment stream is a constant $c_g(s) = c^* = v(w^*)$ for $s \geq t$. That is, the contract is terminated at time t , and the agent is retired with a constant payment c^* after time t .*

From Proposition 2 we see that the lower boundary is a “stationary boundary”, in the sense that the continuation values do not change after hitting it. When a value pair reaches the stationary boundary the agent is retired and receives a constant payment after retirement. The upper boundary is an “extreme boundary” in the sense that the only way to implement an expected payoff pair on the extreme boundary is to make the continuation value and temptation value processes move along the tangent direction on the extreme boundary. Intuitively, if there exists $[c_g, a_g, a_b^c, \beta_g, \beta_b^c]$ such that the expected payoff pair moves inward, then the payoff pair should not be on the boundary of the credible set. The tangential movement on

the extreme boundary can be seen from optimality equation (17). Indeed, first we note that pair (β_b^c, β_g) is in the tangent direction of the extreme boundary, because $\beta_g = \beta_b^c U_W(W_b^c)$, and vector $(-U_W(W_b^c), 1)$ is the normal vector of the extreme boundary. In other words, the volatility terms of the two value processes move on the tangent line. Moreover, denoting $l(W_b^c, W_g) = W_g - U(W_b^c)$ the level function, optimality equation (17) implies that $dl(W_b^c(t), W_g(t)) = 0$ after (W_b^c, W_g) reaches the upper boundary. Hence, (W_b^c, W_g) moves tangentially on the boundary of the zero level set of the level function $l(W_b^c, W_g)$, until it reaches $(0, 0)$ or $(\theta_b u(c_M), \theta_g u(c_M))$. This tangential movement of (W_b^c, W_g) on the upper boundary is consistent with the curvature characterization in Sannikov (2007b).

Corollary 1 *There exists w^* in $(0, \theta_b u(c_M))$ such that $U_W(w^*) = \frac{\theta_g}{\theta_b}$. On the extreme boundary, if $W_b^c < w^*$, the optimal compensation is 0, otherwise optimal compensation is c_M . Moreover, we have $a_g \geq a_b^c$ on the extreme boundary. For all $W \in (0, \theta_b u(c_M))$, we have $U_W(W) > 1$ and $\beta_g \geq \beta_b^c$.*

For concreteness, we described \mathcal{E} as the credible set from the good manager's perspective. However, it is also the credible set from the bad manager's perspective. This is because \mathcal{E} depends on the dynamics of $(W_b^c(t), W_g(t))$, not on who takes the contract.

4.2. Contract design

We now discuss the principal's problem. Denote the principal's value function by $J^g(W_b^c, W_g)$ if the good manager takes the contract. It is dependent on two state variables: the good manager's continuation value process and the bad manager's temptation process. We denote the first-order derivatives with respect to W_g and W_b^c by J_2^g and J_1^g , and the second-order derivatives by J_{11}^g , J_{12}^g and J_{22}^g . Recalling (16), the optimality equation is:

$$J^g(W_b^c, W_g) = \max_{c_g, a_g, a_b^c, \beta_g, \beta_b^c} a_g - c_g + \frac{r\sigma^2}{2} \left[\beta_g^2 J_{22}^g + 2\beta_g \beta_b^c J_{12}^g + (\beta_b^c)^2 J_{11}^g \right] \quad (18)$$

$$[W_g - \theta_g u(c_g) + g(a_g)] J_2^g + [W_b^c - \theta_b u(c_g) + g(a_b^c) + \beta_b^c (a_g - a_b^c)] J_1^g.$$

such that β_b^c enforces a_b^c and β_g enforces a_g .

Moreover, the principal's value function is defined on credible set \mathcal{E} , and the boundary conditions depend on the behavior of the optimal contract on the boundary. First, notice that condition (8) is reduced to choos-

ing the optimal initial value for the bad manager's value process. Next, as stated in the above proposition, on the extreme boundary the terms of the optimal vector $(c_g, a_g, a_b^c, \beta_g, \beta_b^c)$ are determined by optimality equation (17) as deterministic functions of (W_b^c, W_g) . Moreover, note that we have $W_g = U(W_b^c)$ on the extreme boundary, hence those terms can be written as deterministic functions of W_g only. We now state the boundary conditions for the principal's value function. By "boundary conditions" we mean the description of the credible set and the properties of the solution at its boundaries.

Lemma 2 *On the stationary (extreme) boundary, $J^g(W_b^c, W_g) = K^{L,g}(W_g) (K^{U,g}(W_g))$, where $K^{L,g}(W_g) = -v\left(\frac{W_g}{\theta_g}\right) = -v\left(\frac{W_b^c}{\theta_b}\right)$, and $K^{U,g}(W_g)$ is the solution to*

$$K^{U,g}(W_g) = a_g(W_g) - c_g(W_g) + K_W^{U,g}(W_g) [W_g - \theta_g u(c_g(W_g)) + g(a_g(W_g))] + \frac{r\sigma^2\beta_g^2(W_g)}{2} K_{WW}^U(W_g), \quad (19)$$

satisfying $K^{U,g}(0) = 0$ and $K^{U,g}(\theta_g u(c_M)) = -c_M$.

Here, as discussed above, vector $(c_g(W_g), a_g(W_g), a_b^c(W_g), \beta_g(W_g), \beta_b^c(W_g))$ is the optimal solution determined by optimality equation (17).

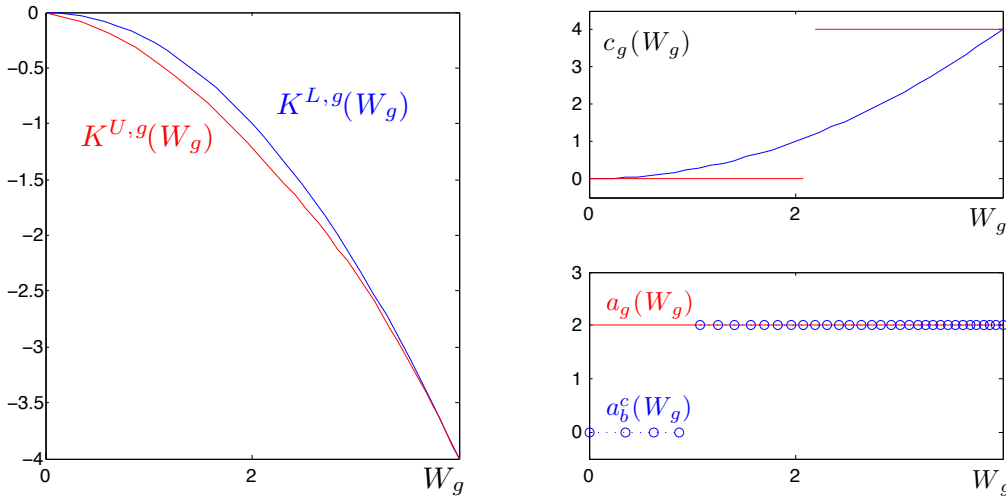


Figure 3

The value function on the boundaries. The graph on the left shows the principal's value function on the upper and lower boundaries. On the right, the first graph shows the optimal payment on the upper (in red) and lower (in blue) boundaries. The second graph shows the optimal effort on the extreme boundary. $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.

Figure 3 shows the principal's value function on the stationary and extreme boundaries, as well as the optimal payment and effort. The effort on the stationary boundary is zero, hence not shown in the figure. As stated in Proposition 2, on the extreme boundary the optimal payment and effort are determined by optimality equation (17), and the manager will get zero payment if and only if $\theta_g - \theta_b U_W(W_b^c) \leq 0$. Also on the extreme boundary, the bad manager will not work if W_g is small, but this does not mean β_b^e is zero, because it still may be better to provide incentives to the good manager to work. Our numerical results also show that, with the fixed continuation value W_g , the principal's value on the stationary boundary (a larger temptation value) is larger than that on the extreme boundary (a smaller temptation value), because the cost to maintain the truthfulness of the bad manager dominates the profit realized by the manager's work on the upper boundary. The extreme and stationary boundaries are inefficient in the sense that the principal cannot generate positive profit at the boundaries.

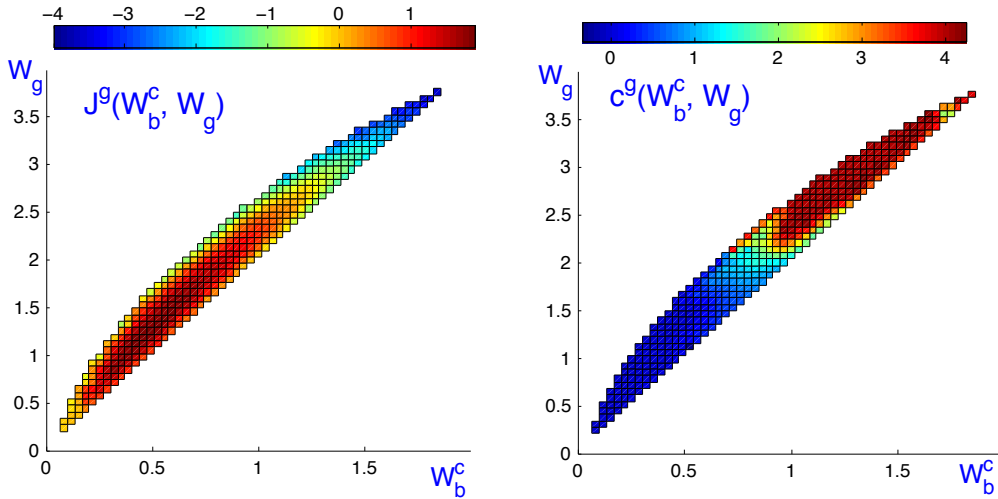


Figure 4
Surface maps of the principal's value function (right) and the good manager's payment (left). $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1, c_M = 4, a_M = 2, g(a_M) = 1, r = 2, \sigma = 1$.

In Figure 4, we present surface maps of the principal's value function and the payment to the good manager. In Sannikov (2008), where only dynamic moral hazard is considered, the principal's value function is non-monotonic in the continuation value of the manager. From Figure 4, we can see that the principal's value function is not only non-monotonic in the continuation value, but also non-monotonic in the temptation value. More precisely, given W_g , the principal's value is low if the temptation value is either very small or very large.

This non-monotonicity of the value function stems from the inefficiency at the boundary of the credible set and has a large impact on the optimal contract. More precisely, from optimality equation (18), we can see that the optimal choice of compensation maximizes

$$-c - [\theta_b J_1^g(W_b^c, W_g) + \theta_g J_2^g(W_b^c, W_g)]u(c). \quad (20)$$

Thus, the agent's compensation is zero when $\theta_b J_1^g(W_b^c, W_g) + \theta_g J_2^g(W_b^c, W_g) \geq 0$. We call such a region of points (W_b^c, W_g) "the probationary domain." Quantity $-\theta_b J_1^g(W_b^c, W_g) - \theta_g J_2^g(W_b^c, W_g)$ is the weighted marginal cost of giving the agent value through the two managers' continuation payoffs. In the probationary domain, where W_g and W_b^c are small, there is no cost in increasing values of W_g and W_b^c , and the principal benefits by doing so. On the other hand, when W_g and W_b^c are large, the managers' continuation value pair has a large likelihood of hitting the (inefficient) boundaries and the principal's value decreases with W_g and W_b^c .

Moreover, the inefficiency of the credible set's boundaries is due to the double-sided income effects. First, when the continuation value of the good manager is sufficiently large, it costs the principal too much to compensate the manager for her effort, which is the inefficiency of the extreme boundary. Second, if the bad manager's temptation value becomes larger whereas the continuation value of the good manager remains the same, it is costly to provide incentives to the bad manager, hence even more costly for the good manager. Thus, it is optimal for the principal to retire the manager if W_b^c is sufficiently large, which is the inefficiency of the stationary boundary.

Because the principal's value function is non-monotonic in both W_g and W_b^c , the shutdown contract may be suboptimal compared to the screening contract, if the reservation utility is small. The principal may prefer to (potentially) hire either manager, by raising the initial value of W_b^c above R , to obtain a greater profit from the good manager's work.

Optimal contract

Having described the boundary conditions, we can now describe the optimal shutdown contract. The following definition adopts the jargon of the repeated games literature.

Definition 4.4 Define set $\mathcal{D}(R) = \{(w_b^c, w_g) \in \mathcal{E}, \text{ such that } w_b^c \leq R \text{ and } w_g \geq R\}$. Set $\mathcal{D}(R)$ is called the “initially and individually rational set ” when the reservation utility is R .

Set $\mathcal{D}(R)$ is the set of expected payoff pairs at time zero, such that the good manager will take the contract, and the bad manager will not.

As in the rest of the paper, we assume that there exists a strict concave solution for optimality equation (18). One numerical example is illustrated by Figure 5.

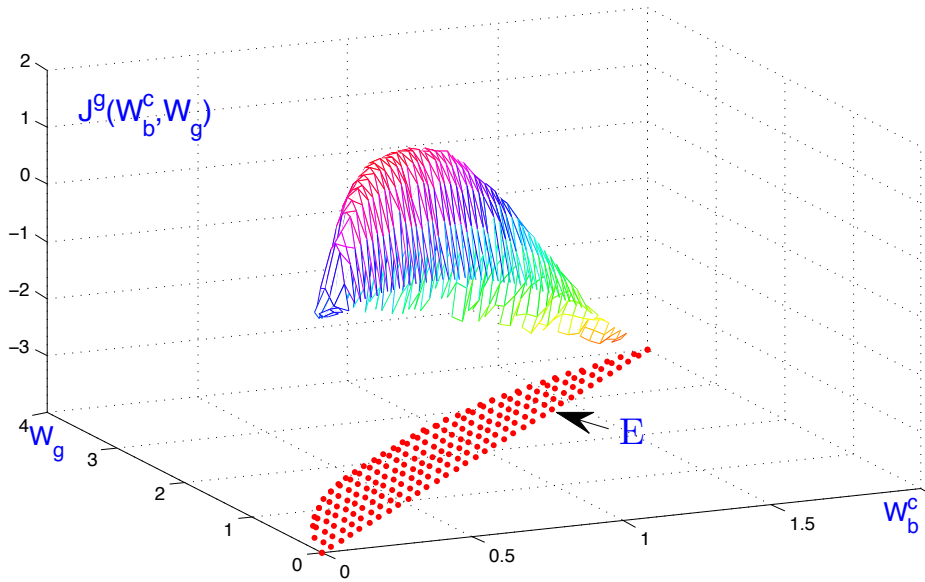


Figure 5

Mesh map of the principal's value function if a good manager is hired. $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1$, $c_M = 4, a_M = 2, g(a_M) = 1, r = 2, \sigma = 1$.

We have the following result.

Proposition 3 The optimal contract is given by $\Psi_g = \{\bar{c}_g(t), \bar{a}_g(t), \bar{a}_b^c(t), \bar{\beta}_g(t), \bar{\beta}_b^c(t)\}_{t \geq 0}$, determined by optimality equation (18) in terms of continuation value process $W_g(t)$ and temptation value process $W_b^c(t)$, which satisfies boundary conditions stated in Lemma 2. The dynamics of $W_g(t)$ and $W_b^c(t)$ follow equations

$$d\bar{W}_g(t) = r[\bar{W}_g(t) + g(\bar{a}_g(t)) - \theta_g u(\bar{c}_g(t))] dt + r\bar{\beta}_g(t)[dY(t) - \bar{a}_g(t)dt], \quad (21)$$

$$d\bar{W}_b^c(t) = r[\bar{W}_b^c(t) + g(\bar{a}_b^c(t)) - \theta_b u(\bar{c}_g(t))] dt + r\bar{\beta}_b^c(t)[dY(t) - \bar{a}_b^c(t)dt], \quad (22)$$

with initial values $\bar{W}_g(0)$ and $\bar{W}_b^c(0)$ satisfying

$$[\bar{W}_b^c(0), \bar{W}_g(0)] \in \arg \max_{[w_b^c, w_g] \in \mathcal{D}(R)} J^g(w_g, w_b^c). \quad (23)$$

When $[\bar{W}_b^c, \bar{W}_g]$ reaches the stationary boundary at $[\theta_b w^*, \theta_g w^*]$, the agent is retired with constant payment $[c_g(t) = v(w^*)]$ thereafter. When $[\bar{W}_b^c, \bar{W}_g]$ reaches the extreme boundary at $[w^*, K^{U,g}(w^*)]$, pair $[\bar{W}_b^c, \bar{W}_g]$ moves thereafter along the upper boundary defined by $W_g = U(W_b^c)$ until it reaches the low retiring value pair $(0, 0)$ or the high retiring value pair $(\theta_b u(c_M), \theta_g u(c_M))$, in which case the agent is retired at zero payment or constant payment c_M , respectively.

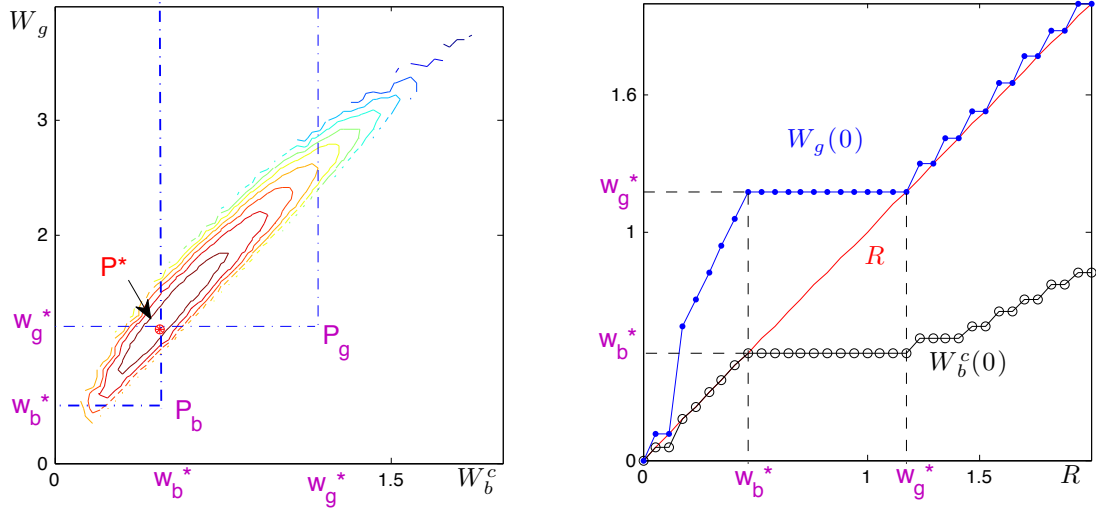


Figure 6
Optimal initial values. The right graph shows how $[W_b^c(0), W_g(0)]$ changes when R increases. The left panel shows a contour map of the principal's value function and shows how set $\mathcal{D}(R)$ changes when R increases. $\mathcal{D}(w_b^*)$ ($\mathcal{D}(w_g^*)$) is the rectangle at the right top of point P_b (P_g). The principal's value function achieves the maximum at (w_b^*, w_g^*) . $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.

Figure 6 provides numerical results for optimal initial values $W_g(0)$ and $W_b^c(0)$ given different reservation utilities R . Note that when $w_b^* \leq R \leq w_g^*$, the initial values are unchanged, set at those levels. This is because $J^g(W_b^c, W_g)$ takes a maximum value at $P^* = (w_b^*, w_g^*)$, when $P^* \in \mathcal{D}(R)$, and w_b^* (w_g^*) is the smallest (largest) reservation value level such that $\mathcal{D}(w_b^*)$ ($\mathcal{D}(w_g^*)$) contains P^* . Another interesting observation is that, although the principal does not want to hire the bad manager, when $R \leq w_b^*$ her participation constraint (8) is as high as possible for the shutdown contract, that is, binding at R . The reason is that the principal value function is not monotonic in W_b^c , and can be increased by raising W_b^c in that region. This implies that

the principal can do better by offering the screening contract instead of the shutdown contract when the reservation value is sufficiently low.

We conclude this section with Figure 7, which describes how the principal's expected profit changes with respect to reservation value R . In the pure moral hazard model, if R is less than the point denoted w_g^0 , it is good for the principal to raise the agent's expected utility, otherwise the manager's continuation value has a large chance of hitting the low retiring value zero. However, when the moral hazard is mixed with adverse selection, the principal's value is also dependent on the temptation process, whose initial value cannot be greater than the reservation utility. Hence the shutdown contract is costly if the reservation value is low.

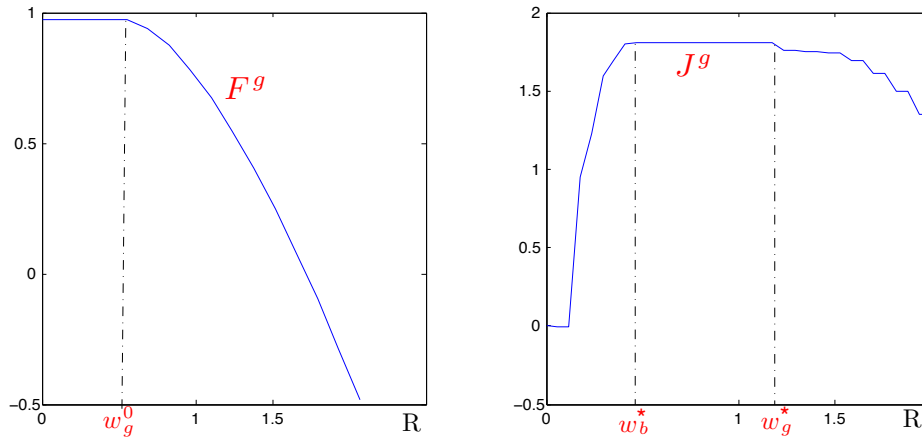


Figure 7
Effect of reservation utility on value functions. The left graph shows how the principal's value changes under pure moral hazard if the good manager is hired. The right panel shows how the principal's value changes under combined moral hazard and adverse selection, for the optimal shutdown contract. $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1, c_M = 4, a_M = 2, g(a_M) = 1, r = 2, \sigma = 1$.

5. Optimal Screening Contract under Adverse Selection and Moral Hazard

A significant feature of Sannikov's (2008) approach is that in the pure moral hazard setting the agent's continuation value is the only state variable. This is no longer true if the agent's type is private information, the continuation value is not sufficient in contract design when the moral hazard is mixed with adverse selection. In fact, Sannikov (2007a) takes a non-standard approach in solving a similar problem, but in which the manager only consumes at a finite horizon.

We indicate now what the optimal solutions should depend on in our framework. We define the continuation value process for the bad agent and the temptation value process for the good agent, as follows.

$$dW_b(t) = r[W_b(t) + g(a_b(t)) - \theta_b u(c_b(t))] dt + r\beta_b(t)[dY(t) - a_b(t)dt], \quad (24)$$

$$dW_g^c(t) = r[W_g^c(t) + g(a_g^c(t)) - \theta_g u(c_b(t))] dt + r\beta_g^c(t)[dY(t) - a_g^c(t)dt], \quad (25)$$

Here, $\beta_b(t)$ enforces $a_b(t)$ and $\beta_g^c(t)$ enforces $a_g^c(t)$. In addition to $J^g(W_b^c, W_g)$, defined previously, we introduce the optimal expected profit $J^b(W_b, W_g^c)$ of the principal when hiring the bad manager. Then, the principal's optimal profit from issuing a screening contract is obtained by maximizing

$$p_g J^g(W_b^c(0), W_g(0)) + p_b J^b(W_b(0), W_g^c(0)).$$

where $[W_b(0), W_g^c(0), W_b^c(0), W_g(0)]$ are initial values.

5.1. Optimality equation

We first need to identify the credible set of $(W_b(t), W_g^c(t))$. Note that the feasible set and the dynamic structure of $(W_b(t), W_g^c(t))$ are the same as those of $(W_b^c(t), W_g(t))$. Hence, the credible set of $(W_b(t), W_g^c(t))$ is also \mathcal{E} . Recall that if (W_b^c, W_g) is on the extreme boundary at time t , the only implementable contract is defined by $(c_g(W_g), a_g(W_g), a_b^c(W_g), \beta_g(W_g), \beta_b^c(W_g))$, which are deterministic functions of W_g and determined by optimality equation (17). In characterizing the contract for the bad manager, if the continuation value and temptation value processes reach the extreme boundary at (W_b, W_g^c) , then the unique contract that keeps the value pair in credible set is the same as that for the good manager with W_g^c replacing W_g : $c_b(W_g^c) = c_g(W_g^c)$, $a_g^c(W_g^c) = a_g(W_g^c)$, $a_b(W_g^c) = a_b^c(W_g^c)$, $\beta_g^c(W_b^c) = \beta_g(W_g^c)$, $\beta_b(W_g^c) = \beta_b^c(W_g^c)$. The difference relative to the shutdown case is that in the screening contract the initial conditions for $(W_b^c(0), W_g(0))$ and $(W_b(0), W_g^c(0))$ have to be such that the managers will only accept the contract designed for their type. In the bad manager's contract, the increment of Brownian motion is $dZ(t) = \frac{1}{\sigma} [dY(t) - a_b(t)dt]$, and the dynamics of $W_g^c(t)$ become

$$dW_g^c(t) = r[W_g^c(t) + g(a_g^c(t)) - \theta_g u(c_b(t)) + \beta_g^c(t)(a_b(t) - a_g^c(t))] dt + \sigma r \beta_g^c(t) dZ(t). \quad (26)$$

Similar to Lemma 2, on the extreme boundary the principal's value function is

$$J^b(W_b, W_g^c)|_{W_g=U(W_b)} = K^{U,b}(W_g^c), \quad (27)$$

where $K^{U,b}(W_g^c)$ is the solution to

$$\begin{aligned} K^{U,b}(W_g^c) = & a_g^c(W_g^c) - c_b(W_g^c) + K_W^{U,b}(W_g^c) [W_g^c - \theta_g u(c_b(W_g^c)) + g(a_g(W_g^c))] \\ & + \frac{r\sigma^2 (\beta_g^c(W_g^c))^2}{2} K_{WW}^U(W_g^c) + K_W^{U,b}(W_g^c) \beta_g^c(a_b(W_g^c) - a_g^c(W_g^c)) \end{aligned} \quad (28)$$

satisfying $K^{U,b}(0) = 0$ and $K^{U,b}(\theta_g u(c_M)) = -c_M$. On the stationary boundary, same as in the contract for the good manager, the agent will be offered a constant payment stream $v(\frac{W_b}{\theta_b}) = v(\frac{W_g^c}{\theta_g})$. Hence the principal's value function on the stationary boundary is

$$J^b(W_b, W_g^c)|_{W_g=L(W_b)} = K^{L,b}(W_g^c) = K^{L,g}(W_g^c). \quad (29)$$

The principal's value function on the extreme boundary when hiring a bad manager may be different from that when hiring a good manager. The difference (as shown in Figure 8) is due to the principal's instantaneous payoff being $a_b(t) - c_b(t)$ if a bad manager is hired, and we have $a_b(t) \leq a_g^c(t)$ on the extreme boundary.

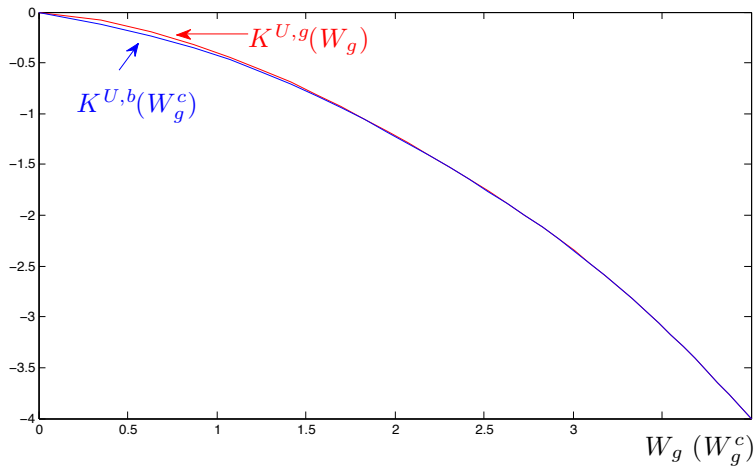


Figure 8
Difference in the principal's value on the upper boundary when hiring a good manager and a bad manager. $u(c) = \sqrt{c}$, $\theta_g = 2$, $\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.

Next, we note that the optimal value function of the principal if the bad manager is hired satisfies

$$J^b(W_b, W_g^c) = \max_{c_b, a_g^c, a_b, \beta_g^c, \beta_b} \left\{ a_b - c_b + \frac{r\sigma^2}{2} \left[(\beta_g^c)^2 J_{22}^b(W_b, W_g^c) + 2\beta_g^c \beta_b J_{12}^b(W_b, W_g^c) + (\beta_b)^2 J_{11}^b(W_b, W_g^c) \right] \right. \\ \left. + [W_g^c - \theta_g u(c_b) + g(a_g^c) + \beta_g^c(a_b - a_g^c)] J_2^b(W_b, W_g^c) + [W_b - \theta_b u(c_b) + g(a_b)] J_1^b(W_b, W_g^c) \right\}, \quad (30)$$

such that β_b enforces a_b and β_g^c enforces a_g^c ,

with boundary conditions (29) and (27).

Definition 5.5 Set $\mathcal{D}^s(R) = \{(w_b^c, w_g), (w_b, w_g^c) \in \mathcal{E} \times \mathcal{E}, \text{ such that } w_b \geq R, w_i \geq w_i^c\}$ is called “initially and individually rational set” for the screening contract when the reservation utility is R .⁴

Let $[\bar{c}_b(t), \bar{a}_g^c(t), \bar{a}_b(t), \bar{\beta}_g^c(t), \bar{\beta}_b(t)]$ be the vector of optimal processes, determined by optimality equation (30) in terms of continuation value process $W_b(t)$ and temptation process $W_g^c(t)$. The following proposition summarizes our results for the screening contract.

Proposition 4 The optimal contract is $\Psi_i = \{\bar{c}_i(t), \bar{a}_i(t)\}_{i=g,b}$ in which Ψ_g depends on the processes in (21) and (22), and Ψ_b depends on the processes

$$dW_b(t) = r[W_b(t) + g(\bar{a}_b(t)) - \theta_b u(\bar{c}_b(t))] dt + r\bar{\beta}_b(t)[dY(t) - \bar{a}_b(t)dt], \quad (31)$$

$$dW_g^c(t) = r[W_g^c(t) + g(\bar{a}_g^c(t)) - \theta_g u(\bar{c}_b(t))] dt + r\bar{\beta}_g^c(t)[dY(t) - \bar{a}_g^c(t)dt], \quad (32)$$

with initial values $\bar{P}(0) = (\bar{W}_b(0), \bar{W}_g^c(0)) \times (\bar{W}_b^c(0), \bar{W}_g(0))$ satisfying

$$\bar{P}(0) \in \arg \max_{(w_b, w_g^c) \times (w_b^c, w_g) \in \mathcal{D}^s(R)} p_g J^g(w_b^c, w_g) + p_b J^b(w_b, w_g^c). \quad (33)$$

The proportions p_b and p_g of good and bad managers in the labor market have no impact on contract dynamics, but they affect the initial values of the continuation and temptation processes, as seen from the following result.

⁴Note that we do not require $w_g \geq R$, because in \mathcal{E} , $w_g \geq w_b$.

Corollary 2 *If $p_g J^g(w_b, w_g) + p_b J^b(w_b, w_g)$ attains the maximum value at interior point (w_b^+, w_g^+) in \mathcal{E} , and if $J_1^g(w_b^+, w_g^+) \geq 0$ and $J_2^b(w_b^+, w_g^+) \geq 0$, then the optimal initial values are $W_b(0) = W_b^c(0) = w_b^+$ and $W_g(0) = W_g^c(0) = w_g^+$, assuming that $R \leq w_b^+$.*

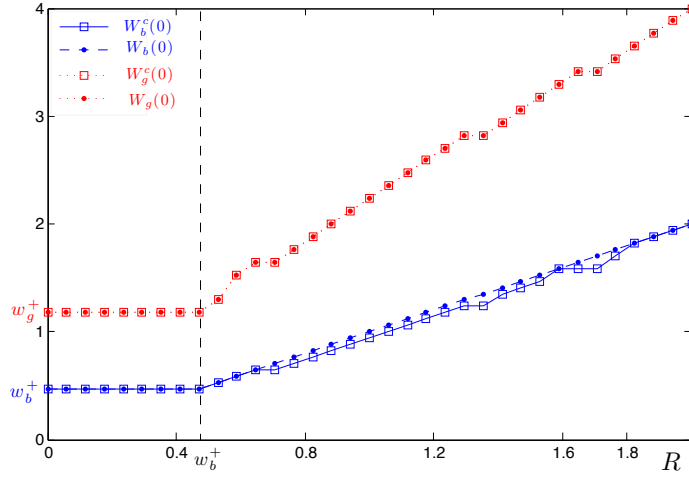


Figure 9

Optimal initial values for the screening contract. $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1, c_M = 4, a_M = 2, g(a_M) = 1, r = 2, \sigma = 1, p_g = 0.3, p_b = 0.7$.

The foregoing corollary is illustrated by Figure 9. If the reservation utility is small, then both managers' expected utilities at time zero are not binding at R . The principal is better off increasing the utilities to the level (w_b^+, w_g^+) . Meanwhile, the optimal screening contract represents a weak separating equilibrium: both managers are indifferent between truth-telling and lying. However, the contract is not a pooling one, the payments and efforts are different. The principal may obtain a strict separating equilibrium by increasing $W_g(0)$ and $W_b(0)$ by a tiny value ε . When the reservation utility is large, the initial value of the bad manager's continuation value $W_b(0)$ is binding at R . The initial value of the good manager's continuation process is equal to the initial value of her temptation process. The binding of $W_b(0)$ at R implies that it is suboptimal for the principal to offer the screening contract when the reservation utility is large. Rather, the shutdown contract should be offered, as seen in Figure 10.

Most of the arguments in favor of the screening contracts in the static models literature is based on the assumption that the labor is in short supply and the principal will suffer a loss if he hires the good manager only. If the market has a sufficient supply of both types of managers, then this argument is no longer valid. Our model shows that if the common reservation utility is low, then it is too costly and inefficient to hire only good managers, because the optimal shutdown contract needs to ensure that the bad manager's initial

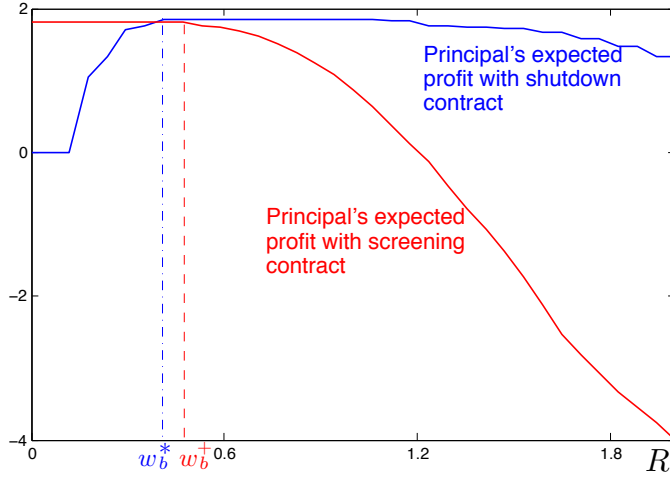


Figure 10

Comparison of shutdown and screening contracts. $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1, c_M = 4, a_M = 2, g(a_M) = 1, r = 2, \sigma = 1, p_g = 0.3, p_b = 0.7$.

temptation value is no larger than the reservation utility, which damages the good manager's incentives. By increasing the bad manager's initial temptation value, the principal's expected profit may increase. In this case, the screening contract is better, with optimally chosen initial value that is not binding at the common reservation value. However, if the reservation utility is high, it becomes too expensive to hire a bad manager. The bad manager's expected utility in the screening contract is binding at the reservation value, which implies that the principal would prefer the bad manager to have a low reservation value. Then, the shutdown contract should be offered, because it specifies the initial value of the temptation process for the bad manager that is lower than the reservation value. In practice, screening contracts are not used for top management positions such as CEO's, who have high reservation utility values. However, these contracts may be optimal for positions with low reservation utility values.

6. Optimal Contracts when Managers are Patient

Discrete-time models of dynamic moral hazard show that first-best efficiency is attainable as the agent's discount rate goes to zero. Sannikov (2008) obtains this result in continuous time. We claim that, in our model, as r goes to 0, the credible set converges to that of the pure adverse selection case. However, the principal's value function in general converges to a value function different from pure adverse selection, because the dishonest manager's private benefit does not vanish as r goes to 0. On the other hand, if we

restrict the set of admissible contracts by a “no private benefit constraint”, the value function also converges to that of pure adverse selection. In this section, we discuss the convergence of the credible set and the value function of the principal when the good manager is hired. The convergence of the principal’s value function when hiring the bad manager can be explored similarly. As is customary, we call the contract derived in our mixed model “the third best contract.”

6.1. Credible set under pure adverse selection

Different from the third best contract, the principal can observe the manager’s effort and the underlying Brownian motion under pure adverse selection setting. Similar to our previous discussion, we consider the shutdown contract for the good manager. The principal’s problem is to choose (c_g, a_g) to maximize

$$rE \left\{ \int_0^\infty e^{-rs} [a_g(s) - c_g(s)] ds \right\},$$

such that

$$rE \left\{ \int_0^\infty e^{-rs} [\theta_g u(c_g(s)) - g(a_g(s))] ds \right\} \geq R, \text{ and } rE \left\{ \int_0^\infty e^{-rs} [\theta_b u(c_g(s)) - g(a_g(s))] ds \right\} \leq R$$

Since the principal can observe the effort, the bad manager also has to apply the good manager’s effort a_g when she chooses to misreport her type at time zero. We can represent the continuation value and the temptation value processes as

$$dW_g(t) = r[W_g(t) - \theta_g u(c_g(t)) + g(a_g(t))] dt + r\beta_g(t)\sigma dZ(t), \quad (34)$$

$$dW_b^c(t) = r[W_b^c(t) - \theta_b u(c_g(t)) + g(a_g(t))] dt + r\sigma\beta_b^c(t) dZ(t). \quad (35)$$

Next we derive the credible set \mathcal{E}^A for $(W_g(t), W_b(t))$. Following the proof of Proposition 2, one can see that the stationary boundary of \mathcal{E}^A is the same as that of credible set in our mixed model. The extreme boundary is given by $w_g = U^A(w_b^c)$, which satisfies

$$U^A(w_b^c) = \max_{c_g, a_g} rE \left[\int_0^\infty e^{-rs} [\theta_g u(c_g(s)) - g(a_g(s))] ds \right]$$

with (35) and $W_b^c(0) = w_b^c$. Thus, $U^A(W_b^c)$ is the solution to

$$U^A(W_b^c) = \max_{c_g, a_g} \theta_g u(c_g) - g(a_g) + U_W^A(W_b^c) [W_b^c - \theta_b u(c_g) + g(a_g)] \quad (36)$$

with boundary condition $U^A(\theta_b u(c_M)) = \theta_g u(c_M)$. The second order derivative term disappears because, optimally, $\beta_b^c = 0$. The solution is simply given by $U^A(W_b^c) = W_b^c + (\theta_g - \theta_b)u(c_M)$, and on this extreme boundary, the optimal payment is c_M and the effort can be arbitrarily chosen by the principal.⁵ Note that $U^A(W_b^c)$ is increasing and concave and $U_W^A = 1$. Also note that credible set \mathcal{E}^A does not depend on discount rate r , which implies that our long-term contract does not improve the efficiency relative to the static model of pure adverse selection model with constant private shock.

Remark 1 *We want to indicate in this remark why first best and pure adverse selection do not have the same efficiency. In the first best setting, the agent's type is known, so that, even if both the good and the bad manager take the same payment stream, the principal can dictate different effort levels. Hence, the continuation value and temptation value processes are*

$$dW_g(t) = r[W_g(t) - \theta_g u(c_g(t)) + g(a_g(t))] dt + r\beta_g(t)\sigma dZ(t), \quad (37)$$

$$dW_b^c(t) = r[W_b^c(t) - \theta_b u(c_g(t)) + g(a_b^c(t))] dt + r\sigma\beta_b^c(t) dZ(t). \quad (38)$$

In the first best, $a_g(t) \neq a_b^c(t)$ in general, because the principal can distinguish the managers. Then, both the lower and the upper boundary of the first best credible set \mathcal{E}^F should be different from those of the pure adverse selection. Indeed, by the definition of the lower boundary, we have

$$L^F(W_b^c) = \min_{c_g, a_g, a_b^c, \beta_g, \beta_b^c} E \left[r \int_0^\infty e^{-rs} (\theta_g u(c_g(s)) - g(a_g(s))) ds \right]$$

subject to (38). Then, the lower boundary $L^F(W_b^c)$ is the solution of

$$L^F(W_b^c) = \min_{c_g, a_g, a_b^c} \{ \theta_g u(c_g) - g(a_g) + U_W^F(W_b^c) [W_b^c - \theta_b u(c_g) + g(a_b^c)] \}$$

subject to $L(\theta_b u(c_M)) = \theta_g u(c_M)$. We see then that the lower boundary of pure adverse selection, given by $L^A(W_b^c) = \frac{\theta_g}{\theta_b} W_b^c$, is not a solution to the above equation. Moreover, from the definition of lower boundary,

⁵The determination of the extreme boundary does not depend on the effort.

we have that pure adverse selection lower boundary is larger than the first-best lower boundary, $L^A(W_b^c) \geq L^F(W_b^c)$. Similarly, the first best upper boundary satisfies

$$U^F(W_b^c) = \max_{c_g, a_g, a_b^c} \{ \theta_g u(c_g) - g(a_g) + U_W^F(W_b^c) [W_b^c - \theta_b u(c_g) + g(a_b^c)] \} \quad (39)$$

with boundary condition $U^F(\theta_b u(c_M)) = \theta_g u(c_M)$. From the definition of $U^F(\cdot)$ and $U^A(\cdot)$, we have $U^F(W_b^c) \geq U^A(W_b^c)$, and $U^A(W_b^c)$ is not a solution of (39). Hence we have a strict inclusion $\mathcal{E}^A \subset \mathcal{E}^F$. That means that the principal can implement more value pairs in the first best setting than under pure adverse selection.

6.2. Credible set and value function when the agents are patient

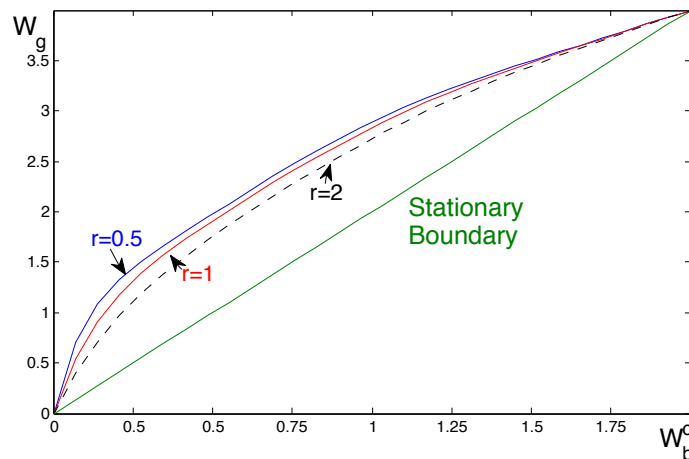


Figure 11
Comparison of Extreme Boundaries with Different Discount Rates. $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1, c_M = 4, a_M = 2, g(a_M) = 1, \sigma = 1$.

Figure 11 presents the credible sets of the mixed model with different discount rates. It appears that as r goes to 0, the upper boundary increases monotonically. However, we are not able to verify this in general. Still, we have the following

Proposition 5 *As discount rate r goes to 0, upper boundary function $U^r(W_b^c)$ locally uniformly converges to $U^A(W_b^c)$ on $(0, \theta_b u(c_M))$.*

Thus, the credible set converges to the one in the pure adverse selection model as r goes to 0 (because the stationary boundary remains the same). Moreover, from Remark 1, we know that the first-best credible set is larger than that of pure adverse selection, hence the credible set of the mixed model cannot converge to that under the first best setting. Hence, in general, the third-best contract cannot obtain the first-best efficiency when r goes to 0, different from Sannikov (2008) in which the moral hazard contract obtains the first best efficiency as r goes to 0.

The stated result is not that surprising, but there is still a question of whether it is possible that the third best contract obtains the pure adverse selection efficiency in the limit? In our setting the inefficiency comes from two sources, one is the agency cost caused by the variation in the agent's income, another is the private benefit $\beta_b^c(a_g - a_b^c)$ of the dishonest manager (as indicated in the dynamics (16)). When r approaches 0, the inefficiency from the variation in the agent's income becomes small relative to the dishonest manager's private benefit, which does not decay as r becomes smaller. In the interior of the credible set a_g and a_b^c may not converge to each other even as r becomes very small. Hence, as r approaches zero, it is unclear what the third best contract converges to.

7. Conclusion

This paper considers a dynamic principal-agent problem with moral hazard that is present continuously, and adverse selection that occurs only at time zero. We derive the optimal contracts for good and bad managers, each of which is based on the honest manager's continuation value and the dishonest manager's temptation value. We find that it may be optimal for the agent to retire early, at varying levels of the manager's continuation value. Different from Sannikov (2008), in which the manager is retired either with zero or the highest payment, in our model retirement may occur at different levels of payment. Another finding is that the principal's value function is a function of two state variables, and is not only non-monotonic in the continuation values, but is also non-monotonic in the temptation values, due to the inefficiency of the credible set's boundary, caused by the double-sided income effects of the managers. We have shown that, when the common reservation utility is high, it is better for the principal to offer the shutdown contract to lower the information rent paid to the good manager. When the reservation utility is low, it is better to offer the screening contract, and raise the expected payoff for the bad manager at time zero so that the good manager can be offered better incentives. Finally, we show that, as the discount rate approaches

zero, the domain of the principal's value function converges to the one of the pure adverse selection model. Nevertheless, the second-best efficiency of pure adverse selection is not attainable in the limit.

Our model can be also applied to investigate financial contracts and capital security design subject to a constant private shocks. Namely, one could extend the model of DeMarzo and Sannikov (2006) by allowing the manager to have private knowledge of the constant quality of the project⁶ Based on the results of this paper, we conjecture that the credible set would consist of two boundaries, a stationary boundary on which the financial contract is terminated, and a reflective boundary, on which the agent is paid. Moreover, our approach could easily be extended to the case of effort taking values in a continuous range. It would be of interest also to extend it to the case in which the agent's type being exposed to repeated persistent shocks and dynamic moral hazard.⁷ We leave these problems for future research.

⁶This would be similar to the model of Sannikov(2007a), but with infinite horizon and instantaneous payment.

⁷Wan (2011) provides a continuous-time model with i.i.d private shocks and dynamic moral hazard in solving optimal executive manager's compensation.

Appendix

A. Proof of Proposition 2

We first investigate the lower boundary. At any time $t \geq 0$, given payment stream $\{c_g(s)\}_{s \geq t}$ defined on (W_g, W_b^c) as in (9) and (10) with $W_b^c(t) > 0$, denoting the bad manager's optimal effort by $\{a_b^c(s)\}_{s \geq t}$, the conditional expected utility at time t is given by

$$e^{-rt} W_b^c(t) = E_t^{a_b^c} \left\{ \int_t^\infty e^{-rs} [\theta_b u(c_g(s)) - g(a_b^c(s))] ds \right\}.$$

Hence, the Brownian motion increment $dZ(t)$ is $\frac{1}{\sigma} [dY(t) - a_b^c(t)dt]$. The value of the good manager if she exerts the same effort as the bad manager is denoted $\hat{W}_g(t)$, and the value of the good manager if she exerts the best effort is denoted $W_g(t)$. Then, we have $W_g(t) \geq \hat{W}_g(t)$ and

$$e^{-rt} W_g(t) \geq e^{-rt} \hat{W}_g(t) = E_t^{a_b^c} \left\{ \int_t^\infty e^{-rs} [\theta_g u(c_g(s)) - g(a_b^c(s))] ds \right\} \geq e^{-rt} W_b^c(t) > 0.$$

Moreover,

$$\frac{W_g(t)}{W_b^c(t)} \geq \frac{E_t^{a_b^c} \left\{ \int_t^\infty e^{-rs} [\theta_g u(c_g(s)) - g(a_b^c(s))] ds \right\}}{E_t^{a_b^c} \left\{ \int_t^\infty e^{-rs} [\theta_b u(c_g(s)) - g(a_b^c(s))] ds \right\}} \geq \frac{\theta_g}{\theta_b}.$$

Thus, in order to show that the lower boundary is given by $L(W_b^c) = \frac{\theta_g}{\theta_b} W_b^c$ on $[0, \theta_b u(c_M)]$ we need to find, for all payoff pairs on this line segment, a strategy that does not drive $(W_b^c(t), W_g(t))$ outside of \mathcal{E} . Such a strategy is constructed next.

Suppose $(W_b^c(t), W_g(t)) = (\theta_b w^*, \theta_g w^*)$. If $w^* = 0$, then the contract will be terminated and both the good and bad managers will retire with zero payment. If $w^* > 0$, then we have

$$\frac{\theta_g}{\theta_b} = \frac{W_g(t)}{W_b^c(t)} \geq \frac{E_t^{a_b^c} \left\{ \int_t^\infty e^{-rs} [\theta_g u(c_g(s)) - g(a_b^c(s))] ds \right\}}{E_t^{a_b^c} \left\{ \int_t^\infty e^{-rs} [\theta_b u(c_g(s)) - g(a_b^c(s))] ds \right\}} \geq \frac{\theta_g}{\theta_b},$$

which implies that the two inequalities hold with equality. The second equality means that the bad manager's effort has to be $a_b^c(s) = 0$ for $s \geq t$. The first inequality then implies that the good manager's best response cannot generate more expected payoff than that if she chooses $a_g(s) = 0$ for $s \geq t$. Moreover,

given $c_g(s)$, zero effort is, in fact, the best effort for the good manager's for $s \geq t$, because it is least costly. Thus, we have

$$e^{-rt}W_g(t) = rE_t^0 \left\{ \int_t^\infty e^{-rs}\theta_g u(c_g(s))ds \right\}$$

and the principal's expected profit at time t is

$$-rE_t^0 \left\{ \int_t^\infty e^{-rs}c_g(s)ds \right\}$$

It is easily verified that this profit, under the constraint that $W_g(t) = \theta_g w^* \geq R$, cannot be higher than if the principal offers $c^*(s) = v(w^*)$, $s \geq t$. Hence, $(W_b^c(s), W_g(s))$ will remain at $(\theta_b w^*, \theta_g w^*)$ for $s \geq t$, and at time t the agent is retired at the constant retirement salary. Thus, we have proved the stated properties on the stationary boundary.

Now we consider the properties of the optimality equation (17) at the extreme boundary. Note that (17) is equivalent to

$$U_{WW}(W) = \min_{\beta_b^c, c_g, a_g, a_b^c} \frac{U(W) - \theta_g u(c_g) + g(a_g) - U_W(W) [W - \theta_b u(c_g) + g(a_b^c) + \beta_b^c (a_g - a_b^c)]}{(\beta_b^c)^2 r\sigma^2/2}, \quad (40)$$

such that β_b^c enforces a_b^c and $\beta_g = \beta_b^c U_W(W)$ enforces a_g , and boundary conditions are satisfied. The proofs of uniqueness, existence and concavity are the same as those in Lemma 1–Lemma 3 in Sannikov (2008) and Lemma 3 in Fong (2009). We now prove the strict monotonicity. Suppose that $\Phi(w)$ is the solution of (40). We want to show that $\Phi'(w) > 0$ for all $w \in [0, \theta_b u(c_M)]$. From the concavity of the solution, we only need to show that $\Phi'(\theta_b u(c_M)) > 0$. Suppose not, then $\Phi'(\theta_b u(c_M)) = \phi_0 \leq 0$. Note that $\Phi(w)$ is also the unique solution of (40) with initial conditions $\Phi(\theta_b u(c_M)) = \theta_g u(c_M)$ and $\Phi'(\theta_b u(c_M)) = \phi_0$. We now consider the solution $\hat{\Phi}(w)$ of (40) with initial conditions $\hat{\Phi}(\theta_b u(c_M)) = \theta_g u(c_M)$ and $\hat{\Phi}'(\theta_b u(c_M)) = 0$. Then we have $\hat{\Phi}(w) = \theta_g u(c_M)$. Note that $\hat{\Phi}'(\theta_b u(c_M)) \geq \phi_0$. Then, it follows from Lemma 2 in Sannikov (2008) that $\Phi(w)$ dominates $\hat{\Phi}$ for any $w \leq \theta_b u(c_M)$, which contradicts $\Phi(0) = 0$. Hence, $\phi_0 > 0$. and $\Phi(w)$ is strictly increasing.

B. Proof of Corollary 1

Recall that the extreme and stationary boundaries connect at $(0,0)$ and $(\theta_b u(c_M), \theta_g u(c_M))$. Hence, there exists a value pair $(w^*, U(w^*))$ on the extreme boundary at which the tangent line is parallel to the stationary boundary, that is $U_W(w^*) = \frac{\theta_g}{\theta_b}$. Moreover, that the optimal payment c on the boundary is either zero or c_m follows then from optimality equation (17). Next, consider function $\hat{U}(W) = W + (\theta_g - \theta_b)u(c_M)$, which is a solution of (17) with initial condition $\hat{U}(\theta_b u(c_M)) = \theta_g u(c_M)$ and $\hat{U}_W(\theta_b u(c_M)) = 1$. Note that $\hat{U}(0) > U(0)$, hence from Lemma 2 of Sannikov (2008), we have $\hat{U}_W(\theta_b u(c_M)) = 1 < U_W(\theta_b u(c_M))$. Then, we have $U_W(W) > 1$ and $\beta_g = U_W(W_b^c) \beta_b^c \geq \beta_b^c$. From the definition of enforcement, we have $a_g \geq a_b^c$.

C. Proof of Lemma 2

First, from Proposition 2 we know that when $(W_b^c(t), W_g(t))$ reaches the stationary (lower) boundary, the good manager is paid a constant $v \left(\frac{W_g}{\theta_g} \right)$, and the principal's value is $K^{L,g}(W_g) = -v \left(\frac{W_g}{\theta_g} \right)$. That is,

$$J^g(W_b^c, W_g)|_{W_g=L(W_b^c)} = K^{L,g}(W_g).$$

On the extreme (upper) boundary, the optimal $(c_g, a_g, a_b^c, \beta_g, \beta_b^c)$ is given by optimality equation (17). Also, because $W_g = U(W_b^c)$, the terms of the contract determined by (17) are deterministic functions of W_g , which we denote $(c_g(W_g), a_g(W_g), a_b^c(W_g), \beta_g(W_g), \beta_b^c(W_g))$. We denote the principal's conditional expected utility on the upper boundary by $K^{U,g}(W_g(t))$:

$$e^{-rt} K^{U,g}(W_g(t)) = E_t \left\{ r \int_t^\infty e^{-rs} [a_g(W_g(s)) - c_g(W_g(s))] ds \right\},$$

subject to $dW_g(t) = r(W_g(t) - \theta_g u(c_g(W_g(t))) + g(a_g(t))) dt + \beta_g(W_g(t)) \sigma dZ(t)$. Hence $e^{-rt} K^{U,g}(W_g(t)) + \int_0^t e^{-rs} [a_g(W_g(s)) - c_g(W_g(s))] ds$ is a martingale. By Feynman-Kac theorem, the principal's value function at the upper boundary satisfies

$$K^{U,g}(W_g) = a_g(W_g) - c_g(W_g) + K_W^{U,g}(W_g) [W_g - \theta_g u(c_g(W_g)) + g(a_g(W_g))] + \frac{r\sigma^2 \beta_g^2(W_g)}{2} K_{WW}^U(W_g),$$

with $K^{U,g}(0) = 0$ and $K^{U,g}(\theta_g u(c_M)) = -c_M$, which was to be shown.

D. Proof of Proposition 3

Let $[\bar{c}_g(t), \bar{a}_g(t), \bar{a}_b^c(t), \bar{\beta}_g(t), \bar{\beta}_b^c(t)]$ be the vector of processes determined by optimality equation (18), contingent on \bar{W}_g and \bar{W}_b^c , with dynamics (21) and (22). The corresponding principal's expected utility is denoted $J^g(\bar{W}_b^c(0), \bar{W}_g(0))$, and the initial conditions are chosen so that

$$(W_b^c(0), W_g(0)) \in \arg \max_{(w_b^c, w_g) \in \mathcal{D}(R)} J^g(w_b^c, w_g). \quad (41)$$

Consider now an arbitrary payment stream $\hat{c} = \{\hat{c}(t)\}_{t \geq 0}$ that satisfies (2) and (3) for the good manager, and (8). Let $\hat{W}_g(t)$ ($\hat{W}_b^c(t)$) be the corresponding continuation value if the good (bad) manager takes the contract, and $(\hat{a}_g(t), \hat{a}_b^c(t), \hat{\beta}_g(t), \hat{\beta}_b^c(t))$ be the corresponding optimal efforts and volatilities of the continuation values, in which $\hat{\beta}_g$ ($\hat{\beta}_b^c$) enforces \hat{a}_g (\hat{a}_b^c). To prevent misreporting, we need to have $(\hat{W}_b^c(0), \hat{W}_g(0)) \in \mathcal{D}(R)$. We claim that this arbitrary payment stream cannot be better for the principal than \bar{c} . Let us introduce the gains process by

$$\hat{G}(t) = \int_0^{t \wedge \tau} e^{-rs} [\hat{a}_g(s) - \hat{c}_g(s)] ds + e^{-rt \wedge \tau} J^g(\hat{W}_b^c(t \wedge \tau), \hat{W}_g(t \wedge \tau)),$$

where τ is the first hitting time of pair $(\hat{W}_b^c(t), \hat{W}_g(t))$ of the boundaries of the credible set. If we can show that $\hat{G}(t)$ is a supermartingale, then, because $\hat{G}(t)$ is bounded, it is also a supermartingale all the way to $t = \infty$, and the principal's expected utility satisfies

$$E \left[\int_0^\infty e^{-rs} [\hat{a}_g(s) - \hat{c}_g(s)] ds \right] \leq J^g(\hat{W}_b^c(0), \hat{W}_g(0)).$$

Furthermore,

$$J^g(\hat{W}_b^c(0), \hat{W}_g(0)) \leq J^g(\bar{W}_b^c(0), \bar{W}_g(0)), \quad \text{in } \mathcal{D}(R)$$

from the definition of $(\bar{W}_b^c(0), \bar{W}_g(0))$ in Proposition 3. The supermartingale property of $\hat{G}(t)$ is straightforward to verify by applying Ito's Lemma on $e^{-rt} J^g(\hat{W}_b^c(t), \hat{W}_g(t))$ and recalling optimality equation (18).

E. Proof of Proposition 4

Consider an alternative contract $(\hat{W}_b^c(0), \hat{W}_g(0), \hat{a}_g, \hat{a}_b^c, \hat{\beta}_g, \hat{\beta}_b^c)$ for the good manager, and $(\hat{W}_b(0), \hat{W}_g^c(0), \hat{a}_g, \hat{a}_b, \hat{\beta}_g^c, \hat{\beta}_b)$ for the bad manager such that

- (a) $(\hat{W}_b^c(0), \hat{W}_g(0)) \times (\hat{W}_b(0), \hat{W}_g^c(0)) \in \mathcal{D}^s(\mathcal{R})$,
- (b) $(\hat{\beta}_g, \hat{\beta}_b^c)$ enforces (\hat{a}_g, \hat{a}_b^c) , $(\hat{\beta}_g^c, \hat{\beta}_b)$ enforces (\hat{a}_g^c, \hat{a}_b) ,
- (c) Once $(\hat{W}_b^c(t), \hat{W}_g(t))$ or $(\hat{W}_b(t), \hat{W}_g^c(t))$ reaches the boundary of the credible set, the alternative contract follows the unique contract on the boundary.

If (a) is not satisfied, the managers will lie or not take the contract. Item (b) is the necessary and sufficient conditions for the agent's incentive compatibility. Item (c) is a necessary property of any incentive compatible contract. Then, as in the proof of Proposition 3, the principal's expected profit when hiring the good (bad) manager cannot be more than $J^g(\hat{W}_b^c(0), \hat{W}_b^c(0))$ ($J^b(\hat{W}_b(0), \hat{W}_g^c(0))$). Hence, the principal's total expected payoff cannot be more than $p_g J^g(\hat{W}_b^c(0), \hat{W}_b^c(0)) + p_b J^b(\hat{W}_b(0), \hat{W}_g^c(0))$, which is not larger than $p_g J^g(\bar{W}_b^c(0), \bar{W}_g(0)) + p_b J^b(\bar{W}_b(0), \bar{W}_g^c(0))$.

F. Proof of Corollary 2

When (IR_b) is not binding, the initial value problem is reduced to

$$\max_{w_g, w_b, \delta_g, \delta_b} p_g J^g(w_b - \delta_b, w_g) + p_b J^b(w_b, w_g - \delta_g),$$

subject to $\delta_b, \delta_g \geq 0$. The Lagrangian for this problem is $p_g J^g(w_b - \delta_b, w_g) + p_b J^b(w_b, w_g - \delta_g) + \lambda_b \delta_b + \lambda_g \delta_g$, with $\lambda_i \geq 0$. First-order conditions are

$$p_g J_1^g(w_b - \delta_b, w_g) + p_b J_1^b(w_b, w_g - \delta_g) = 0, \quad p_g J_2^g(w_b - \delta_b, w_g) + p_b J_2^b(w_b, w_g - \delta_g) = 0,$$

$$-p_g J_1^g(w_b - \delta_b, w_g) + \lambda_b = 0, \quad -p_b J_2^b(w_b, w_g - \delta_g) + \lambda_g = 0.$$

Because $p_g J^g(w_b, w_g) + p_b J^b(w_b, w_g)$ attains the maximum value at interior point (w_b^+, w_g^+) , and $J_1^g(w_b^+, w_g^+) \geq 0$, $J_2^b(w_b^+, w_g^+) \geq 0$, the solution is $\delta_b = \delta_g = 0$, $w_b = w_b^+$ and $w_g = w_g^+$, with $p_g J_1^g(w_b^+, w_g^+) = \lambda_b$, $p_b J_2^b(w_b^+, w_g^+) = \lambda_g$.

G. Proof of Proposition 5

First we need the following result.

Lemma A.1 *We have $U^A(W_b^c) \geq U^r(W_b^c)$ for all r .*

Proof. Recall that $U^A(W_b^c) = W_b^c + (\theta_g - \theta_b)u(c_M)$, and it is the solution of

$$U^A(W_b^c) = \max_{c_g, a_g} \{ \theta_g u(c_g) - g(a_g) + U_W^A(W_b^c) [W_b^c - \theta_b u(c_g) + g(a_g)] \}. \quad (42)$$

Next, for the mixed model with discount rate r , denote the optimal payment, efforts and intensity processes on the upper boundary $U^r(W_b^c)$ by $(c_g^r(W_b^c), a_g^r(W_b^c), \beta_g^r(W_b^c))$ and $(a_b^{c,r}(W_b^c), \beta_b^{c,r}(W_b^c))$, such that $\beta_g^r(W_b^c)$ ($\beta_b^{c,r}(W_b^c)$) enforces $a_g^r(W_b^c)$ ($a_b^{c,r}(W_b^c)$). We have

$$\begin{aligned} U^r(W_b^c) &= rE \left[\int_0^\infty e^{-rs} [\theta_g u(c_g^r(s)) - g(a_g^r(s))] ds \right] \\ &= rE \left[\int_0^\infty e^{-rs} [\theta_g u(c_g^r(s)) - g(a_g^r(s))] ds \right] + \lim_{t \rightarrow \infty} E [e^{-rt} U^A(W_b^c(t))], \end{aligned} \quad (43)$$

because $\lim_{t \rightarrow \infty} E [e^{-rt} U^A(W_b^c(t))] = 0$. Then, applying Ito's rule on $e^{-rt} U^A(W_b^c(t))$ we have

$$\begin{aligned} &E [e^{-rt} U^A(W_b^c(t)) - U^A(w_b)] \\ &= E \left[\int_0^t [-re^{-rs} U^A(W_b^c(s)) + re^{-rs} U_W^A(W_b^c(s) [W_b^c(s) - \theta_b u(c_g^r(s)) + g(a_g^r(s)) + \beta_b^{c,r}(a_g^r(s) - a_b^r(s))]] ds \right] \end{aligned}$$

Combining this with (43), we get, taking $t = \infty$,

$$\begin{aligned}
U^r(w_b) - U^A(w_b) &= rE \left[\int_0^\infty e^{-rs} [\theta_g u(c_g^r(s)) - g(a_g^r(s)) - U^A(W_b^c(s))] ds \right] \\
&\quad + E \left[\int_0^\infty r e^{-rs} U_W^A(W_b^c) [W_b^c(s) - \theta_b u(c_g^r(s)) + g(a_b^{c,r}) + \beta_b^{c,r} (a_g^r(s) - a_b^r(s))] ds \right] \\
&\leq rE \left[\int_0^\infty e^{-rs} [\theta_g u(c_g^r(s)) - g(a_g^r(s)) - U^A(W_b^c(s)) + U_W^A(W_b^c) [W_b^c(s) - \theta_b u(c_g^r(s)) + g(a_g^r(s))]] ds \right] \\
&\leq 0
\end{aligned}$$

The first inequality is due to the fact that $U_W^A(W_b^c) > 0$ and $g(a_b^{c,r}(s)) + \beta_b^{c,r}(a_g^r(s) - a_b^r(s)) \leq g(a_g^r(s))$ because $\beta_b^{c,r}(s)$ enforces $a_b^{c,r}(s)$. The last inequality is due to optimality equation (42). Thus, we have shown $U^r(w_b) \leq U^A(w_b)$. ■

To prove Proposition 5, in addition to this lemma, it is sufficient to find a lower bound for $U^r(w_b^c)$ that converges to $U^A(w_b)$. We denote by $\tilde{U}^r(w_b^c)$ the upper boundary of the credible set when the principal only considers payment streams $\{c_g(t)\}_{t \geq 0}$ that induce both managers to take the same, maximal effort level (a_M). We claim that

$$\tilde{U}^r(w_b^c) = \bar{U}^r(w_b^c) := \max_{c_g, \beta_b^c} E \left\{ \int_0^\infty r e^{-rs} [\theta_g u(c_g(s)) - g(a_M)] ds \right\}$$

subject to

$$dW_b^c(t) = r[W_b^c(t) - \theta_b u(c_g(t)) + g(a_M)] dt + r\sigma\beta_b^c(t) dZ(t), \quad W_b^c(0) = w_b^c.$$

in which $\beta_b^c(t)$ enforces a_M . If we can show that $\bar{U}_W^r(w_b^c) \geq 1$, then, as before, one can show that a_m will be chosen by the good manager, too, and hence $\bar{U}_W^r(w_b^c) = \tilde{U}_W^r(w_b^c)$. By applying the dynamic programming principle, we have

$$\bar{U}^r(W) = \max_{c_g} \left\{ \theta_g u(c_g) - g(a_M) + \bar{U}_W^r(W) [W - \theta_b u(c_g) + g(a_M)] + \frac{r\sigma^2}{2} (\gamma(a_M))^2 \bar{U}_{WW}^r(W) \right\}, \quad (44)$$

satisfying $\bar{U}^r(0) = 0$, $\bar{U}^r(\theta_b u(c_M)) = \theta_g u(c_M)$. Using the same proof as in Proposition 2, we can show that $\bar{U}^r(W)$ is concave, increasing and that, indeed, $\bar{U}_W^r(W) \geq 1$. Thus, $\bar{U}(W_b^c) = \tilde{U}(W_b^c)$.

Next, note that $U^A(W)$ is also the solution of

$$U^A(W_b^c) = \max_{c_g} \left\{ \theta_g u(c_g) - g(a_M) + U_W^A(W_b^c) [W - \theta_b u(c_g) + g(a_M)] \right\}. \quad (45)$$

satisfying $U^A(\theta_b u(c_M)) = \theta_g u(c_M)$, because a_g in (42) cancels out. Then, mimicking the proof of Lemma A.1, we get the first inequality in

Lemma A.2 $U^A(W_b^c) \geq U^r(W_b^c) \geq \tilde{U}^r(W_b^c)$.

The second inequality follows from the definitions of $U^r(W_b^c)$ and $\tilde{U}^r(W_b^c)$. Finally, Proposition 5 will be proved, if we prove

Lemma A.3 $\tilde{U}^r(W)$ converges locally uniformly to $U^A(W)$.

Proof. Denote

$$H^r(W, J, J', J'') = -J + \max_{c_g} \left\{ \theta_g u(c_g) - g(a_M) + J' [W - \theta_b u(c_g) + g(a_M)] + \frac{r\sigma^2}{2} (\gamma(a_M))^2 J'' \right\}$$

Then $\tilde{U}^r(W)$ is the solution of $H^r(W, \tilde{U}^r(W), \tilde{U}_{WW}^r(W), \tilde{U}_W^r(W)) = 0$, and $U^A(W)$ is the solution of

$$H^0(W, U^A(W), U_W^A(W), U_W^A(WW)) = 0. \quad (46)$$

Lions (1983) proves that, under the conditions that $H^r(W, J, J', J'')$ is continuous and $H^r(W, J, J', J'')$ converges to $H^0(W, J, J', J'')$ locally uniformly, the limit of solutions is also a solution to the limit equation. That is, if $\tilde{U}^r(W)$ converges locally uniformly to some function $\tilde{U}^0(W)$ on $(0, \theta_g u(c_M)]$, then $\tilde{U}^0(W)$ is also a solution of (46). From the uniqueness of the solution to (46), we have $\tilde{U}^0(W) = U^A(W)$. Thus, we only need to prove the existence of the limit of (a subsequence of) $\tilde{U}^r(W)$. Because function $\tilde{U}^r(W)$ is concave and uniformly bounded on $(0, \theta_g u(c_M)]$, it is locally Lipschitz and the derivative $\tilde{U}_W^r(W)$ is locally bounded. Therefore, from Arzela-Ascoli theorem, there exists a subsequence of functions $\tilde{U}^{r_n}(W)$ and a function $U^0(W)$ such that $\tilde{U}^{r_n}(W)$ locally uniformly converges to $U^0(W)$ on $(0, \theta_g u(c_M)]$. ■

References

1. Abreu, D., Pearce, D., Stacchetti, E. (1990) "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring." *Econometrica*, Vol. 58, 1041-1063.
2. Biais, B., Mariotti, T., Plantin, G., Rochet, J.C. (2007) "Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications," *Review of Economic Studies*, Vol. 74, 345-390.
3. Cvitanić, J., Zhang, J. (2007) "Optimal Compensation with Adverse Selection and Dynamic Actions," *Mathematics and Financial Economics*, Vol. 1, 21-55.
4. Cvitanić, J., Wan, X., Zhang, J. (2008) "Continuous-Time Principal-Agent Problems with Hidden Action and Lump-Sum Payment," *Applied Mathematics and Optimization*, Vol. 59, 99-146.
5. DeMarzo, P.M., Sannikov, Y. (2006) "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *Journal of Finance*, Vol. 61, 2681-2724.
6. Fong, K.G. (2009) "Evaluating Skilled Experts: Optimal Scoring Rules for Surgeons," Working Paper.
7. Giat, Y., S.T. Hackman, Subramanian, A. (2010) "Investment under Uncertainty, Heterogeneous Beliefs and Agency Conflicts." *The Review of Financial Studies* Vol. 23, 1360-1404.
8. Green, E. (1987) "Lending and the Smoothing of Uninsurable Income," in E. Prescott and N. Wallace, Eds., *Contractual Arrangements for Intertemporal Trade*, Minneapolis: University of Minnesota Press.
9. He, Z. (2009) "Optimal Executive Compensation When Firm Size Follows Geometric Brownian Motion," *Review of Financial Studies*, Vol. 22, 859-892.
10. Holmstrom, B., Milgrom, P. (1987) "Aggregation and Linearity in the Provision of Intertemporal Incentives," *Econometrica*, Vol. 55, 303-328.
11. Laffont, J.J. and Martimort, D. (2002) "The Theory of Incentives: The Principal-agent Model," Princeton University Press.
12. Lions, P.L. (1983) "Optimal Control of Diffusion Processes and Hamilton Jacobi Equations, Part II," *Comm. Partial Differential Equations*, Vol. 8, 1229-1276.
13. Prat, J. Jovanovic, B. "Dynamic incentive contracts under parameter uncertainty". Working paper, NYU.

14. Sannikov, Y. (2007a) "Agency Problems, Screening and Increasing Credit Lines," working paper.
15. Sannikov, Y. (2007b) "Games with Imperfectly Observable Actions in Continuous Time," *Econometrica*, Vol. 75, 1285-1329.
16. Sannikov, Y. (2008) "A Continuous-Time Version of the Principal-Agent Problem." *Review of Economic Studies*, Vol. 75, 957-984.
17. Schättler, H., and Sung, J. (1993) "The First-Order Approach to Continuous-Time Principal-Agent Problem with Exponential Utility." *Journal of Economic Theory* Vol. 61 331–371.
18. Schättler, H., J. Sung. (1997) "On Optimal Sharing Rules in Discrete and Continuous-Times Principal-Agent Problems with Exponential Utility." *Journal of Economic Dynamics and Control* Vol. 21 551–574.
19. Spear, S. and Srivastava, S. (1987) "On Repeated Moral Hazard with Discounting." *Review of Economic Studies*, Vol. 54, 599-617.
20. Sung, J. (1995) "Linearity with Project Selection and Controllable Diffusion Rate in Continuous-Time Principal-Agent Problems," *RAND Journal of Economics*, Vol. 26, 720-743.
21. Sung, J. (1997) "Corporate Insurance and Managerial Incentives." *Journal of Economic Theory* Vol. 74 297–332.
22. Sung, J. (2005) "Optimal Contracts under Adverse Selection and Moral Hazard: A Continuous-Time Approach," *Review of Financial Studies*, Vol. 18, 1021-1073.
23. Wan, X. (2011) "Dynamic Agency, Costly Project Search and Repeated Private Shocks," working paper.
24. Williams, N. (2006) "On Dynamic Principal-Agent Problems in Continuous Time," working paper.
25. Williams, N. (2009) ""Persistent Private Information." working paper.
26. Zhang, Y. (2009) "Dynamic Contracting with Persistent Shocks," *Journal of Economic Theory*, Vol. 144, 635-75.