

# Risk Aversion and Equilibrium Optimal Portfolios in Large Markets\*

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## Abstract

We consider a simple continuous-time economy, populated by a large number of agents, more risk averse than the log agent, with heterogeneous risk aversion densely covering an interval. Even though the dividend is a geometric Brownian motion, the equilibrium investment opportunity set is stochastic and optimal portfolios are highly non-trivial and non-myopic. We present closed form asymptotic expressions for the optimal portfolios when the horizon, or the volatility of terminal dividend becomes large. The non-myopic component of the optimal portfolios is always positive and monotone decreasing in time. For each moment in time, there is a threshold risk aversion such that the non-myopic component is increasing (decreasing) in risk aversion for risk aversion below (above) the threshold. The threshold risk aversion is monotone decreasing in time and approaches the value of two at the terminal horizon. The phenomena we obtain are markedly different from the corresponding results in 2-agent economies.

**Keywords:** optimal portfolio, equilibrium, heterogeneous agents.

**JEL Classification.** D53, G11, G12

# 1 Introduction

It is well known (see, e.g., Merton (1971)) that, when the interest rate and the Sharpe ratio of the stock are constant (or deterministic functions of time), the optimal portfolio is myopic and instantaneously mean-variance efficient, as in the Markowitz model. Things get much more complicated when the Sharpe ratio is stochastic because the optimal portfolio contains a non-myopic component, responsible for hedging against (or, taking advantage of) future fluctuations of the investment opportunity set. There are a few explicitly solvable partial equilibrium models (see, e.g., Kim and Omberg (1996), Wachter (2002), Liu (2007)), but almost nothing is known about non-myopic optimal portfolios in general equilibrium.<sup>1</sup>

In this paper we consider an analogue of the benchmark partial equilibrium dynamic model, a la Merton (1971), with the traders having heterogeneous CRRA preferences, and the dividends following a geometric Brownian motion. We assume that the number of agents is large and their risk aversions densely cover an interval. This is a natural assumption when the market is sufficiently developed. Even though the dividend is a geometric Brownian motion, the equilibrium investment opportunity set is stochastic and optimal portfolios are highly non-trivial and non-myopic.

We present closed form asymptotic expressions for dynamic optimal portfolios when the horizon, or the volatility of terminal dividend becomes large.<sup>2</sup> The non-myopic component of optimal portfolios is always positive and monotone decreasing in time. For each moment in time there is a threshold risk aversion, such that the non-myopic component is increasing (decreasing)

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<sup>1</sup>A somewhat related literature on the so-called portfolio turnpikes studies partial equilibrium optimal portfolio choice when the horizon is large. See, e.g., Dybvig, Rogers and Back (1999) and Guasoni and Robertson (2008).

<sup>2</sup>It follows directly from the scaling properties of the Brownian motion that sending volatility to infinity is the same as sending time to infinity.

in risk aversion for risk aversion below (above) the threshold. The threshold risk aversion is monotone decreasing in time and approaches the value of two at terminal horizon. In particular, the non-myopic component is always increasing in risk aversion for risk aversions below 2.

Using tools from Malliavin calculus (see, for example, Detemple, Garcia and Rindisbacher 2003), we derive expressions for the equilibrium optimal portfolios, as well as the drift and the volatility of the stock price in terms of the risk aversion of the representative agent. Using the fact that the representative agent exhibits decreasing relative risk aversion, we show that the stock volatility is always larger than the volatility of the dividend, giving a new perspective on the volatility puzzle. Furthermore, optimal portfolios are decreasing in risk aversion and the non-myopic (hedging) component is always positive, even in the very long run.

There are several papers that analyze similar risk sharing problems with heterogeneous risk attitudes. Dumas (1989) analyzes numerically a continuous time production economy with two agents having different risk aversions. Wang (1996) studies the equilibrium yield curve in a continuous-time economy with two agents for special values of risk aversions. Basak and Cuoco (1998) consider a restricted participation model with two agents having different risk preferences. Benninga and Mayshar (2000) study option pricing in a one-period economy with heterogeneous CRRA agents. Moreover, models with heterogeneous beliefs and asymmetric information have been studied, as in Basak (2005) and Jouini and Napp (2009) and Biais, Bossaerts and Spatt (2009).

Some of the phenomena we find are completely new and depend on the relative values of the risk aversions of the agents, and hence having many agents in the economy can result in a very different qualitative properties in

equilibrium compared to having two agents only, the case which is usually studied in the literature.

We describe the setup in Section 2, derive a representation and basic properties of optimal portfolios in Section 3, and obtain the closed form asymptotic portfolios in Section 4. Section 5 concludes, while the proofs are presented in Appendix.

## 2 Setup and Notation

### 2.1 The Model

We consider a standard setting similar to that of Wang (1996). The economy has a finite horizon and evolves in continuous time. Uncertainty is described by a one-dimensional, standard Brownian motion  $B_t$ ,  $t \in [0, T]$  on a complete probability space  $(\Omega, \mathcal{F}_T, P)$ , where  $\mathcal{F}$  is the augmented filtration generated by  $B_t$ . There is a single share of a risky asset in the economy, the stock, which pays a terminal dividend

$$D = D_T = e^{\rho T + \sigma B_T}.$$

We also assume that a zero coupon bond with instantaneous constant risk-free rate  $r$  is available in zero net supply.<sup>3</sup>

Let now  $\{\gamma_k, k \in N\}$  be an infinite sequence of real numbers, which is everywhere dense in an interval  $[1, \Gamma]$ .<sup>4</sup> For each fixed  $K$ , let  $G_K = \{\gamma_i, i = 1, \dots, K\}$ .

Consider now an economy  $\mathcal{E}_K$ , populated by  $K$  competitive agents, indexed by their risk aversion  $\gamma \in G_K$ . Agents behave rationally, and are

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<sup>3</sup>The assumption of constant  $r$  is introduced only for simplicity of exposition.

<sup>4</sup>E.g., the set of all rational numbers in that interval. All results directly extend to the case when the lower end of the interval is any  $\gamma \geq 1$ .

heterogeneous in risk preferences. Agent with risk aversion  $\gamma$  is initially endowed with  $\psi_\gamma^K > 0$  shares of stock, and the total supply of the stock is normalized to one,

$$\sum_{\gamma \in G_K} \psi_\gamma^K = 1.$$

Agent  $\gamma$  chooses portfolio strategy,  $\pi_{\gamma t}^K = \pi_{\gamma t}$ <sup>5</sup>, the portfolio weight in the risky asset, as to maximize the CRRA expected utility

$$E \left[ \frac{W_{\gamma T}^{1-\gamma}}{1-\gamma} \right]$$

of its final wealth  $W_{\gamma T}$ , where the wealth  $W_{\gamma t}$  of agent  $\gamma$  evolves as

$$dW_{\gamma t} = W_{\gamma t}(rdt + \pi_{\gamma t}(S_t^{-1}dS_t - rdt)).$$

Here,  $S_t = S_t^K$  is the equilibrium stock price. The instantaneous drift and volatility of the stock price  $S_t$  are denoted by  $\mu_t^K = \mu_t$  and  $\sigma_t^K = \sigma_t$  respectively,

$$S_t^{-1}dS_t = \mu_t dt + \sigma_t dB_t.$$

We now make the following

**Definition 1.** For an adapted process  $X_t^K$  (such as, e.g.,  $\sigma_t^K$ ,  $\mu_t^K$  or  $\pi_{\gamma t}^K$ ) and any fixed  $\lambda \in (0, 1)$ , we define the processes

$$X_{\sup}^K(\lambda) = \limsup_{T \rightarrow \infty} X_{\lambda T}, \quad X_{\inf}^K(\lambda) = \liminf_{T \rightarrow \infty} X_{\lambda T}$$

and

$$X^K(\lambda) = \lim_{T \rightarrow \infty} X_{\lambda T}$$

if the limit exists.

Clearly, knowing the limit  $X^K(\lambda)$  allows us to uncover the behavior of

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<sup>5</sup>We suppress index  $K$  when it is clear that we are considering a fixed  $K$ .

$X_t^K$  because, for large  $T$ ,

$$X_t^K \approx X^K(t/T).$$

The goal of this paper is to study the behavior of optimal portfolios  $\pi_\gamma(\lambda)$ .

**Remark 1.** Some of our results hold if we allow agents with risk aversion less than one in the economy. In particular, the asymptotic behavior of the drift and the volatility of the stock price, that we compute below, is not influenced by those agents. However, we have not been able to compute explicitly the optimal portfolios for such agents, so we decided to exclude them, for simplicity of exposition.

## 2.2 Equilibrium

**Definition 2.** *We say that the market is in equilibrium if the agents behave optimally and both the risky asset market and the risk-free market clear.*

It is well known that the above financial market is complete, if the volatility process  $\sigma_t$  of the stock price is almost everywhere strictly positive.<sup>6</sup> When the market is complete, there exists a unique stochastic discount factor (SDF)  $M = M_T$  such that the stock price is given by

$$S_t = e^{r(t-T)} \frac{E_t[MD]}{E_t[M]}.$$

Because of the market completeness, equilibrium allocation is Pareto-efficient and can be characterized as an Arrow-Debreu equilibrium. See, e.g. Duffie (1986), Wang (1996). Because the endowments are co-linear (all agents hold shares of the same single stock), the equilibrium is in fact unique, up to a multiplicative factor, and unique if we fix the risk-free rate. See, e.g., Dana (1995), Dana (2001).<sup>7</sup>

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<sup>6</sup>This follows also from Proposition 4 below.

<sup>7</sup>Since the endowment is neither bounded away from zero nor from infinity, some

It is well known (see, for example, Cvitanić and Zapatero (2004)) that in this complete market setting the optimal terminal wealth is of the form

$$W_{\gamma T} = (\lambda_{\gamma} M)^{-b}$$

where

$$b = \gamma^{-1}$$

is the relative risk tolerance of agent  $k$ , and  $\lambda_{\gamma}$  is determined via the *budget constraint*

$$E[(\lambda_{\gamma} M)^{-b} M] = W_{\gamma 0} = \psi_{\gamma} S_1 = \psi_{\gamma} E[DM].$$

We formalize this in

**Proposition 1.** *The equilibrium allocation is given by*

$$W_{\gamma T} = \frac{\psi_{\gamma} E[DM]}{E[M^{1-b}]} M^{-b}$$

and equilibrium SDF  $M$  solves the equation

$$\sum_{\gamma \in G_K} \frac{\psi_{\gamma} E[DM]}{E[M^{1-b}]} M^{-b} = D. \quad (1)$$

### 3 Bounds and Representation for Portfolio Strategies

To prove the above results, we will use the notion of a representative agent. Since the market is complete, it is well known (see, e.g., Cvitanić and Zapatero (2004)) that the prices in our heterogeneous economy coincide with those in an artificial economy, populated by a single, representative agent with 

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additional care is needed to show the existence of equilibrium. See, e.g., Dana (2001) and Malamud (2008).

a utility function  $U$ , and the equilibrium stochastic discount factor equals the marginal utility of the representative agent, evaluated at the aggregate endowment,

$$M = U'(D). \quad (2)$$

That is, the function  $U'(x)$  is the unique solution to the equation

$$\sum_{\gamma \in G_K} \psi_\gamma E[DM] E[M^{1-b}]^{-1} (U'(x))^{-b} = x. \quad (3)$$

Let

$$\gamma^U(x) = -\frac{x U''(x)}{U'(x)}$$

be the relative risk aversion of the representative agent.

**Proposition 2.** *The relative risk aversion  $\gamma^U(x)$  is monotone decreasing in  $x$  and satisfies*

$$\max_{G_K} \gamma = \lim_{x \rightarrow +0} \gamma^U(x) \geq \gamma^U(x) \geq \lim_{x \rightarrow \infty} \gamma^U(x) = \min_{G_K} \gamma \quad (4)$$

In the case of a one period economy, Proposition 2 was proved by Benninga and Mayshar (2000). The proof for our continuous-time economy is analogous to theirs, and we present it in Appendix for the reader's convenience.

It turns out that the stock volatility and the optimal portfolio weights can be expressed in terms of conditional expected values involving  $M$ ,  $D$  and  $\gamma^U(D)$ , as follows:

**Proposition 3.** *The drift and volatility of the stock price are given by*

$$\begin{aligned} \mu_t &= r + \sigma \frac{E_t[M\gamma^U(D)]}{E_t[M]} \sigma_t \\ \sigma_t &= \sigma \left( 1 - \frac{E_t[MD\gamma^U(D)]}{E_t[MD]} + \frac{E_t[M\gamma^U(D)]}{E_t[M]} \right) \end{aligned} \quad (5)$$

and the optimal portfolio of agent  $\gamma$  is given by

$$\pi_{\gamma t} = \left( (b-1) \frac{E_t[M^{1-b}\gamma^U(D)]}{E_t[M^{1-b}]} + \frac{E_t[M\gamma^U(D)]}{E_t[M]} \right) \sigma_t^{-1}$$

where  $b = \gamma^{-1}$ .

Note that the optimal portfolio of an agent with risk aversion 1 (the agent with log utility) is always myopic and is given by

$$\pi_{1t} = \frac{\mu_t - r}{\sigma_t^2}.$$

The next proposition provides bounds for the price drift and volatility and, consequently, for the log optimal portfolio  $\pi_{1t}$ .

**Proposition 4.** *The price volatility is always larger than the dividend volatility, and bounded from above as follows, for all  $t$ :*

$$\sigma \leq \sigma_t \leq \sigma \left( 1 + \max_{G_K} \gamma - \min_{G_K} \gamma \right). \quad (6)$$

The instantaneous Sharpe ratio satisfies

$$\sigma \min_{G_K} \gamma \leq \frac{\mu_t - r}{\sigma_t} \leq \sigma \max_{G_K} \gamma \quad (7)$$

and therefore

$$\frac{\min_{G_K} \gamma}{1 + \max_{G_K} \gamma - \min_{G_K} \gamma} \leq \pi_{1t} = \frac{\mu_t - r}{\sigma_t^2} \leq \max_{G_K} \gamma.$$

If  $\max_{G_K} \gamma - \min_{G_K} \gamma > 0$ , then all inequalities are strict.

The bound (7) has a very clear interpretation. In the “individual” economy of agent  $\gamma$ , the “individual” equilibrium Sharpe ratio is given by  $\sigma \gamma$ . Proposition 4 shows that, when risk aversions are heterogeneous, the Sharpe ratio stays between the minimal and the maximal individual Sharpe ratios. As we will see below (Theorem 1), both bounds (6) and (7) are asymptotically

sharp.

The economic mechanism responsible for bound (6) is the fact that the representative agent exhibits decreasing relative risk aversion. This property forces the equilibrium stock price to respond to aggregate fluctuations in a highly non-linear way, driving the volatility up. Note also that bound (6) has direct implications for the well known volatility puzzle, that is, that, empirically, the volatility of the stock prices is significantly higher than the volatility of the dividends. We see that in our economy *heterogeneous risk aversion drives the price volatility up*. The size of the ratio  $\sigma_t/\sigma$  is then determined by the size  $\max_{G_K} \gamma - \min_{G_K} \gamma$  of heterogeneity.

Recall that, when both drift and volatility of  $S_t$  are constant, the optimal partial equilibrium portfolio of any CRRA agent is myopic (i.e., it only depends on instantaneous stock parameters), and is given by

$$\pi_{\gamma t}^{\text{myopic}} = \frac{\mu_t - r}{\gamma \sigma_t^2} = \gamma^{-1} \pi_{1t}.$$

Since the equilibrium Sharpe ratio is stochastic, optimal portfolios may contain a non-myopic component. The following proposition provides some basic properties of the optimal portfolios.

**Proposition 5.** *Optimal portfolio  $\pi_{\gamma t}$  is monotone decreasing in  $\gamma$  and satisfies*

$$\pi_{1t} > \pi_{\gamma t} > \pi_{\gamma t}^{\text{myopic}}$$

The fact that the stock holding is monotone decreasing in risk aversion is intuitively clear. Interestingly enough, the proof of this result is non-trivial and we do not know whether this monotonicity also holds when there are agents in the economy whose risk aversions are below one.

We can always decompose

$$\pi_{\gamma t} = \pi_{\gamma t}^{\text{myopic}} + \pi_{\gamma t}^{\text{hedging}},$$

where  $\pi_{\gamma t}^{\text{hedging}}$  is the hedging component, arising because the investment opportunity set is stochastic. Proposition 5 shows that the hedging component  $\pi_{\gamma}^{\text{hedging}}$  is always positive for risk aversion above one.<sup>8</sup> Furthermore, by Proposition 4,  $\pi_{1t} > 0$  and therefore Proposition 5 implies that there is no short-selling by any agent and moreover

$$\pi_{\gamma t} \leq \max_{G_K} \gamma.$$

## 4 The Infinite Population Limit

### 4.1 Drift and Volatility of the Stock Price

We start with the following result. Let

$$\begin{aligned} \mathcal{X} &= \{(\gamma_1, \dots, \gamma_K) \in \mathbb{R}^K \mid \exists \{i, j, k, l\} : \gamma_i + \gamma_j = 2(1 + \gamma_k(1 - \gamma_l^{-1}))\} \\ \mathcal{Y} &= \{(\gamma_1, \dots, \gamma_K) \in \mathbb{R}^K \mid \exists \{i, j, k\} : \gamma_i + \gamma_j = 2(1 + \gamma_k)\}. \end{aligned} \tag{8}$$

Clearly,  $\mathcal{X}$ ,  $\mathcal{Y}$  are finite unions of smooth hyper-surfaces and therefore have measure zero.

**Proposition 6.** *For each fixed  $K$ , we have:*

- Let  $G_K = \{\gamma_1, \dots, \gamma_K\}$  with  $(\gamma_1, \dots, \gamma_K) \in \mathbb{R}^K \setminus \mathcal{Y}$ . There exists a set  $C_K \subset (0, 1)$ ,  $|C_K| = K - 1$  such that the limits  $\sigma^K(\lambda)$ ,  $\mu^K(\lambda)$  exist for all  $\lambda \in (0, 1) \setminus C_K$  and are deterministic functions of  $\lambda$ ;
- Let  $G_K = \{\gamma_1, \dots, \gamma_K\}$  with  $(\gamma_1, \dots, \gamma_K) \in \mathbb{R}^K \setminus \mathcal{X}$ . Then, for each  $\gamma \in G_K$ , there exists a set  $D_{K,\gamma} \subset (0, 1)$ ,  $|D_{K,\gamma}| \leq 2(K - 1)$  such that the limit  $\pi_{\gamma}^K(\lambda)$  exists for all  $\lambda \in (0, 1) \setminus D_{K,\gamma}$  and is a deterministic function of  $\lambda$ .

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<sup>8</sup>Since the non-myopic component is always nonnegative, the agent is actually taking advantage of the fluctuations, rather than hedging against them, but we still keep the name ‘‘hedging component’’.

The limit functions  $X^K(\lambda) = \sigma^K(\lambda)$ ,  $\mu^K(\lambda)$  and  $\pi_\gamma^K(\lambda)$  are provided explicitly in Appendix. Naturally, they are model dependent: optimal portfolio of each agent depends on risk aversions of all other agents in the economy. The main result of this paper is that, when the number of agents is sufficiently large, the optimal portfolios become model independent. Furthermore, both  $X_{\text{sup}}^K$  and  $X_{\text{inf}}^K$  converge uniformly to the same limit as  $K \rightarrow \infty$  (recall definition (1)). In particular, the gap  $X_{\text{sup}}^K - X_{\text{inf}}^K$  converges to zero as  $K \rightarrow \infty$ , and not only for generic values. We will need the following

**Definition 3.** *Given a function  $X(\lambda)$ , we say that  $(X_{\text{sup}}^K, X_{\text{inf}}^K)(\lambda)$  uniformly converges to a function  $X(\lambda)$  as  $K \rightarrow \infty$  if*

$$\lim_{K \rightarrow \infty} \sup_{\lambda \in (0,1)} (|X_{\text{sup}}^K(\lambda) - X(\lambda)| + |X_{\text{inf}}^K(\lambda) - X(\lambda)|) = 0.$$

*Similarly, if  $X^K(\gamma, \lambda)$  depends on  $\gamma$ , we say that  $X^K(\gamma, \lambda)$  uniformly converges to a function  $X(\gamma, \lambda)$  as  $K \rightarrow \infty$  if*

$$\lim_{K \rightarrow \infty} \sup_{\gamma \in G_K} \sup_{\lambda \in (0,1)} (|X_{\text{sup}}^K(\gamma, \lambda) - X(\lambda)| + |X_{\text{inf}}^K(\lambda) - X(\lambda)|) = 0.$$

We first obtain the asymptotic behavior of the drift and volatility of the stock price.

**Theorem 1.** *When  $K \rightarrow \infty$ , the asymptotic drift and volatility  $(\mu_{\text{sup}}^K, \mu_{\text{inf}}^K)(\lambda)$  and  $(\sigma_{\text{sup}}^K, \sigma_{\text{inf}}^K)(\lambda)$  uniformly converge to*

$$\mu(\lambda) = \begin{cases} r + (1 + \lambda^{-1})^2 \sigma^2, & \lambda \geq (\Gamma - 1)^{-1} \\ r + \Gamma^2 \sigma^2, & \lambda < (\Gamma - 1)^{-1} \end{cases}$$

and (9)

$$\sigma(\lambda) = \begin{cases} \sigma(1 + \lambda^{-1}), & \lambda \geq (\Gamma - 1)^{-1} \\ \sigma \Gamma, & \lambda < (\Gamma - 1)^{-1} \end{cases}$$

respectively. Consequently, the myopic component  $\pi_\gamma^{K, \text{myopic}}$  converges to

$$\pi_\gamma^{\text{myopic}}(\lambda) = \gamma^{-1}$$

as  $K \rightarrow \infty$ .

The following result is a direct consequence of Theorem 1.

**Corollary 1.** *In the limit  $T \rightarrow \infty$ , the instantaneous drift, the volatility and the Sharpe ratio of the stock are monotone decreasing in  $t = \lambda T$ .*

The fact that the drift, volatility and the Sharpe ratio are decreasing is not surprising. It is a well known conventional wisdom (see, e.g., Rubinstein (1991), Blume and Easley (1992), Evstigneev et al (2006)) that the agent, maximizing logarithmic utility of terminal wealth will have the highest wealth growth rate and dominate the whole economy in the long run. It is possible to show that this is also true in our model.<sup>9</sup> By Proposition 3, the size of drift and volatility are determined by the representative agent's risk aversion  $\gamma^U(x)$ . As time goes by, the contribution to  $\gamma^U$  of the agents with risk aversion far away from 1 becomes smaller and the average risk aversion decreases and drives the stock characteristics down.

A surprising consequence of Theorem 1 is that, even though  $\sigma_t$  and  $\mu_t$  decrease in  $\lambda = t/T$ , they never converge to those that would appear in the economy populated solely by the log agent. In fact, in such a homogeneous log economy, we would have  $\mu_t = r + \sigma^2$  and  $\sigma_t = \sigma$ . But, by Theorem 1, as  $\lambda \rightarrow 1$ , that is, as  $t \rightarrow T$ , we have  $\sigma(\lambda) \rightarrow 2\sigma$  and  $\mu(\lambda) \rightarrow r + 4\sigma^2$ . These are drift and volatility that would appear in equilibrium in a homogeneous economy populated by a single agent with risk aversion two.<sup>10</sup> The intuition

<sup>9</sup>See, Cvitanic and Malamud (2009).

<sup>10</sup>This is very different from the results of Kogan, Ross, Wang and Westerfield (2006), who find that drift and volatility always coincide with those determined by the surviving agent when  $\lambda$  is sufficiently close to 1.

behind this interesting phenomenon is that, for each finite  $T$ , there are small probability events in which the log agent does not dominate the economy. In those events, the price behavior is determined by agents with higher risk aversion and this pushes equilibrium drift and volatility up.

## 4.2 Optimal Portfolios

We are now ready to formulate the main result of this paper.

**Theorem 2.** *We have*

*if  $\lambda > (\Gamma - 1)^{-1}$  then*

$$\pi_\gamma(\lambda) = \gamma^{-1} + \frac{\gamma - 1}{(\lambda + 1)\gamma(1 + \lambda(\gamma - 1))} ;$$

*if  $\lambda < (\Gamma - 1)^{-1}$  then*

$$\pi_\gamma(\lambda) = \gamma^{-1} + (\gamma - 1) \frac{(\Gamma - 1)(1 + \lambda(\gamma - 1)) - (\gamma - 1)}{\Gamma\gamma(1 + \lambda(\gamma - 1))}.$$

The result of Theorem 2 is somewhat surprising. Indeed, Theorem 1 suggests that, for sufficiently large  $T$ , the market price of risk  $(\mu_t - r)/\sigma_t$  can be approximated by

$$\frac{\mu_t - r}{\sigma_t} \approx \frac{\mu(t/T) - r}{\sigma(t/T)}.$$

That is, asymptotically, the market price of risk is a deterministic function of  $t$  and therefore, the asymptotic optimal portfolio should be myopic, whereas, by Theorems 1 and 2, the non-myopic component

$$\pi_\gamma^{\text{hedging}}(\lambda) = \pi_\gamma(\lambda) - \gamma^{-1}$$

of the optimal portfolio is always strictly positive. The reason is that, for fixed (even very large)  $T$ , large deviations from the behavior predicted by Theorem 1 still occur. Even though the probability of such deviations is very small,

they might have dramatic impact on the behavior of prices, especially in those states where aggregate consumption is low and so the marginal utility of all agents is very high. In fact, in our economy, the agents are taking advantage of these deviations, by investing an additional fraction of their wealth into stock. This phenomenon is related to the well-known “peso problem”: the possibility of disastrous events (even if the probability of those events is very low) may have very strong effects on the equilibrium outcomes, such as, for example, the equity premium. See, e.g., Danthine and Donaldson (2002).

The bounds, established in the appendix, imply that the size of the deviation

$$|\pi_{\gamma t} - \pi_{\gamma}(t/T)|$$

is controlled by the size of  $|B_t/T|$ . By the strong law of large numbers for the Brownian motion,  $B_t/T \rightarrow$  almost surely as  $T \rightarrow \infty$  for  $t = \lambda T$ . Thus, we can interpret Theorem 2 as a law of large numbers for the optimal portfolio. On average,  $B_{\lambda T}/T \approx 0$  and therefore,

$$\pi_{\gamma t} \approx \pi_{\gamma}(t/T).$$

However, for each finite  $T$ , there will be deviations from the law of large numbers, that are, to the main order, normally distributed, just like there are deviations from the standard law of large numbers, controlled by the central limit theorem.

We summarize the basic monotonicity properties of the optimal portfolios in

**Corollary 2.** *Let  $\lambda > (\Gamma - 1)^{-1}$ . Then,*

- *the hedging portfolio*

$$\pi_{\gamma}^{\text{hedging}}(\lambda)$$

*is monotone decreasing in  $\lambda$  for each fixed  $\gamma$ ;*

- for each fixed  $\lambda$ ,  $\pi_\gamma^{\text{hedging}}(\lambda)$  is monotone increasing in  $\gamma$  for

$$\gamma < 1 + \lambda^{-1/2}$$

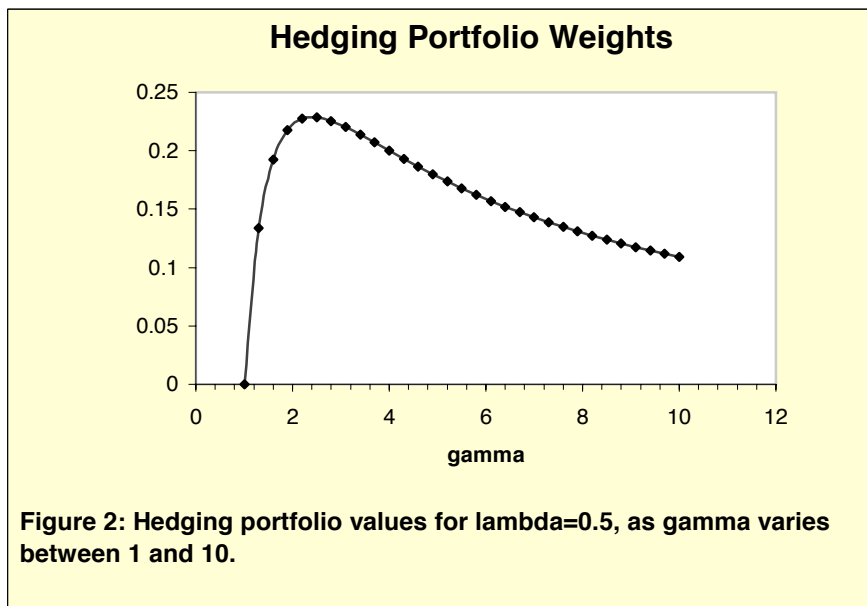
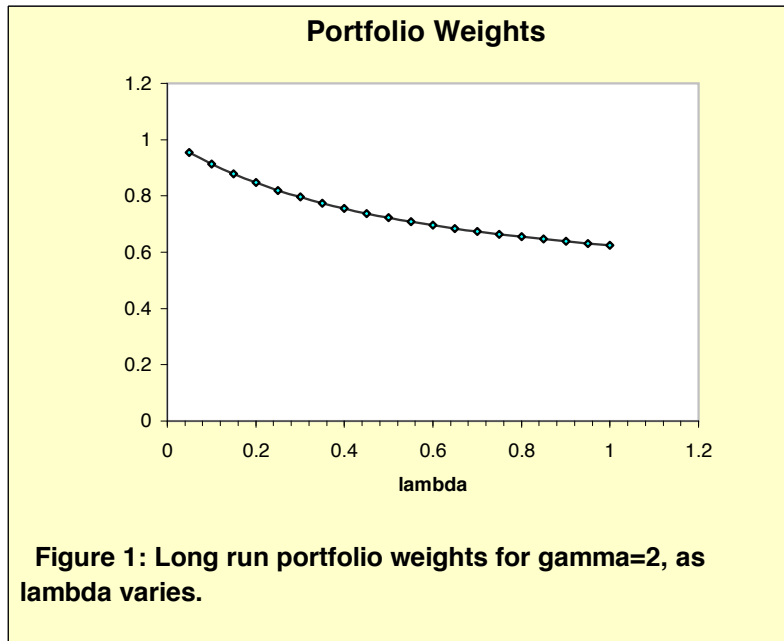
and is monotone decreasing for  $\gamma > 1 + \lambda^{-1/2}$ .

Note that even though, for  $\lambda > (\Gamma - 1)^{-1}$  the hedging component is decreasing in  $\lambda$ , it does not vanish as  $\lambda$  converges to one - for agents with  $\gamma > 1$  there is still positive hedging even in the very long run, in which the log-agent dominates. This is related to the fact that  $\mu_t$  and  $\sigma_t$  also do not converge to the value corresponding to the economy populated by the log agent only.

**Remark 2.** The behavior is different for  $\lambda < (\Gamma - 1)^{-1}$ . In that case, in contrast to Corollary 2, the hedging component is monotone increasing in  $\lambda = t/T$ . For each fixed  $\lambda < (\Gamma - 1)^{-1}$ , the behavior is similar to that of Corollary 2: the hedging component is increasing in risk aversion for risk aversions below a certain threshold, depending on  $\Gamma$  and decreasing afterwards.

Note, however that the quantity  $\Gamma$  is difficult to interpret. In fact, it is not unreasonable to assume that there are agents with arbitrarily large risk aversion in the economy, but their mass  $\sum_{\gamma > \Gamma} \psi(\gamma)$  is very small. In this sense, the case  $\lambda < (\Gamma - 1)^{-1}$  of small  $\lambda$  is not very interesting.

We illustrate the above results by graphing the portfolio weights in Figures 1 and 2.



## 5 Conclusions

We show that even a simple equilibrium model with heterogeneous agents is able to produce highly non-trivial portfolio strategies. The equilibrium

drift and the volatility of the price change as time goes by, leading to the portfolio weights which are not myopic, unlike in the partial equilibrium Merton-Black-Scholes model. When the horizon is large, the non-myopic (hedging) component is an explicitly given, deterministic function of time and risk aversion, which does not depend on *any parameters of the model*. The hedging component is strictly positive for risk aversions higher than one.

Surprisingly, at the times closest to the final time horizon, at which the log agent dominates, the drift and volatility are determined by the agent with risk aversion two. The size of the price volatility is influenced by the overall dividend volatility and the difference between the highest and the lowest risk aversion.

The main novel message coming out of these results is that the equilibrium with many agents is qualitatively different from the equilibrium with two (or a small number of) agents only, the most represented case in the existing literature. Thus, one has to be careful about drawing conclusions from the two-agent case.

It would be of significant interest to extend our framework to intermediate consumption and also heterogeneous beliefs, which are the subjects of our ongoing research.

## Appendix

### A Bounds and Representation of Optimal Portfolios, Drift and Volatility

*Proof of Proposition 2.* Let

$$z_\gamma = \psi_\gamma E[DM] E[M^{1-b}]^{-1}.$$

Differentiating (3), we get

$$U''(D) = -\frac{1}{\sum_{\gamma \in G_K} z_\gamma b M^{-b-1}} \Rightarrow \gamma^U(D) = \frac{D}{\sum_{\gamma \in G_K} z_\gamma b M^{-b}}$$

and therefore, using

$$\sum_{\gamma \in G_K} z_\gamma M^{-b} = D,$$

we get

$$\begin{aligned} -\frac{d}{dD} \gamma^U(D) &= -\frac{1}{\sum_{\gamma \in G_K} z_\gamma b M^{-b}} + \frac{D \sum_{\gamma \in G_K} z_\gamma b^2 M^{-b}}{\left(\sum_{\gamma \in G_K} z_\gamma b M^{-b}\right)^3} \\ &= \left(\sum_{\gamma \in G_K} z_\gamma b M^{-b}\right)^{-3} \left(\sum_{\gamma \in G_K} z_\gamma b^2 M^{-b} \sum_{\gamma \in G_K} z_\gamma M^{-b} - \left(\sum_{\gamma \in G_K} z_\gamma b M^{-b}\right)^2\right). \end{aligned} \quad (10)$$

Applying the Cauchy-Schwarz inequality

$$\sum_{\gamma \in G_K} x_\gamma \sum_{\gamma \in G_K} y_\gamma \geq \left(\sum_{\gamma \in G_K} (x_\gamma y_\gamma)^{1/2}\right)^2$$

to the numbers

$$x_\gamma = z_\gamma b^2 M^{-b}, \quad y_\gamma = z_\gamma M^{-b}$$

we get the first statement. The second is straightforward to check. Q.E.D.

*Proof of Proposition 3.* Recall that price  $S_t$  and the wealth of agent  $k$  satisfy

$$\log S_t = \log E_t[MD] - \log E_t[M]$$

$$\log W_{\gamma t} = \log \left( \frac{\psi_\gamma E[DM]}{E[M^{1-b}]} E_t[M^{1-b}] \right) - \log E_t[M].$$

We get the volatility  $\sigma_t$  as the Malliavin derivative  $\mathcal{D}_t \log S_t$  and we get  $\sigma_t \pi_{\gamma t}$

as the Malliavin derivative  $\mathcal{D}_t \log W_{\gamma t}$ .<sup>11</sup> Thus, we have

$$\pi_{\gamma t} = \frac{\mathcal{D}_t \log W_{\gamma t}}{\mathcal{D}_t \log S_t}. \quad (11)$$

We will now calculate the Malliavin derivatives. Since  $D = e^{\rho T + \sigma B_T}$ , we have

$$\mathcal{D}_t D = \sigma D$$

and therefore, using (2)-(3), we get

$$\mathcal{D}_t M = U''(D) \mathcal{D}_t D = \sigma D U''(D).$$

Using the identity

$$\mathcal{D}_t E_t[X] = E_t[\mathcal{D}_t X]$$

we can compute

$$\mathcal{D}_t \log W_{\gamma t} = \frac{1-b}{E_t[M^{1-b}]} E_t[M^{-b} \mathcal{D}_t M] - \frac{1}{E_t[M]} E_t[\mathcal{D}_t M]$$

and

$$\begin{aligned} \mathcal{D}_t \log S_t &= \frac{1}{E_t[MD]} E_t[DD_t M + \sigma DM] - \frac{1}{E_t[M]} E_t[\mathcal{D}_t M] \\ &= \sigma + \frac{E_t[DD_t M]}{E_t[DM]} - \frac{E_t[\mathcal{D}_t M]}{E_t[M]}. \end{aligned} \quad (12)$$

It remains to show the expression for the drift. By the above,

$$\frac{dE_t[MD]}{E_t[MD]} = U_t dW_t, \quad \frac{dE_t[M]}{E_t[M]} = V_t dW_t$$

where, by (12),

$$U_t = \frac{\mathcal{D}_t E_t[MD]}{E_t[MD]} = \sigma \left( 1 - \frac{E_t[\gamma^U(D) DM]}{E_t[DM]} \right)$$

---

<sup>11</sup>For an expedient introduction to Malliavin derivatives see Detemple, Garcia and Rindisbacher (2003).

and

$$V_t = \frac{\mathcal{D}_t E_t[M]}{E_t[M]} = -\sigma \frac{E_t[\gamma^U(D)M]}{E_t[M]}.$$

Applying Ito's formula, we get

$$d \log S_t = r dt + d \log \frac{E_t[MD]}{E_t[M]} = \frac{1}{2}(2r + V_t^2 - U_t^2)dt + (U_t - V_t)dW_t.$$

Therefore,

$$\mu_t = r + \frac{1}{2}(V_t^2 - U_t^2 + (U_t - V_t)^2) = r + V_t(V_t - U_t)$$

and the claim follows. Q.E.D.

The following result is well known.

**Lemma 1.** *If both  $g(x)$  and  $h(x)$  are increasing (or both decreasing) then*

$$E[g(Z)]E[h(Z)] \leq E[g(Z)h(Z)].$$

*If both  $g, h$  are strictly increasing (or both strictly decreasing), then the inequality is also strict unless  $Z$  is constant almost surely. If one function is increasing and the other is decreasing, then the inequality reverses.*

**Lemma 2.** *We have*

$$\frac{E_t[M\gamma^U(D)]}{E_t[M]} \geq \frac{E_t[MD\gamma^U(D)]}{E_t[MD]}.$$

*Furthermore,*

$$\frac{E_t[M^{1-x}\gamma^U(D)]}{E_t[M^{1-x}]}$$

*is monotone decreasing in  $x$ .*

*Proof of Lemma 2.* Introduce the new measure

$$dQ = \frac{M}{E[M]}dP.$$

Then,

$$\frac{E_t[M\gamma^U(D)]}{E_t[M]} = E_t^Q[\gamma^U(D)], \quad \frac{E_t[MD\gamma^U(D)]}{E_t[MD]} = \frac{E_t^Q[D\gamma^U(D)]}{E_t^Q[D]}.$$

Lemma 1 and Proposition 2 imply

$$E_t^Q[\gamma^U(D)] E_t^Q[D] \geq E_t^Q[D\gamma^U(D)]$$

and the claim follows.

Let now  $x > y$  and consider the new measure

$$dQ = \frac{M^{1-x}}{E_t[M^{1-x}]} dP.$$

Then,

$$\frac{E_t[M^{1-x}\gamma^U(D)]}{E_t[M^{1-x}]} = E_t^Q[\gamma^U(D)]$$

and

$$\frac{E_t[M^{1-y}\gamma^U(D)]}{E_t[M^{1-y}]} = \frac{E_t^Q[M^{x-y}\gamma^U(D)]}{E_t^Q[M^{x-y}]}.$$

Since  $M$  is monotone decreasing in  $D$ , the variable  $M^{x-y}$  is also decreasing in  $D$  and the claim follows from Lemma 1 and Proposition 2. Q.E.D.

*Proof of Proposition 4.* The fact that  $\sigma_t \geq \sigma$  follows directly from Lemma 2 and Proposition 3. The remaining inequalities follow from the bounds

$$\min_{G_K} \gamma \leq \gamma^U(D) \leq \max_{G_K} \gamma,$$

established in Proposition 2. Q.E.D.

*Proof of Proposition 5.* We use the representation for optimal portfolios, derived in Proposition 3. If  $\gamma_1 > \gamma_2 > 1$ , then  $b_1 < b_2 < 1$  and Lemma 2 implies that

$$(1 - b_1) \frac{E_t[M^{1-b_1}\gamma^U(D)]}{E_t[M^{1-b_1}]} \geq (1 - b_2) \frac{E_t[M^{1-b_2}\gamma^U(D)]}{E_t[M^{1-b_2}]}$$

that is  $\pi_{\gamma,t}$  is monotone decreasing in  $\gamma$ . Similarly, by Lemma 2,

$$\frac{E_t[M^{1-b}\gamma^U(D)]}{E_t[M^{1-b}]} < \frac{E_t[M\gamma^U(D)]}{E_t[M]}$$

and therefore

$$(b-1)\frac{E_t[M^{1-b}\gamma^U(D)]}{E_t[M^{1-b}]} + \frac{E_t[M\gamma^U(D)]}{E_t[M]} \geq b\frac{E_t[M\gamma^U(D)]}{E_t[M]}$$

since  $\gamma > 1$ , which is what had to be proved.

Q.E.D.

## B Technical Lemmas

Everywhere in the sequel we use

$$\gamma_1 < \cdots < \gamma_K$$

to denote the elements of the set  $G_K$  of risk aversions, reordered to be increasing. Similarly, we will use the notation

$$b_i = \gamma_i^{-1} \quad \text{and} \quad \psi_i = \psi_{\gamma_i}.$$

**Lemma 3.** *Let  $\Gamma \geq 1$  be such that  $\Gamma b_i > 1$  for all  $i$  and  $\gamma \leq 1$  be such that  $\gamma b_i \leq 1$  for all  $i$ . Then,*

$$\begin{aligned} & \left( \sum_i D^{-\gamma_i/\gamma} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i/\gamma} \right)^\gamma \\ & \leq M \leq \left( \sum_i D^{-\gamma_i/\Gamma} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i/\Gamma} \right)^\Gamma. \end{aligned} \quad (13)$$

*Proof.* Let

$$z_i = \psi_i E[DM]/E[M^{1-b_i}].$$

Then, the equilibrium equation is

$$\sum_i z_i M^{-b_i} = D.$$

Suppose that

$$M > \left( \sum_i D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma} \right)^\Gamma.$$

Then,

$$\begin{aligned} \sum_i z_i M^{-b_i} D^{-1} &< \sum_i z_i \left( \sum_i D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma} \right)^{-\Gamma/\gamma_i} \\ &= \sum_i \left( \frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}} \right)^{\Gamma/\gamma_i}. \end{aligned} \quad (14)$$

Since  $\Gamma > \gamma_i$  for all  $i$ , we get

$$\left( \frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}} \right)^{\Gamma/\gamma_i} < \frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}$$

and therefore

$$\sum_i z_i M^{-b_i} D^{-1} < \sum_i \frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}} = 1$$

which is a contradiction. The estimate from below follows by the same argument. Q.E.D.

**Lemma 4.** *Let  $M$  be the equilibrium SDF. If  $\gamma_i < 1$  then*

$$1 \leq \frac{E[DM]^{1-\gamma_i} E[M^{1-b_i}]^{\gamma_i}}{E[D^{1-\gamma_i}]} \leq \psi_i^{\gamma_i-1}.$$

*If  $\gamma_i > 1$  then*

$$\psi_i^{\gamma_i-1} \leq \frac{E[DM]^{1-\gamma_i} E[M^{1-b_i}]^{\gamma_i}}{E[D^{1-\gamma_i}]} \leq 1.$$

*Proof.* The utility of agent  $i$ 's optimal wealth is given by

$$\begin{aligned} \frac{1}{1-\gamma_i} E[W_{iT}^{1-\gamma_i}] &= \frac{1}{1-\gamma_i} \psi_i^{1-\gamma_i} \left( \frac{E[DM]}{E[M^{1-b_i}]} \right)^{1-\gamma_i} E[M^{1-b_i}] \\ &= \frac{1}{1-\gamma_i} \psi_i^{1-\gamma_i} E[DM]^{1-\gamma_i} E[M^{1-b_i}]^{\gamma_i}. \end{aligned} \quad (15)$$

The utility from just consuming its endowment (the terminal dividend of its initial portfolio) is

$$\frac{1}{1-\gamma_i} E[(\psi_i D)^{1-\gamma_i}] = \frac{1}{1-\gamma_i} \psi_i^{1-\gamma_i} E[D^{1-\gamma_i}].$$

Furthermore, by definition, in equilibrium we must have  $W_{iT} \leq D$  and therefore

$$\frac{1}{1-\gamma_i} E[(\psi_i D)^{1-\gamma_i}] \leq \frac{1}{1-\gamma_i} E[W_{iT}^{1-\gamma_i}] \leq \frac{1}{1-\gamma_i} E[D^{1-\gamma_i}].$$

Multiplying both sides by  $1-\gamma_i$  and using (15), we get the result. Q.E.D.

**Lemma 5.** *The function*

$$f_a(x_1, \dots, x_K) = \left( \sum_i x_i^{1/a} \right)^a$$

*is jointly concave in  $(x_1, \dots, x_K)$  if  $a \geq 1$  and jointly convex if  $a < 1$ . Furthermore,  $f_a$  is monotone increasing in  $a$ .*

*Proof.* Let  $a > 1$ . The Hessian  $H(f)$  of the function  $f$  is given by

$$\begin{aligned}
H(f) &= (\partial^2 f / \partial x_i \partial x_j)_{i,j=1}^n \\
&= (1 - a^{-1}) \left( \sum_i x_i^{1/a} \right)^{a-2} \left( \left( x_i^{1/a-1} x_j^{1/a-1} \right)_{i,j} \right. \\
&\quad \left. - \left( \sum_i x_i^{1/a} \right) \text{diag}(x_i^{1/a-2})_i \right) \\
&= (1 - a^{-1}) f^{1-2/a} \text{diag}(x_i^{a^{-1}/2-1})_i \left( \left( x_i^{a^{-1}/2} x_j^{a^{-1}/2} \right)_{i,j} \right. \\
&\quad \left. - f^{1/a} I \right) \text{diag}(x_i^{a^{-1}/2-1})_i.
\end{aligned}$$

Here,  $I$  is the identity matrix. The matrix

$$A = \left( x_i^{a^{-1}/2} x_j^{a^{-1}/2} \right)_{i,j}$$

equals  $f^{1/a}$  times the orthogonal projection onto the vector  $(x_i^{a^{-1}/2})_i$ . Thus,  $\|A\| = f^{1/a}$  and the matrix

$$\left( x_i^{a^{-1}/2} x_j^{a^{-1}/2} \right)_{i,j} - f^{1/a} I$$

is negative definite. Therefore,  $H(f)$  is also negative definite. The case  $a < 1$  is similar.

It remains to prove monotonicity. Let  $a < b$ . Then,

$$\frac{\left( \sum_i x_i^{1/a} \right)^a}{\left( \sum_i x_i^{1/b} \right)^b} = \left( \sum_i \left( \frac{x_i}{\left( \sum_i x_i^{1/b} \right)^b} \right)^{1/a} \right)^a = \left( \sum_i \left( \frac{x_i^{1/b}}{\left( \sum_i x_i^{1/b} \right)} \right)^{b/a} \right)^a.$$

Since

$$\frac{x_i^{1/b}}{\left( \sum_i x_i^{1/b} \right)} < 1$$

for all  $i$  and  $b/a > 1$ , we get

$$\left( \frac{x_i^{1/b}}{\left( \sum_i x_i^{1/b} \right)} \right)^{b/a} < \frac{x_i^{1/b}}{\left( \sum_i x_i^{1/b} \right)}$$

and hence

$$\left( \sum_i \left( \frac{x_i^{1/b}}{\left( \sum_i x_i^{1/b} \right)} \right)^{b/a} \right)^a < \left( \sum_i \frac{x_i^{1/b}}{\left( \sum_i x_i^{1/b} \right)} \right)^a = 1.$$

Q.E.D.

**Lemma 6.** *For any positive random variable  $Z$ , we have*

$$\begin{aligned} \left( \sum_i E_t[Z D^{-\gamma_i}]^{1/\gamma} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i/\gamma} \right)^\gamma &\leq E_t[Z M] \\ &\leq \left( \sum_i E_t[Z D^{-\gamma_i}]^{1/\Gamma} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i/\Gamma} \right)^\Gamma. \end{aligned} \quad (16)$$

*Proof.* Using Lemma 3, we get

$$\begin{aligned} E_t \left[ \left( \sum_i (Z D^{-\gamma_i})^{1/\gamma} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i/\gamma} \right)^\gamma \right] &\leq E_t[Z M] \\ &\leq E_t \left[ \left( \sum_i (Z D^{-\gamma_i})^{1/\Gamma} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i/\Gamma} \right)^\Gamma \right]. \end{aligned} \quad (17)$$

By Lemma 5,

$$\begin{aligned} \left( \sum_i (E_t[X_i])^{1/\gamma} \right)^\gamma &\leq E_t \left[ \left( \sum_i X_i^{1/\gamma} \right)^\gamma \right] \\ &\leq E_t \left[ \left( \sum_i X_i^{1/\Gamma} \right)^\Gamma \right] \leq \left( \sum_i (E_t[X_i])^{1/\Gamma} \right)^\Gamma \end{aligned} \quad (18)$$

for any positive random variables  $X_i$ . Applying (18) to the random variables

$$X_i = Z D^{-\gamma_i} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i},$$

we get the stated result. Q.E.D.

**Lemma 7.** *Let  $t = \lambda T$ ,  $\lambda \in (0, 1)$ . Then, for any random variable  $Z$  such that  $\epsilon \leq Z \leq \epsilon^{-1}$  for some  $\epsilon > 0$ , we have*

$$\lim_{T \rightarrow \infty} \frac{E_t[Z D M]}{(\psi_1 E[DM] E[M^{1-b_1}]^{-1})^{\gamma_1} E_t[Z D^{1-\gamma_1}]} = 1.$$

*Proof.* By (16),

$$\begin{aligned} & \left( \sum_i E_t[Z D^{1-\gamma_i}]^{1/\gamma} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i/\gamma} \right)^\gamma \leq E_t[Z D M] \\ & \leq \left( \sum_i E_t[Z D^{1-\gamma_i}]^{1/\Gamma} (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i/\Gamma} \right)^\Gamma. \end{aligned} \quad (19)$$

Dividing (19) by

$$E_t[Z D^{1-\gamma_1}] (\psi_1 E[DM] E[M^{1-b_1}]^{-1})^{\gamma_1}$$

and observing that

$$\frac{E_t[Z D^{1-\gamma_i}] (\psi_i E[DM]/E[M^{1-b_i}])^{\gamma_i}}{E_t[Z D^{1-\gamma_1}] (\psi_1 E[DM]/E[M^{1-b_1}]^{-1})^{\gamma_1}} = \frac{E_t[Z D^{1-\gamma_i}] E[DM]^{1-\gamma_1} E[M^{1-b_1}]^{\gamma_1}}{E_t[Z D^{1-\gamma_1}] E[DM]^{1-\gamma_i} E[M^{1-b_i}]^{\gamma_i}}$$

we arrive at

$$\begin{aligned} & \left( 1 + \sum_{i \neq 1} \left( \epsilon^2 \frac{\psi_i^{\gamma_i} E_t[D^{1-\gamma_i}] E[DM]^{1-\gamma_1} E[M^{1-b_1}]^{\gamma_1}}{\psi_1^{\gamma_1} E_t[D^{1-\gamma_1}] E[DM]^{1-\gamma_i} E[M^{1-b_i}]^{\gamma_i}} \right)^{1/\gamma} \right)^\gamma \\ & \leq \frac{E_t[Z D M]}{(\psi_1 E[DM] E[M^{1-b_1}]^{-1})^{\gamma_1} E_t[Z D^{1-\gamma_1}]} \\ & \leq \left( 1 + \sum_{i \neq 1} \left( \epsilon^{-2} \frac{\psi_i^{\gamma_i} E_t[D^{1-\gamma_i}] E[DM]^{1-\gamma_1} E[M^{1-b_1}]^{\gamma_1}}{\psi_1^{\gamma_1} E_t[D^{1-\gamma_1}] E[DM]^{1-\gamma_i} E[M^{1-b_i}]^{\gamma_i}} \right)^{1/\Gamma} \right)^\Gamma. \end{aligned} \quad (20)$$

By Lemma 4,

$$K_1 \frac{E[D^{1-\gamma_1}]}{E[D^{1-\gamma_i}]} \leq \frac{E[DM]^{1-\gamma_1} E[M^{1-b_1}]^{\gamma_1}}{E[DM]^{1-\gamma_i} E[M^{1-b_i}]^{\gamma_i}} \leq K_2 \frac{E[D^{1-\gamma_1}]}{E[D^{1-\gamma_i}]}$$

for some  $K_1, K_2 > 0$  and therefore

$$\begin{aligned} & \left( 1 + \tilde{K}_1 \sum_{i \neq 1} \left( \frac{E_t[D^{1-\gamma_i}] E[D^{1-\gamma_1}]}{E_t[D^{1-\gamma_1}] E[D^{1-\gamma_i}]} \right)^{1/\gamma} \right)^\gamma \\ & \leq \frac{E_t[Z D M]}{(\psi_1 E[DM] E[M^{1-b_1}]^{-1})^{\gamma_1} E_t[Z D^{1-\gamma_1}]} \\ & \leq \left( 1 + \tilde{K}_2 \sum_{i \neq 1} \left( \frac{E_t[D^{1-\gamma_i}] E[D^{1-\gamma_1}]}{E_t[D^{1-\gamma_1}] E[D^{1-\gamma_i}]} \right)^{1/\Gamma} \right)^\Gamma. \quad (21) \end{aligned}$$

Furthermore,

$$\frac{E_t[D^{1-\gamma_i}] E[D^{1-\gamma_1}]}{E_t[D^{1-\gamma_1}] E[D^{1-\gamma_i}]} = e^{\sigma(\gamma_1 - \gamma_i)B_t} e^{\frac{1}{2}\sigma^2(|1-\gamma_1|^2 - |1-\gamma_i|^2)t}$$

converges to zero almost surely and the proof is complete. Q.E.D.

Define the intervals

$$\Pi_K = \left( 0, \frac{2}{\gamma_K + \gamma_{K-1}} \right),$$

$$\Pi_1 = \left( \frac{2}{\gamma_1 + \gamma_2}, 1 \right)$$

and

$$\Pi_i = \left( \frac{2}{\gamma_i + \gamma_{i+1}}, \frac{2}{\gamma_{i-1} + \gamma_i} \right)$$

for  $i = 2, \dots, K-1$ .

**Lemma 8.** *Let  $t = \lambda T$ . For any*

$$\lambda \in \Pi_i$$

and any random variable  $Z$  such that  $\epsilon \leq Z \leq \epsilon^{-1}$  for some  $\epsilon > 0$ , we have

$$\lim_{T \rightarrow \infty} \frac{E_t[Z M]}{(\psi_i E[DM] E[M^{1-b_i}]^{-1})^{\gamma_i} E_t[Z D^{-\gamma_i}]} = 1.$$

*Proof.* By (16),

$$\begin{aligned} & \left( \sum_j E_t[Z D^{-\gamma_j}]^{1/\gamma} (\psi_j E[DM]/E[M^{1-b_j}])^{\gamma_j/\gamma} \right)^\gamma \leq E_t[Z M] \\ & \leq \left( \sum_j E_t[Z D^{-\gamma_j}]^{1/\Gamma} (\psi_j E[DM]/E[M^{1-b_j}])^{\gamma_j/\Gamma} \right)^\Gamma. \end{aligned} \quad (22)$$

In complete analogy with (21), we get

$$\begin{aligned} & \left( 1 + \epsilon^2 K_1 \sum_{j \neq i} \left( \frac{E_t[D^{-\gamma_j}] E[D^{1-\gamma_i}]}{E_t[D^{-\gamma_i}] E[D^{1-\gamma_j}]} \right)^{1/\gamma} \right)^\gamma \\ & \leq \frac{E_t[Z M]}{(\psi_i E[DM] E[M^{1-b_i}]^{-1})^{\gamma_i} E_t[Z D^{-\gamma_i}]} \\ & \leq \left( 1 + \epsilon^{-2} K_2 \sum_{j \neq i} \left( \frac{E_t[D^{-\gamma_j}] E[D^{1-\gamma_i}]}{E_t[D^{-\gamma_i}] E[D^{1-\gamma_j}]} \right)^{1/\Gamma} \right)^\Gamma. \end{aligned} \quad (23)$$

However, we have

$$\frac{E_t[D^{-\gamma_j}] E[D^{1-\gamma_i}]}{E_t[D^{-\gamma_i}] E[D^{1-\gamma_j}]} = e^{\sigma(\gamma_i - \gamma_j) B_t} e^{\frac{1}{2} \sigma^2 T ((\gamma_j^2 - \gamma_i^2)(1-\lambda) + (1-\gamma_i)^2 - (1-\gamma_j)^2)}$$

and the claim follows if

$$\gamma_j^2(1-\lambda) - (1-\gamma_j)^2 < \gamma_i^2(1-\lambda) - (1-\gamma_i)^2 \quad (24)$$

for all  $j \neq i$ . Now, the function

$$h(\gamma) = \gamma^2(1-\lambda) - (1-\gamma)^2 = -\lambda\gamma^2 + 2\gamma - 1$$

is a concave parabola, attaining maximum at the vertex  $1/\lambda$ . Therefore, (24) holds if  $\gamma_i$  is the closest to  $1/\lambda$  among all risk aversions. Let  $1 < i < K$ .

Then,

$$\frac{\gamma_{i-1} + \gamma_i}{2} < \frac{1}{\lambda} < \frac{\gamma_i + \gamma_{i+1}}{2}$$

and therefore, since  $\gamma_{i-1} < \gamma_i < \gamma_{i+1}$ , the claim is immediate. The cases  $i = 1, K$  are analogous. Q.E.D.

Denote by  $J(1 - \alpha)$  the set of agents in the economy  $\mathcal{E}_K$ , whose risk aversion is the closest to  $1 - \alpha$ . Clearly,  $J(1 - \alpha)$  may consist of at most two agents. Furthermore, for generic risk aversions,  $J(1 - \alpha)$  is a singleton.

**Lemma 9.** *Let  $t = \lambda T$  for some  $\lambda \in [0, 1)$ . If  $J(1 - \alpha)$  is a singleton, then*

$$\lim_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)D^\alpha]}{E_t[D^\alpha]} = \gamma_{J(1-\alpha)}.$$

In general,

$$\begin{aligned} \min_{l \in J(1-\alpha)} \gamma_l &\leq \liminf_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)D^\alpha]}{E_t[D^\alpha]} \\ &\leq \limsup_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)D^\alpha]}{E_t[D^\alpha]} \leq \max_{l \in J(1-\alpha)} \gamma_l \quad (25) \end{aligned}$$

*Proof.* We have

$$\begin{aligned} &\gamma^U(D) - \gamma_{J(1-\alpha)} \\ &= \frac{D}{\sum_i \psi_i E[DM] E[M^{1-b_i}]^{-1} b_i M^{-b_i}} - \gamma_{J(1-\alpha)} \\ &= \frac{\sum_i \psi_i E[DM] E[M^{1-b_i}]^{-1} M^{-b_i}}{\sum_i \psi_i E[DM] E[M^{1-b_i}]^{-1} b_i M^{-b_i}} - \gamma_{J(1-\alpha)} \frac{\sum_i \psi_i E[DM] E[M^{1-b_i}]^{-1} b_i M^{-b_i}}{\sum_i \psi_i E[DM] E[M^{1-b_i}]^{-1} b_i M^{-b_i}} \\ &= \frac{\sum_{i \neq J(1-\alpha)} \psi_i E[DM] E[M^{1-b_i}]^{-1} (1 - b_i \gamma_{J(1-\alpha)}) M^{-b_i}}{\sum_i \psi_i E[DM] E[M^{1-b_i}]^{-1} b_i M^{-b_i}} \\ &= \frac{\sum_{i \neq J(1-\alpha)} (1 - b_i \gamma_{J(1-\alpha)}) W_{iT}}{\sum_i b_i W_{iT}}. \end{aligned} \tag{26}$$

We have

$$\min_i b_i D \leq \sum_i b_i W_{iT} \leq \max_i b_i D$$

and therefore

$$|\gamma^U(D) - \gamma_{J(1-\alpha)}| \leq K \sum_{i \neq J(1-\alpha)} W_{iT} D^{-1}.$$

Thus, it suffices to show that

$$\frac{E_t[W_{iT} D^{\alpha-1}]}{E_t[D^\alpha]} \rightarrow 0$$

for all  $i \neq J(1-\alpha)$ .

Applying Lemma 4, we get

$$E[DM]^{1-\gamma_i} E[M^{1-b_i}]^{\gamma_i} \geq K E[D^{1-\gamma_i}]$$

that is

$$E[DM] E[M^{1-b_i}]^{-1} \leq E[DM]^{b_i} K^{-b_i} E[D^{1-\gamma_i}]^{-b_i}.$$

Similarly,

$$(\psi_i E[DM] / E[M^{1-b_{J(1-\alpha)}}])^{\gamma_{J(1-\alpha)}} \leq K E[DM] E[D^{1-\gamma_{J(1-\alpha)}}]^{-1}.$$

By Lemma 3,

$$M \geq (\psi_i E[DM] / E[M^{1-b_{J(1-\alpha)}}])^{\gamma_{J(1-\alpha)}} D^{-\gamma_{J(1-\alpha)}}$$

and hence,

$$\begin{aligned} W_{iT} &= \psi_i E[DM] E[M^{1-b_i}]^{-1} M^{-b_i} \\ &\leq \psi_i E[DM] E[M^{1-b_i}]^{-1} ((\psi_i E[DM] / E[M^{1-b_{J(1-\alpha)}}])^{\gamma_{J(1-\alpha)}} D^{-\gamma_{J(1-\alpha)}})^{-b_i} \\ &\leq K \frac{E[D^{1-\gamma_{J(1-\alpha)}}]^{b_i}}{E[D^{1-\gamma_i}]^{b_i}} D^{\gamma_{J(1-\alpha)} b_i}. \end{aligned} \tag{27}$$

Therefore,

$$W_{iT} D^{-1} \leq K e^{\frac{1}{2} \sigma^2 T b_i ((1-\gamma_{J(1-\alpha)})^2 - (1-\gamma_i)^2)} e^{\sigma (\gamma_{J(1-\alpha)} b_i - 1) B_T}.$$

Introduce now a new family of measures

$$dQ_t = \frac{D_t^\alpha dP_t}{E[D_t^\alpha]}$$

of  $\mathcal{F}_t$ . It is not difficult to see that this family of measures is time consistent and hence, by the Kolmogorov theorem (Oksendal (2003), p.11), there is a unique measure  $dQ$  on infinite-horizon paths generating them on finite horizon  $\sigma$ -algebras  $\mathcal{F}_t$ . Under this measure,  $\sigma B_T$  will also be a Brownian motion, but with a drift. Namely,

$$E^Q[e^{\sigma B_T}] = \frac{E[e^{(\alpha+1)\sigma B_T}]}{E[e^{\alpha\sigma B_T}]} = e^{\frac{1}{2}\sigma^2 T(2\alpha+1)} = e^{\alpha\sigma^2 T + \frac{1}{2}\sigma^2 T}$$

and hence it has a drift  $\alpha\sigma^2$ . Thus, under this measure  $\sigma B_T = \alpha\sigma^2 T + \sigma B_T^Q$  and

$$\begin{aligned} & e^{\frac{1}{2}\sigma^2 T b_i((1-\gamma_{J(1-\alpha)})^2 - (1-\gamma_i)^2)} e^{\sigma(\gamma_{J(1-\alpha)} b_i - 1) B_T} \\ &= e^{\frac{1}{2}\sigma^2 T (b_i((1-\gamma_{J(1-\alpha)})^2 - (1-\gamma_i)^2) + 2\alpha(\gamma_{J(1-\alpha)} b_i - 1))} e^{\sigma(\gamma_{J(1-\alpha)} b_i - 1) \sigma B_T^Q}. \end{aligned} \quad (28)$$

The most important observation is that

$$\begin{aligned} & b_i((1-\gamma_{J(1-\alpha)})^2 - (1-\gamma_i)^2) + 2\alpha(\gamma_{J(1-\alpha)} b_i - 1) < 0 \\ & \Leftrightarrow (1-\gamma_{J(1-\alpha)})^2 - (1-\gamma_i)^2 + 2\alpha(\gamma_{J(1-\alpha)} - \gamma_i) < 0. \end{aligned} \quad (29)$$

In fact,

$$\begin{aligned} & (1-\gamma_{J(1-\alpha)})^2 + 2\alpha\gamma_{J(1-\alpha)} = (\gamma_{J(1-\alpha)} - (1-\alpha))^2 - (1-\alpha)^2 + 1 \\ & < (\gamma_i - (1-\alpha))^2 - (1-\alpha)^2 + 1 = (1-\gamma_i)^2 + 2\alpha\gamma_i \end{aligned} \quad (30)$$

which follows from the definition of  $J(1-\alpha)$ .

Let now

$$\epsilon = \frac{1}{2}\sigma^2 (b_i(-(1-\gamma_{J(1-\alpha)})^2 + (1-\gamma_i)^2) - 2\alpha(\gamma_{J(1-\alpha)} b_i - 1)) > 0.$$

Let also  $\gamma_{J(1-\alpha)} b_i > 1$  (the case  $\gamma_{J(1-\alpha)} b_i < 1$  is completely analogous). Then,

since  $W_{iT}D^{-1} \leq 1$ ,

$$\begin{aligned} E_t^Q[W_{iT}D^{-1}] &= E_t^Q[W_{iT}D^{-1} I_{\sigma B_T^Q < \epsilon T / (2(\gamma_{J(1-\alpha)} b_i - 1))}] \\ &+ E_t^Q[W_{iT}D^{-1} I_{\sigma B_T^Q > \epsilon T / (2(\gamma_{J(1-\alpha)} b_i - 1))}] \leq K e^{-\epsilon T/2} + E_t^Q[I_{\sigma B_T^Q > \epsilon T / (2(\gamma_{J(1-\alpha)} b_i - 1))}]. \end{aligned} \quad (31)$$

Let

$$\delta = \sigma^{-1} \epsilon / (2(\gamma_{J(1-\alpha)} b_i - 1)) > 0.$$

Then,

$$\begin{aligned} E_t^Q[I_{\sigma B_T^Q > \epsilon T / (2(\gamma_{J(1-\alpha)} b_i - 1))}] &= \text{Prob}_t^Q[\sigma(B_T^Q - B_t^Q) \geq T\sigma(\delta - B_t^Q/T)] \\ &= (1 - N(\sigma\sqrt{T}(\delta - B_t^Q/(t\lambda^{-1})))/(\sigma\sqrt{1-\lambda})). \end{aligned} \quad (32)$$

By the Law of Large Numbers,  $B_t^Q/t \rightarrow 0$  as  $t \rightarrow \infty$  and therefore

$$\delta - \frac{B_t^Q}{t\lambda^{-1}} \rightarrow \delta > 0$$

almost surely. Hence,

$$\sqrt{T}(\delta - B_t^Q/(t\lambda^{-1})) / (\sigma\sqrt{1-\lambda}) \rightarrow +\infty$$

as  $T \rightarrow \infty$  and

$$N(\sigma\sqrt{T}(\delta - B_t^Q/(t\lambda^{-1})) / (\sigma\sqrt{1-\lambda})) \rightarrow 1.$$

Therefore,

$$E_t[I_{\sigma B_T^Q > \epsilon T / (2(\gamma_{J(1-\alpha)} b_i - 1))}] \rightarrow 0$$

and the claim follows.

Finally, in the case when there are two elements  $\gamma_l$  and  $\gamma_m$  in the set  $J(1-\alpha)$ , define

$$\gamma_T(1-\alpha) = \frac{W_{lT} + W_{mT}}{b_l W_{lT} + b_m W_{mT}}.$$

Then, using the same arguments as above, we can show that

$$\lim_{T \rightarrow \infty} \frac{E_t[(\gamma^U(D) - \gamma_T(1 - \alpha)) D^\alpha]}{E_t[D^\alpha]} = 0$$

almost surely. Clearly,

$$\min\{\gamma_l, \gamma_m\} \leq \gamma_T(1 - \alpha) \leq \max\{\gamma_l, \gamma_m\}$$

and the claim follows. Q.E.D.

## C Proofs of Main Results

### C.1 Drift and Volatility for the Case of Finite $K$

**Proposition 7.** (1) For any  $\lambda \in (0, 1]$ , we have

$$\lim_{T \rightarrow \infty} \frac{E_t[MD\gamma^U(D)]}{E_t[MD]} = \gamma_1$$

(2) For any  $\lambda \in \Pi_j$

$$\begin{aligned} \min_{l \in J(1+\gamma_j)} \gamma_l &\leq \liminf_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)M]}{E_t[M]} \\ &\leq \limsup_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)M]}{E_t[M]} \leq \max_{l \in J(1+\gamma_j)} \gamma_l. \end{aligned} \quad (33)$$

Finally, if  $\lambda = 2/(\gamma_{j-1} + \gamma_j)$ , then

$$\begin{aligned} \min_{l \in J(1+\gamma_j) \cup J(1+\gamma_{j-1})} \gamma_l &\leq \liminf_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)M]}{E_t[M]} \\ &\leq \limsup_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)M]}{E_t[M]} \leq \max_{l \in J(1+\gamma_j) \cup J(1+\gamma_{j-1})} \gamma_l. \end{aligned} \quad (34)$$

*Proof.* Since  $\gamma^U(D)$  is bounded away from zero and infinity, Lemmas 7 and 9 together yield that

$$\frac{E_t[MD\gamma^U(D)]}{E_t[MD]} = \gamma_1.$$

Similarly, for  $\lambda \in \Pi_j$ , Lemmas 8 and 9 imply the first claim of item (2). Finally, if  $\lambda = 2/(\gamma_j + \gamma_{j-1})$ , we define

$$\gamma_T^{(j)} = \frac{\sum_{l \in J(1+\gamma_j) \cup J(1+\gamma_{j-1})} W_{lT}}{\sum_{l \in J(1+\gamma_j) \cup J(1+\gamma_{j-1})} b_l W_{lT}}.$$

Then, using the same arguments as above, we can show that

$$\lim_{T \rightarrow \infty} \frac{E_t[(\gamma^U(D) - \gamma_T^{(j)}) M]}{E_t[M]} = 0$$

almost surely, and the claim follows since

$$\min_{l \in J(1+\gamma_j) \cup J(1+\gamma_{j-1})} \gamma_l \leq \gamma_T^{(j)} \leq \max_{l \in J(1+\gamma_j) \cup J(1+\gamma_{j-1})} \gamma_l.$$

Q.E.D.

Using Propositions 3 and 7, we immediately get

**Proposition 8.** For any  $\lambda \in \Pi_m$ ,

$$\sigma^K(\lambda) = \sigma(1 + \gamma_{J(1+\gamma_m)} - \gamma_1), \quad \mu^K(\lambda) = r + \sigma \sigma^K(\lambda) \gamma_{J(1+\gamma_m)}$$

if the set  $J(1 + \gamma_m)$  is a singleton.

## C.2 Optimal Portfolios for the Case of Finite $K$

**Lemma 10.** Assume  $\gamma_i > 1$ . Let  $\Gamma \geq 1$  be such that  $\Gamma b_k / (1 - b_i) > 1$  for all  $k$  and  $\gamma \leq 1$  be such that  $\gamma b_k / (1 - b_i) \leq 1$  for all  $k$ . Then,

$$\begin{aligned} & \left( \sum_k D^{-\gamma_k(1-b_i)/\gamma} (\psi_k E[DM] / E[M^{1-b_k}])^{\gamma_k(1-b_i)/\gamma} \right)^\gamma \leq M^{1-b_i} \\ & \leq \left( \sum_k D^{-\gamma_k(1-b_i)/\Gamma} (\psi_k E[DM] / E[M^{1-b_k}])^{\gamma_k(1-b_i)/\Gamma} \right)^\Gamma. \end{aligned} \quad (35)$$

*Proof.* The proof is completely analogous to that of Lemma 3 after rewriting

the equilibrium equation as

$$\sum_k z_k (M^{1-b_i})^{-b_k/(1-b_i)} = D$$

with

$$z_k = \psi_k E[DM]/E[M^{1-b_k}].$$

Q.E.D.

Following the same arguments as in the proof of (16), we get

**Lemma 11.** *For any positive random variable  $Z$ , we have*

$$\begin{aligned} & \left( \sum_k E_t [Z D^{-\gamma_k(1-b_i)}]^{1/\gamma} (\psi_k E[DM]/E[M^{1-b_k}])^{(1-b_i)\gamma_k/\gamma} \right)^\gamma \\ & \leq E_t [Z M^{1-b_i}] \\ & \leq \left( \sum_k E_t [Z D^{-\gamma_k(1-b_i)}]^{1/\Gamma} (\psi_k E[DM]/E[M^{1-b_k}])^{(1-b_i)\gamma_k/\Gamma} \right)^\Gamma. \end{aligned} \quad (36)$$

Now, given an  $i \in \{1, \dots, K\}$ , we define the intervals  $\Theta_1^i, \dots, \Theta_i^i$  as follows: we set

$$\Theta_1^i = \left( \frac{2 - (\gamma_1 + \gamma_2)b_i}{(\gamma_1 + \gamma_2)(1 - b_i)}, 1 \right)$$

and, for  $j \in \{1, \dots, i-1\}$ ,

$$\Theta_j^i = \left( \frac{2 - (\gamma_j + \gamma_{j+1})b_i}{(\gamma_j + \gamma_{j+1})(1 - b_i)}, \frac{2 - (\gamma_{j-1} + \gamma_j)b_i}{(\gamma_{j-1} + \gamma_j)(1 - b_i)} \right)$$

and, finally,

$$\Theta_i^i = \left( 0, \frac{2 - (\gamma_{i-1} + \gamma_i)b_i}{(\gamma_{i-1} + \gamma_i)(1 - b_i)} \right).$$

We can now prove with the following

**Lemma 12.** *For any  $\lambda \in \Theta_j^i$ ,*

$$E_t [M^{1-b_i}] \sim (\psi_j E[DM] E[M^{1-b_j}]^{-1})^{\gamma_j(1-b_i)} E_t [D^{-\gamma_j(1-b_i)}]$$

and

$$E_t[M^{1-b_i}\gamma^U(D)] \sim (\psi_j E[DM] E[M^{1-b_j}]^{-1})^{\gamma_j(1-b_i)} E_t[\gamma^U(D) D^{-\gamma_j(1-b_i)}].$$

*Proof.* In complete analogy with (23), we get

$$\begin{aligned} & \left( 1 + K_1 \sum_{k \neq j} \left( \frac{E_t[D^{-\gamma_k(1-b_i)}] E[D^{1-\gamma_j}]^{1-b_i}}{E_t[D^{-\gamma_j(1-b_i)}] E[D^{1-\gamma_k}]^{1-b_i}} \right)^{1/\gamma} \right)^\gamma \\ & \leq \frac{E_t[M^{1-b_i}]}{(\psi_j E[DM] E[M^{1-b_j}]^{-1})^{\gamma_j(1-b_i)} E_t[D^{-\gamma_j(1-b_i)}]} \\ & \leq \left( 1 + K_1 \sum_{k \neq j} \left( \frac{E_t[D^{-\gamma_k(1-b_i)}] E[D^{1-\gamma_j}]^{1-b_i}}{E_t[D^{-\gamma_j(1-b_i)}] E[D^{1-\gamma_k}]^{1-b_i}} \right)^{1/\Gamma} \right)^\Gamma. \quad (37) \end{aligned}$$

Now,

$$\frac{E_t[D^{-\gamma_k(1-b_i)}] E[D^{1-\gamma_j}]^{1-b_i}}{E_t[D^{-\gamma_j(1-b_i)}] E[D^{1-\gamma_k}]^{1-b_i}} = e^{\frac{1}{2}\sigma^2 T(1-b_i) \left( (1-\lambda)(\gamma_k^2 - \gamma_j^2)(1-b_i) + (1-\gamma_j)^2 - (1-\gamma_k)^2 \right)}$$

and we need to show that

$$(1-\lambda)\gamma_k^2(1-b_i) - (1-\gamma_k)^2 < (1-\lambda)\gamma_j^2(1-b_i) - (1-\gamma_j)^2.$$

The function

$$f(\gamma) = (1-\lambda)\gamma^2(1-b_i) - (1-\gamma)^2 = -(1 - (1-\lambda)(1-b_i))\gamma^2 + 2\gamma - 1$$

is a quadratic parabola with the vertex at  $\gamma^v = 1/(1 - (1-\lambda)(1-b_i))$ .

Therefore, the inequality holds if and only if

$$|\gamma_k - \gamma^v| > |\gamma_j - \gamma^v|$$

and the claim follows from the definition of the interval  $\Theta_j^i$ . Q.E.D.

Now, repeating the same arguments as in the proof of Proposition 7, we can prove the following

**Lemma 13.** For any  $\lambda \in \Theta_j^i$ ,

$$\begin{aligned} \min_{l \in J(1+\gamma_j(1-b_i))} \gamma_l &\leq \liminf_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)M^{1-b_i}]}{E_t[M^{1-b_i}]} \\ &\leq \limsup_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)M^{1-b_i}]}{E_t[M^{1-b_i}]} \leq \max_{l \in J(1+\gamma_j(1-b_i))} \gamma_l. \end{aligned} \quad (38)$$

Finally, if  $\lambda = \frac{2-(\gamma_{j-1}+\gamma_j)b_i}{(\gamma_{j-1}+\gamma_j)(1-b_i)}$ , then

$$\begin{aligned} \min_{l \in J(1+\gamma_j(1-b_i)) \cup J(1+\gamma_{j-1}(1-b_i))} \gamma_l &\leq \liminf_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)M^{1-b_i}]}{E_t[M^{1-b_i}]} \\ &\leq \limsup_{T \rightarrow \infty} \frac{E_t[\gamma^U(D)M^{1-b_i}]}{E_t[M^{1-b_i}]} \leq \max_{l \in J(1+\gamma_j(1-b_i)) \cup J(1+\gamma_{j-1}(1-b_i))} \gamma_l. \end{aligned} \quad (39)$$

We can summarize our findings in

**Proposition 9.** Let  $t = \lambda T$  and fix an  $i > 1$ . Then, for any  $\lambda \in \Theta_j^i \cap \Pi_m$ ,

$$\pi_{\gamma_i}^K(\lambda) = \frac{(b_i - 1)\gamma_{J(1+\gamma_j(1-b_i))} + \gamma_{J(1+\gamma_m)}}{1 - \gamma_1 + \gamma_{J(1+\gamma_m)}}.$$

if the sets  $J(1 + \gamma_m)$  and  $J(1 + \gamma_j(1 - b_i))$  are singletons. Finally, for  $i = 1$ ,

$$\pi_{\gamma_1}(\lambda) = 1.$$

Proposition 6 directly follows from Propositions 8 and 9 since the sets  $J(1 + \gamma_m)$  and  $J(1 + \gamma_j(1 - b_i))$  are singletons under the assumptions of Proposition 6.

### C.3 The Infinite Population Limit

*Proof of Theorem 1.* As risk aversions  $\gamma_i$  become dense in the interval  $[1, \Gamma]$ , the set  $J(1 - \alpha)$  converges to  $1 - \alpha$  if  $1 - \alpha \in [1, \Gamma]$  and to  $\Gamma$  if  $\alpha \leq 1 - \Gamma$ . Clearly, the convergence is uniform. As the distance between  $\gamma_j$  and  $\gamma_{j-1}$

goes to zero, we have that  $\lambda \in \Pi_i$  implies that  $\gamma_i \approx \lambda^{-1}$ . Thus, as  $K \rightarrow \infty$ ,

$$\gamma_{J(1+\gamma_m)} \rightarrow \begin{cases} 1 + \lambda^{-1}, & \lambda \geq (\Gamma - 1)^{-1} \\ \Gamma, & \lambda < (\Gamma - 1)^{-1} \end{cases}$$

for  $\lambda \in \Pi_m$  and the claim about  $\sigma_{\text{sup,inf}}^K$  follows from Proposition 7. Q.E.D.

*Proof of Theorem 2.* Consider first the generic position case when all sets  $J$  in question are singletons. By Proposition 9, for any  $\lambda \in \Theta_j^i \cap \Pi_m$ , we have

$$\pi_\gamma^K(\lambda) = \pi_\gamma^{K, \text{myopic}}(\lambda) + \pi_\gamma^{K, \text{hedging}}(\lambda)$$

with

$$\pi_\gamma^{K, \text{myopic}}(\lambda) = \lim_{T \rightarrow \infty} \frac{\mu^K(\lambda) - r}{\gamma \sigma^K(\lambda)^2} = \frac{b \gamma_{J(1+\gamma_m)}}{1 - \gamma_1 + \gamma_{J(1+\gamma_m)}}$$

and therefore, the hedging component is given by

$$\pi_\gamma^{K, \text{hedging}}(\lambda) = (1 - b) \frac{\gamma_{J(1+\gamma_m)} - \gamma_{J(1+\gamma_j(1-b))}}{1 - \gamma_1 + \gamma_{J(1+\gamma_m)}}.$$

As  $K$  increases and the set  $G_K$  of risk aversions becomes dense in  $[1, \Gamma]$ , we have that  $\lambda \in \Theta_j^i$  implies that

$$\lambda \approx \frac{2 - 2\gamma_j b_i}{2\gamma_j(1 - b_i)} \Leftrightarrow \gamma_j \approx \frac{1}{b_i + \lambda(1 - b_i)}.$$

Since  $\gamma_i, \gamma_j \leq \Gamma$ ,

$$1 + \gamma_j(1 - b_i) \leq 1 + \Gamma(1 - \Gamma^{-1}) = \Gamma.$$

Consequently, for sufficiently large  $K$ ,

$$\gamma_{J(1+\gamma_j(1-b_i))} \approx 1 + \gamma_j(1 - b_i) \approx 1 + \frac{1 - b_i}{b_i + \lambda(1 - b_i)}.$$

Thus, for  $\lambda \geq (\Gamma - 1)^{-1}$ ,

$$\begin{aligned}
\pi_{\gamma}^{K, \text{hedging}}(\lambda) &= (1 - b) \frac{\gamma^{J(1+\gamma_m)} - \gamma^{J(1+\gamma_j(1-b))}}{1 - \gamma_1 + \gamma^{J(1+\gamma_m)}} \\
&\approx (1 - b) \frac{\lambda^{-1} - \frac{1-b}{b+\lambda(1-b)}}{1 + \lambda^{-1}} \\
&= \frac{b(1-b)}{(\lambda+1)(b+\lambda(1-b))} = \frac{\gamma-1}{(\lambda+1)\gamma(1+\lambda(\gamma-1))}.
\end{aligned} \tag{40}$$

Clearly, the convergence is uniform and Proposition 9 implies the required for arbitrary  $\gamma_i$  and all  $\lambda \in (0, 1)$ .

Q.E.D.

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