Mathematical Social Sciences 44 (2002) 235–252
mathematical social sciences
www.elsevier.com/locate/econbase

Group order preservation and the proportional rule for the adjudication of conflicting claims

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Received 1 May 2001; received in revised form 1 April 2002; accepted 1 May 2002

Abstract

We investigate the existence of rules for the adjudication of conflicting claims satisfying 'group order preservation': given two groups of claimants, suppose that the sum of the claims of the members of the first group is greater than or equal to the sum of the claims of the members of the second group. Then, similar inequalities should hold for the sums of the awards to the members of the two groups, and for the sums of the losses incurred by the members of two groups. The property is easily satisfied. We then combine it with two others. First is 'claims continuity': the chosen awards vector should vary continuously with the claims vector. Second is 'consistency': the awards vector chosen for each problem should be 'in agreement' with the awards vector chosen for each problem derived from it by imagining some of the claimants receiving their awards and leaving. We show that only the proportional rule satisfies all three requirements. This characterization holds even if the number of potential claimants is as low as 3. We also offer a version of the characterization for a variant of the model in which the set of claimants is modelled as a continuum.

Keywords: Conflicting claims; Group order preservation; Replication invariance; Consistency; Proportional rule

JEL classification: C79; D63; D74

1. Introduction

When a firm goes bankrupt, the problem arises of dividing its liquidation value among its creditors. More generally, we consider the problem of dividing a good over which

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0165-4896/02/$ – see front matter
PII: S0165-4896(02)00038-0

agents have conflicting claims. Our objective is to contribute to the search for well-behaved methods of solving such conflicts.

Such methods are called 'division rules' or simply 'rules'.[1] A basic requirement on rules is that they should respect the ordering of claims: if the claim of one agent is greater than or equal to the claim of another agent, his award should be greater than or equal to this other agent's award. A parallel inequality for losses also makes sense: the loss incurred by the agent with the larger claim should be greater than or equal to that incurred by the agent with the smaller claim. We propose here to generalize this idea of 'order preservation' to groups of claimants, and to study the implications of this more demanding requirement: given any two groups, if the sum of the claims of the members of the first group is greater than or equal to the sum of the claims of the members of the second group, then a similar inequality should hold for the sums of the awards to the members of the two groups; here too, we require a parallel inequality for the sums of the losses incurred by the members of the two groups. We refer to this two-part property as 'group order preservation'.[2] A special form of it, 'equal treatment of equal groups', is the requirement that if the sums of the claims of the members of two groups are equal, the sums of their awards should be equal.

Group order preservation is not very restrictive. For each problem, we identify a non-empty polygonal area of awards vectors at which the property is satisfied. Although it is not compatible with two properties that have been extensively discussed in the literature, 'invariance under claims truncation' and 'minimal rights first', as we show next by means of simple examples, it is compatible with an important requirement designed to ensure the good behavior of rules in situations where the population of claimants may change. This requirement, called 'consistency', is that the awards vector chosen for each problem should be 'in agreement' with the awards vector chosen for each problem obtained by imagining an arbitrary subgroup of claimants leaving with their awards, and re-evaluating the situation from the viewpoint of the remaining claimants. Consistency has played an important role recently in game theory and resource allocation theory. We show that if imposed together with the very mild requirement that the chosen awards vector should vary continuously with the claims vector, 'claims continuity', then equal treatment of equal groups and consistency are satisfied by only one rule, the proportional rule. We give a direct proof of this result, and we also derive it from a characterization of the proportional rule based on equal treatment of equal groups, claims continuity and 'replication invariance', the requirement that the awards vector chosen for a replicated problem should be the replica of the awards vector chosen for the problem that is replicated (this result is proved independently by Ching and Kakkar, 2000). This corollary is obtained by noting that equal treatment of equals and consistency together imply replication invariance.

Two additional requirements of relevance here are 'converse consistency', the requirement that if an awards vector for some problem is such that its restriction to each two-claimant group would be chosen for the problem of dividing between them the sum of their awards, then it should be chosen. The second one is 'division invariance', a

[1] For a survey of the literature devoted to this subject, see Thomson (1995).
[2] This property is formulated by Thomson (1988a).

converse of replication invariance. Characterizations of the proportional rule can be obtained on the basis of these properties too, due to logical relations between them and replication invariance. These relations hold because in our context, rules are single-valued. We also show that claims continuity is redundant if group order preservation is imposed instead of equal treatment of equal groups.

The results just described are proved in a model in which the population of potential agents is unbounded above. Next, we turn our attention to situations where instead it is finite. We show that in that case—in fact, this number may be as low as three—the proportional rule is the only one to satisfy claims continuity, equal treatment of equal groups, and consistency. To understand how these results are related, note that in the three-claimant case, replication invariance has of course little force since the only problems that can be replicated are trivial one-claimant problems; then, replication invariance amounts to equal treatment of equals, a property that is satisfied by all standard rules. If there are more than three claimants, replication invariance starts having bite but the proof techniques for our characterizations involving this property requires replications of arbitrary orders. The bound on the population of potential claimants that we impose forces us to develop a different proof, which seems to have no parallel in the existing literature on consistency.

We also offer a characterization of the proportional rule on the basis of equal treatment of equal groups alone for a variant of the model in which the set of claimants is modelled as a continuum.

In the concluding section, we relate our results to known characterizations of the proportional rule.

2. Claims problems and rules to solve them

Since some of our requirements involve comparing the choices made for problems involving different populations of claimants, we need to cast our analysis in a sufficiently general framework for such comparisons to be possible. There is a set $I \subset \mathbb{N}$, where $\mathbb{N}$ designates the natural numbers, of 'potential' claimants. This set could be $\mathbb{N}$ itself or a finite subset of $\mathbb{N}$. However, when I is infinite, at any given time, only a finite number of claimants are present. Let $\mathcal{N}$ be the class of finite and nonempty subsets of I. Given $N \in \mathcal{N}$, a ***claims problem*** with claimant set N is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\Sigma_N c_i \geq E$.[3] Let $\mathscr{C}^N$ be the class of all problems. A ***rule*** is a function defined on the union of all $\mathscr{C}^N$'s, where N ranges over $\mathcal{N}$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathscr{C}^N$ a point of $\mathbb{R}^N_+$ satisfying $0 \leqq x \leqq c$ and whose coordinates add up to E, a property to which we refer as efficiency. Such a point is an ***awards vector*** for (c, E). Let $X(c, E)$ be the set of awards vectors for (c, E). Let R be our generic notation for rules.

The most prominent rule in practice as well as in the theoretical literature is the proportional rule, for which awards are proportional to claims. The idea of proportionality as a principle of justice is prominent in Aristotle's writings.

[3]By the notation $\mathbb{R}^N_+$ we mean the Cartesian product $\mathbb{R}^{|N|}_+$ in which each dimension is indexed by a member of N. Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

Proportional rule, *P*. For each $N \in \mathcal{N}$, and each $(c, E) \in \mathscr{C}^N$, $P(c, E) \equiv \lambda c$, where λ is chosen so as to achieve efficiency.

Our starting point is the following property of rules: if the claim of one agent is greater than or equal to that of another agent, his award should be greater than or equal to this other agent's award. Also, the loss he incurs should be greater than or equal to that other agent's loss (Aumann and Maschler, 1985).

Order preservation. For each $N \in \mathcal{N}$, each $(c, E) \in \mathscr{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i \geq c_j$, then $R_i(c, E) \geq R_j(c, E)$. Also, $c_i - R_i(c, E) \geq c_j - R_j(c, E)$.

Note that *order preservation* implies that if two agents have equal claims their awards should be equal:

Equal treatment of equals. For each $N \in \mathcal{N}$, each $(c, E) \in \mathscr{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $R_i(c, E) = R_j(c, E)$.

We propose here to generalize the idea of *order preservation* to groups of claimants, by comparing the aggregate claims they hold. If the sum of the claims of the members of one group is greater than or equal to the sum of the claims of the members of another group, the sum of the awards to the members of the first group should be greater than or equal to the sum of the awards to the members of the second group. We require a parallel inequality between the sums of the losses incurred by the two groups:[4]

Group order preservation. For each $N \in \mathcal{N}$, each $(c, E) \in \mathscr{C}^N$, and each pair $\{N', N''\}$ of subsets of N, if $\Sigma_{N'} c_i \geq \Sigma_{N''} c_i$, then $\Sigma_{N'} R_i(c, E) \geq \Sigma_{N''} R_i(c, E)$. Also, $\Sigma_{N'}(c_i - R_i(c, E)) \geq \Sigma_{N''}(c_i - R_i(c, E))$.

By considering 'groups' containing only one claimant each, we see immediately that *group order preservation* implies *order preservation*. *Group order preservation* amounts to adding to the *order preservation* inequalities a number of other inequalities pertaining to groups. In the Euclidean space to which awards vectors belong, and for each problem, the set of awards vectors meeting all the constraints is a polygonal region. To illustrate, let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathscr{C}^N$ be such that $c_1 \leq c_2 \leq c_3$. Then, when groups are compared, the set of awards vectors satisfying all the inequalities is obtained from the set of awards vectors satisfying the order preservation inequalities by adding two more (Fig. 1):[5]

[4]In many branches of game theory and the theory of resource allocation, properties defined for individuals often have counterparts for groups. For instance, in the theory of fair allocation, to the property of 'no-envy' corresponds a property of 'group no-envy' expressing the requirement that not only individuals but also groups should be treated fairly.

[5]There are only two additional inequalities, since the comparison of agent 1's claim to the sum of agents 2 and 3's claims and that of agent 2's claim to the sum of agents 1 and 3's claims does not add any restriction not already implied by *equal treatment of equals*.

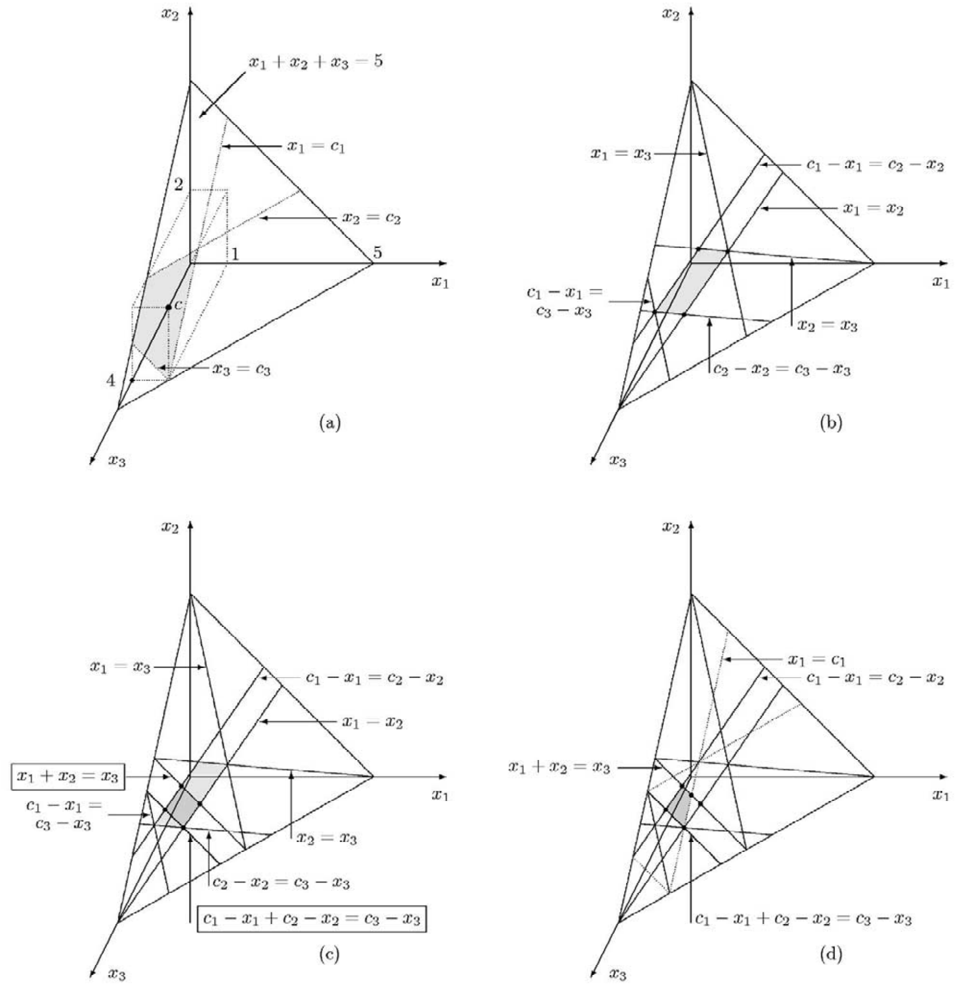

Fig. 1. Strengthening order preservation to group order preservation in the three-claimant case. In each panel, claims are fixed and equal to (1, 2, 4) and the amount to divide is 5. The shaded area of panel (a) indicates the set of awards vectors and that of panel (b) the set of efficient vectors meeting the order preservation inequalities. The entire shaded area of panel (c) is the same as that of panel (b) and the darker region is its subset of efficient vectors meeting the two additional inequalities coming from *group order preservation* (they are printed in rectangles). The shaded area of panel (d) is the set of awards vectors meeting the *group order preservation* inequalities (here, only the binding constraints are labelled).

(i) If $c_1 + c_2 \geq c_3$, then $x_1 + x_2 \geq x_3$ and $c_1 - x_1 + c_2 - x_2 \geq c_3 + x_3$;
(ii) If $c_1 + c_2 \leq c_3$, then $x_1 + x_2 \leq x_3$ and $c_1 - x_1 + c_2 - x_2 \geq c_3 + x_3$.

This set is non-empty as it contains the proportional awards vector, as is easily verified. Constructing rules satisfying *group order preservation* is then easy because for each problem, one can select from it in an arbitrary fashion. It is therefore puzzling that although all of the rules that have been discussed in the literature *preserve order*, only one satisfies *group order preservation*, the proportional rule. Our characterizations will help us understand why.

It is even easier to construct rules satisfying the group counterpart of *equal treatment of equals*: two groups whose aggregate claims are equal should receive equal aggregate awards:

Equal treatment of equal groups. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair $\{N', N''\}$ of subsets of N, if $\Sigma_{N'} c_i = \Sigma_{N''} c_i$, then $\Sigma_{N'} R_i(c, E) = \Sigma_{N''} R_i(c, E)$.

In the two-claimant case, *group order preservation* is equivalent to *order preservation*, and *equal treatment of equal groups* is equivalent to *equal treatment of equals*.

Our next property is the requirement that the names of claimants should be immaterial.

Anonymity. For each pair $\{N, N'\}$ of elements of $\mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each bijection π: $N \to N'$, and each $i \in N$, $R_{\pi(i)}\big((c_{\pi(i)})_{i \in N}, E\big) = R_i(c, E)$.

Next we define two requirements that have played an important role in the literature and enquire about their compatibility with our new properties. First is the requirement that the chosen awards vector should be unaffected if each claim is truncated by the amount to divide:

Invariance under claims truncation. For each $(c, E) \in \mathcal{C}^N$, $R(c, E) = R(t(c, E), E)$, where for each $i \in N$, $t_i(c, E) = \min\{c_i, E\}$.

The following example shows that *equal treatment of equal groups* is incompatible with invariance under claims truncation: Let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be defined by $(c, E) \equiv (1, 1, 2; 1)$. If a rule R satisfies *equal treatment of equal groups*, then $x_1 + x_2 = x_3$, where $x \equiv R(c, E)$. If R also satisfies *invariance under claims truncation*, then $x = R(1, 1, 1; 1)$. However, by *equal treatment of equal groups* (here, *equal treatment of equals* would suffice), $R_1(1, 1, 1; 1) = R_2(1, 1, 1; 1) = R_3(1, 1, 1; 1)$.

Another interesting requirement is that the following two procedures should be equivalent: either (i) the amount available is divided directly, or (ii) it is divided in two rounds; first, each claimant is assigned his ***minimal right***, defined to be the difference between the amount available and the sum of the claims of the other agents (or 0 if this difference is negative); then claims are adjusted down by these first-round awards and whatever is left is divided by applying the rule. Let $m_i(c, E) \equiv \max\{E - \Sigma_{N \setminus i} c_j, 0\}$ and $m(c, E) \equiv (m_i(c, E))_{i \in N}$.

Minimal rights first. For each $(c, E) \in \mathcal{C}^N$, we have $R(c, E) = m(c, E) + R(c - m(c, E), E - \Sigma m_i(c, E))$.[6]

[6]Note that the pair $(c - m(c, E), E - \Sigma m; (c, E))$ is a well-defined problem.

This property too is incompatible with *equal treatment of equal groups*, as shown by the following example: Let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathscr{C}^N$ be defined by $(c, E) \equiv (1, 1, 2; 3)$. If a rule R satisfies *equal treatment of equal groups*, then $x_1 + x_2 = x_3$, where $x \equiv R(c, E)$. Note that $m(c, E) = (0, 0, 1)$. If R also satisfies *minimal rights first*, then $x = m(c, E) + R(1–0, 1–0, 2–1; 3–1) = (0, 0, 1) + (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, where the equality $R(1–0, 1–0, 2–1; 3–1) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ is obtained by *equal treatment of equal groups* (here too, *equal treatment of equals* would suffice).[7]

In spite of the two incompatibilities just uncovered between *equal treatment of equal groups* and our two invariance properties, this property and in fact, its stronger version, *group order preservation*, are not very restrictive, as we noted above. The permissiveness of a property is in general a good thing because there are always more demands that one would like to make on rules. We will take advantage of it to look for rules also satisfying certain requirements having to do with possible changes in the population of claimants. Our main one is that if some claimants leave with their awards and the situation is re-evaluated from the viewpoint of those who remain, the rule should assign to them the same awards as initially.[8] The problem these claimants face is the ***reduced problem associated with the awards vector chosen for the initial problem and the subgroup they constitute***. Their claims are what they were initially and the amount to divide between them is the difference between the amount available initially and the sum of the awards to the claimants who left; alternatively it is the sum of the awards intended for them, the claimants who stay.[9]

Consistency. For each $N \in \mathscr{N}$, each $(c, E) \in \mathscr{C}^N$, and each $N' \subset N$, if $x \equiv R(c, E)$, then $x_{N'} = R(c_{N'}, \Sigma_{N'} x_i)$.

Many rules are *consistent*. The proportional rule is one of them, as is easy to check.

A variant of *consistency* that is often considered, ***bilateral consistency***, is obtained by limiting attention to subgroups of two remaining claimants (that is, adding the requirement $|N'| = 2$ to the previous statement).

Our next requirement is that if an awards vector for some problem is such that its restriction to each subgroup of two claimants is the awards vector chosen for the associated reduced problem they face, then it should be chosen for the problem involving the entire population of claimants.[10]

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[7] Without going into the details, we note that this conclusion can also be obtained from the previous incompatibility between *equal treatment of equal groups* and *invariance under claims invariance*, by observing that *minimal rights first* and *invariance under claims invariance* are 'dual' properties, and *equal treatment of equal groups* is a 'self-dual' property. For the concept of duality, see Aumann and Maschler (1985), Herrero and Villar (1998a, 2001), and Thomson (1995).

[8] The many applications that have been made of the idea of consistency are surveyed by Thomson (2001).

[9] Note that since we require rules to be such that for each $i \in N$, $x_i \in [0, c_i]$, then the sum of the claims of the remaining claimants is still greater than or equal to the amount that remains to divide, so the pair $(c_{N'}, \Sigma_{N'} x_i)$ is indeed a well-defined problem.

[10] The version of the requirement obtained by writing the hypotheses for all proper subgroups of the initial set of claimants, instead of only for the two-claimant subgroups, turns out to be equivalent to it.

Converse consistency. For each $N \in \mathcal{N}$, each $(c, E) \in \mathscr{C}^N$, and each $x \in X(c, E)$, if $[$for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = R(c_{N'}, \Sigma_{N'} x_i)]$, *then* $x = R(c, E)$.

To define the next requirement, we first need to explain how to replicate a problem. Let $N \in \mathcal{N}$ and $(c, E) \in \mathscr{C}^N$ be given. Let $k \in \mathbb{N}_+$. By a ***k-replica of* (*c*, *E*)** we mean a problem in which each of the members of N has $k-1$ clones—these are agents whose claims are equal to his—and in which the amount available is k times what it was initially. Formally, if N' designates the set of claimants in the replicated problem, we have $N' \supset N$, $|N'| = k|N|$, and there is a partition of N' into $|N|$ groups of k agents indexed by $i \in N$, $(N^i)_{i \in N}$, such that for each $i \in N$ and each $j \in N^i$, $c_j = c_i$. The requirement is that the awards vector chosen for a replicated problem should be the replica of the awards vector chosen for the problem that is replicated.

Replication invariance. For each $N \in \mathcal{N}$, each $(c, E) \in \mathscr{C}^N$, each $N' \supset N$, and each $(c', E') \in \mathscr{C}^{N'}$, if (c', E') is a k-replica of (c, E), with associated partition $(N^i)_{i \in N}$, then for each $i \in N$ and each $j \in N^i$, $R_j(c', E') = R_i(c, E)$.

Given a problem (c, E), we will use the shorthand notation $k*x$ for a k-replica of some awards vector x for (c, E).

Next is the converse of *replication invariance*. It says that if the awards vector chosen for a replicated problem happens to be a replicated awards vector (of the same order), then the awards vector that is being replicated should be the awards vector chosen for the problem that is being replicated.[11]

Division invariance. For each $N \in \mathcal{N}$, each $(c, E) \in \mathscr{C}^N$, each $N' \supset N$, and each $(c', E') \in \mathscr{C}^{N'}$, if (c', E') is a k-replica of (c, E), with associated partition $(N^i)_{i \in N}$, and $R(c', E')$ is a corresponding k-replica of an awards vector for (c, E), then for each $i \in N$ and each $j \in N^i$, $R_j(c', E') = R_i(c, E)$.

Our final requirement is that for each amount to divide, the chosen awards vector should be a continuous function of the claims vector. It is hard to imagine circumstances in which this requirement would not be desirable, and in fact, *all* rules that have been considered in the literature on claims resolution are not only *claims continuous*, but in fact 'fully' *continuous* (that is, they respond in a continuous manner to simultaneous changes in all components of the problem).

Claims continuity. For each $N \in \mathcal{N}$, each sequence $\{(c^\nu, E^\nu)\}_{\nu=1}^{\infty}$ of elements of $\mathscr{C}^N$, and each $(c, E) \in \mathscr{C}^N$, if $(c^\nu, E^\nu) \to (c, E)$ and for each $\nu \in \mathbb{N}$, $E^\nu = E$, then $R(c^\nu, E^\nu) \to R(c, E)$.

The following relations between the various properties we just defined will be critical.

[11] It is the counterpart of a property introduced by Thomson (1988b) for the problem of fair division.

The first lemma is a straightforward consequence of the definitions, but we give the proof in Appendix A for completeness.[12]

Lemma 1. (Elevator lemma). *If a bilaterally consistent rule coincides with a conversely consistent rule in the two-person case, then coincidence occurs in general.*

Lemma 2. (*a*) *If a rule satisfies equal treatment of equals and consistency, it satisfies division invariance.*

(*b*) *If a rule satisfies equal treatment of equals and division invariance, it satisfies replication invariance.*

(*c*) *If a rule satisfies anonymity and converse consistency, it is invariant under replication of a two-claimant problem.*

Proof. In the proof of each of the three assertions, let R designate a rule satisfying the hypotheses.

(a) Let $N \in \mathcal{N}$, $(c, E) \in \mathscr{C}^N$, and $k \in \mathbb{N}$. Let (c', E') be a k-replica of (c, E) with agent set N'. By *equal treatment of equals*, the clones of each member of N receive equal awards at $R(c', E')$, so $R(c', E') = k{*}x$ for some $x \in X(c, E)$. By *consistency*, $(k{*}x)_N = R(c'_N, \Sigma_N x_i)$. Since $(k{*}x)_N = x$ and $R(c'_N, \Sigma_N x_i) = R(c, E)$, we are done.

(b) Let $N \in \mathcal{N}$, $(c, E) \in \mathscr{C}^N$, and $k \in \mathbb{N}$. Let (c', E') be a k-replica of (c, E) with agent set N'. By *equal treatment of equals*, the clones of each member of N receive equal awards at $R(c', E')$. Let $k{*}x$ designate the awards vector chosen for (c', E'). Now, the hypothesis of *division invariance* applies to $k{*}x$ in (c', E'). This property says that $x \equiv R(c, E)$. Altogether, we obtain $R(c', E') = k{*}R(c, E)$, as required by *replication invariance*.

(c) Let $N \in \mathcal{N}$ be such that $|N| = 2$, $(c, E) \in \mathscr{C}^N$, and $k \in \mathbb{N}$. Let (c', E') be a k-replica of (c, E) with agent set N'. Let $x \equiv R(c, E)$. We need to show that $R(c', E') = k{*}x$. Let $y \equiv k{*}x$. Let $\bar{N} \subset N'$ with $\bar{N} = 2$, say $\bar{N} = \{\ell, \ell'\}$. There are two cases. Either $\bar{N}$ consists of two clones of some claimant in N, in which case $y_\ell = y_{\ell'}$, but then by *equal treatment of equals*, implied by *anonymity*, $(y_\ell, y_{\ell'}) = R(c_\ell, c_{\ell'}, y_\ell + y_{\ell'})$. Or $\bar{N}$ contains a clone of each of the claimants in N, and here by *anonymity*, $(y_\ell, y_{\ell'}) = R(c_\ell, c_{\ell'}, y_\ell + y_{\ell'})$. Thus y satisfies the hypotheses of *converse consistency* for R in (c', E'), and since R is *conversely consistent*, $y = k{*}x = R(c', E')$. □

Obviously, *replication invariance* implies *division invariance*. The reason why these properties are not equivalent, however, is that a rule may satisfy *division invariance* by never meeting the hypotheses of the property (except in the trivial case when the amount to divide is 0, when it has to, by definition of a rule). Then, there is no reason why *replication invariance* should be met. Indeed, consider a rule that selects for each replicated problem in which the amount to divide is positive, an awards vector at which at least two claimants with equal claims receive unequal awards. Then the rule satisfies

[12] The lemma, which is a fundamental tool in the theory of consistency, appears in this form in Thomson (2001).

division invariance. Yet, it violates the conclusion of *replication invariance* in every non-trivial problem in which the hypotheses of the axiom apply.

3. Characterizations of the proportional rule

Our main results are characterizations of the proportional rule. Except for one, they pertain to the case when the population of potential agents is unbounded. We will soon present an argument that implies our first result, but for a reader not familiar with the theory of *consistency*, we include the proof as it is representative of the sort of reasoning that is typical in that literature. The logic is as follows: to show that a rule assumed to satisfy our axioms is the proportional rule, and starting from a typical claims problem, we introduce new agents and add resources in such a way that (i) the awards vector chosen for the resulting ‘augmented problem’ is its proportional awards vector, and (ii) the reduction of the augmented problem with respect to the initial set of agents and this awards vector is the initial problem (this implies that in the augmented problem the sum of what the new agents are awarded should be equal to the added resources.) The difficulty in carrying out this sort of argument is figuring out exactly how many agents to bring in, what their claims should be, and what resources to add.

Theorem 1. *Suppose* $|I| = \infty$. *The proportional rule is the only rule satisfying claims continuity, equal treatment of equal groups, and consistency.*

Proof. We have already seen that P satisfies the three properties listed in the theorem. Conversely, let R be a rule satisfying the properties. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ be such that there exist $a > 0$, $\gamma \in \mathbb{N}^N$ such that $c = a\gamma$. Let $i \in N$ be such that for each $j \in N$, $c_i \geq c_j$, or equivalently, $\gamma_i \geq \gamma_j$. Let $\alpha \in \mathbb{R}_+$ be such that $(c, E) = \alpha c$.

We now augment the initial problem by introducing γ_i new agents, each with a claim equal to a, and increase the resource so that at the proportional awards vector of the augmented problem, all claimants receive the fraction α of their claims. Let $\bar{N} \in \mathcal{N}$ be such that $\bar{N} \cap N = \emptyset$ and $|\bar{N}| = \gamma_i$. Let $(c', E') \in \mathcal{C}^{N \cup \bar{N}}$ be defined by $c'_N \equiv c$, for each $\ell \in \bar{N}$, $c'_\ell \equiv a$, and $E' \equiv E + \alpha\gamma_i a$. Let $y \equiv R(c', E')$. By *equal treatment of equal groups*, for each pair $\{\ell, \ell'\} \subset \bar{N}$, $y_\ell = y_{\ell'}$,[13] so there exists $\bar{\alpha} \in \mathbb{R}_+$ such that for each $\ell \in \bar{N}$, $y_\ell = \bar{\alpha} a$. For each $j \in N$, let $N^j \subset \bar{N}$ be a group of γ_j claimants, and note that $c_j = \Sigma_{N^j}\ c_\ell$. By *equal treatment of equal groups*, $y_j = \Sigma_{N^j}\ y_\ell = \gamma_j \bar{\alpha} a$. Thus, $(y_j, y_{N^j}) = \bar{\alpha} a(\gamma_j, e_{N^j})$, where e_{N^j} is the vector of ones in $\mathbb{R}^{N^j}$. Since this conclusion holds for each $j \in N$, we obtain $y = \bar{\alpha} c'$. By feasibility, $\bar{\alpha} = \alpha$. By *consistency*, $y_N = R(c'_N, \Sigma_N\ \alpha c_i)$. To conclude for this case, recall that $y_N = x$ and note that $(c'_N, \Sigma_N\ \alpha c_i) = (c, E)$.

Finally, we consider the case when there are no a and γ as specified in the first

[13] *Equal treatment of equals* would suffice here.

paragraph of the proof. Any such problem can be approximated by a sequence of problems for which α and γ do exist. We then appeal to claims continuity. □

A natural question is whether substituting *bilateral consistency* for *consistency* in Theorem 1 would enlarge the class of admissible rules. The answer is no. Indeed, if a rule satisfies *claims continuity*, *equal treatment of equal groups*, and *bilateral consistency*, then we can conclude as in the proof of Theorem 1 that in the two-claimant case, it coincides with the proportional rule. Since the proportional rule is *conversely consistent*, it follows by the Elevator Lemma that coincidence holds in general.

Theorem 1 as well as two other characterizations of the proportional rule can also be obtained as corollaries of the following result, using the Elevator Lemma and Lemma 2. This result is proved independently by Ching and Kakkar (2000).

Theorem 2. *Suppose $|I|=\infty$. The proportional rule is the only rule satisfying claims continuity, equal treatment of equal groups, and replication invariance.*

Proof. We have already seen that P satisfies *claims continuity* and *equal treatment of equals groups*, and it obviously satisfies *replication invariance*. Conversely, let R be a rule satisfying the axioms listed in the theorem. Let $N \in \mathcal{N}$ and $(c, E) \in \mathscr{C}^N$ be such that there exist $a > 0$, $\gamma \in \mathbb{N}^N$ such that $c = \gamma a$. Let k be a common multiple of the γ_i, when i runs over N.

We now replicate k times the problem (c, E) and designate by (c', E') the replicated problem. Let $y \equiv R(c', E')$. For each $\{i, j\} \subset N$, there exists a subset of the agent set in the replicated problem—let us call it N'—that contains γ_j clones of claimant i and γ_i clones of claimant j. For each $\ell \in \{i, j\}$, let $N^{\ell} \subset N'$ be the set of agents in N' who are clones of agent ℓ. Then $\Sigma_{\ell \in N^i} c_{\ell} = \Sigma_{\ell \in N^j} c_{\ell}$, so that by *equal treatment of equal groups*, $\Sigma_{\ell \in N^i} y_{\ell} = \Sigma_{\ell \in N^j} y_{\ell}$. By *equal treatment of equal groups*, (this time, *equal treatment of equals* would suffice), for each $\{\ell, \ell'\} \subset N^i$ and each $\{\ell, \ell'\} \subset N^j$, we have $y_{\ell} = y_{\ell'}$. It follows that for some $\bar{\alpha} \in \mathbb{R}_+$, $y_{N'} = y_{N^i \cup N^j} = \bar{\alpha}(c_i, \ldots, c_i, c_j, \ldots, c_j)$. This argument applying for each $\{i, j\} \subset N$, we obtain that for some $\bar{\alpha} \in \mathbb{R}_+$, $x = \bar{\alpha} c$. By efficiency, $y = k * P(c, E)$, and by *replication invariance*, $R(c, E) = P(c, E)$.

If there are no a and γ as specified in the first paragraph of the proof, we conclude by an approximation argument as in Theorem 1, and appeal to *claims continuity*. □

Theorem 3. *Suppose $|I|=\infty$. Then, the proportional rule is the only rule satisfying*

(a) *Claims continuity, equal treatment of equal groups, and consistency*
(b) *Claims continuity, equal treatment of equal groups, and division invariance*
(c) *Claims continuity, equal treatment of equal groups, anonymity, and converse consistency*.

Proof. (a) This statement is a consequence of parts (a) and (b) of Lemma 2 and Theorem 2.

(b) This statement is a consequence of part (b) of Lemma 2 and Theorem 2.

(c) Let R be a rule satisfying *anonymity* and *converse consistency*. Part (c) of Lemma

2 and Theorem 2 imply that $R = P$ in the two-claimant case. Since P is *consistent* and R *conversely consistent*, we then conclude by the Elevator Lemma that $R = P$ in general. □

Our next result is a characterization of the proportional rule that does not involve *claims continuity*, although *equal treatment of equal groups* is strengthened to *group order preservation* (note however that in the uniqueness proof, we only invoke the first part of this property).

Theorem 4. *Suppose* $|I| = \infty$. *The proportional rule is the only rule satisfying group order preservation and replication invariance.*

Proof. We have already seen that P satisfies the properties listed in the theorem. Conversely, let R be a rule satisfying the properties, and suppose by way of contradiction that there exist $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ such that $R(c, E) \neq P(c, E)$. Obviously $E > 0$. Let $x \equiv R(c, E)$. Let $i \in N$ be such that for each $j \in N$, $c_i \geq c_j$. By *group order preservation* (*order preservation* would suffice here), $x_i > 0$. Then, there exists $\ell \in N$ such that $x_\ell / x_i \neq c_\ell / c_i$. Equivalently, there exists $t > 0$ such that

$$\frac{x_\ell}{x_i} \not\in \left]\frac{c_\ell}{c_i} - t, \frac{c_\ell}{c_i} + t\right[.$$

Let m: $\mathbb{N} \to \mathbb{N}$ be defined by:

$$m(n) \equiv \max\left\{m' \in \mathbb{N}: \frac{m'}{n} \leq \frac{c_\ell}{c_i}\right\}$$

Note that the function m is nowhere decreasing. Furthermore (*):

$$\frac{m(n)}{n} \leq \frac{c_\ell}{c_i} < \frac{m(n) + 1}{n}.$$

Finally,

$$\frac{m(n)}{n} \to \frac{c_\ell}{c_i} \quad \text{and} \quad \frac{m(n) + 1}{n} \to \frac{c_\ell}{c_i}.$$

Hence, there exists n large enough so that

$$\frac{m(n)}{n}, \frac{m(n) + 1}{n} \in \left]\frac{c_\ell}{c_i} - t, \frac{c_\ell}{c_i} + t\right[.$$

Let $N' \in \mathcal{N}$ be such that $|N'| = (n + 1)|N|$ and $(c', E') \in \mathcal{C}^{N'}$ be an $(n + 1)$-replica of (c, E). In N', there are n clones of agent ℓ, and since $m(n) + 1 \leq n + 1$, there are at least $m(n) + 1$ clones of agent i. By (*):

$$m(n)c_i \leq nc_\ell < (m(n) + 1)c_i$$

By *group order preservation* applied to a group consisting of n clones of agent ℓ and

$m(n)$ clones of agent i, and then to a group consisting of n clones of agent ℓ and $m(n)+1$ clones of agent i, we obtain:

$$m(n)R_i(c', E') \leq nR_\ell(c', E') < (m(n)+1)R_i(c', E').$$

By *replication invariance*, $R_i(c', E') = x_i$ and $R_j(c', E') = x_j$, so that this double inequality can be written as $m(n)x_i \leq nx_\ell < (m(n)+1)x_i$. Dividing through by nx_i yields:

$$\frac{m(m)}{n} \leq \frac{x_\ell}{x_i} < \frac{m(n)+1}{n}.$$

Thus,

$$\frac{x_\ell}{x_i} \in \left]\frac{c_\ell}{c_i} - t, \frac{c_\ell}{c_i} + t\right[,$$

in contradiction to the way t was defined. □

The proofs of Theorems 1–4 require the population of agents to be unbounded above. In applications, natural bounds often apply however, and the question then is whether other rules would become available. If there are never more than two claimants, this is certainly the case. For *consistency* to come into play, there should be at least three claimants. Also, if there are no more than three claimants, *replication invariance* has no bite. Indeed, only one-claimant 'groups' can be replicated then, and in that case, the axiom amounts to *equal treatment of equals*, which is satisfied by all standard rules. It is therefore of interest that if *consistency* is imposed instead of *replication invariance*, a characterization of the proportional rule is recovered. This is the content of the next theorem, which interestingly, relies on an induction proof that does not seem to have counterparts in the existing literature on *consistency*. We consider two-claimant problems in which the larger claim is an integer multiple of the smaller claim and the induction is on this coefficient of multiplicity.

Theorem 5. *Suppose* $|I| \geq 3$. *The proportional rule is the only rule satisfying claims continuity, equal treatment of equal groups, and consistency.*

Proof. We have already seen that P satisfies the properties listed in the theorem. Conversely, let R be a rule satisfying the properties. We show that $R = P$ in two main steps. The first one deals with the two-claimant case. The second one extends the conclusion obtained for the two-claimant case to all other cases.

Step 1: For each $N \in \mathcal{N}$ with $|N| = 2$, and each $(c, E) \in \mathscr{C}^N$, $R(c, E) = P(c, E)$.

To fix the idea, set $N \equiv \{i, j\}$.

Substep 1: If $c_i = mc_j$ for some $m \in \mathbb{N}$, then $R_i(c, E) = mR_j(c, E)$.

We establish the claim by induction on m. If $m = 1$, it follows from *equal treatment of equal groups* (*equal treatment of equals* would suffice here).

Now, let $n \geq 1$, and suppose that for each $m \in \mathbb{N}$ such that $m < n$, the claim is true. We will establish that it is true for $m = n$. Suppose then that $c_i = nc_j$. Let $\ell \in I \backslash \{i, j\}$ (this can be done since $|I| \geq 3$), $N' \equiv \{i, j, \ell\}$, and $(c', E') \in \mathscr{C}^{N'}$ be defined by $c'_i \equiv c_i$,

$c'_j \equiv c_j$, $c'_\ell \equiv (n-1)c_j$, and $E' \equiv E + [(n-1)/(n+1)]\,E$. Let $y = R(c', E')$. Note that $c'_i = c'_j + c'_\ell$, so that by *equal treatment of equal groups*, $y_i = y_j + y_\ell$. By *consistency* and the induction hypothesis, $y_\ell = (n-1)\,y_j$. These last two equalities give $y_i = ny_j$. By efficiency, we then obtain $y = (1, n, n-1)\,[E/(n+1)]$. By *consistency*, $(y_i, y_j) = R(c'_N, y_i + y_j)$ and since $(c'_N, y_i + y_j) = (c, E)$, we conclude $R_i(c, E) = nR_j(c, E)$, as claimed.

Substep 2: If $c_i = qc_j$ for some $q \in \mathbb{Q}$, then $R_i(c, E) = qR_j(c, E)$.

We have $c_i = (r/m)\,c_j$ for some $r, m \in \mathbb{N}$. Let $\ell \in I \setminus \{i, j\}$ (this can be done since $|I| \geq 3$) and $N' \equiv \{i, j, \ell\}$. Let $(c', E') \in \mathscr{C}^{N'}$ be defined by $c'_i \equiv c_i$, $c'_j \equiv c_j$, $c'_\ell \equiv (1/m)\,c_j$ and $E' \equiv E + [1/(r+m)]\,E$. Let $y = R(c', E')$. By *consistency* and Step 1, $y_i = ry_\ell$ and $y_j = my_k$. By efficiency, we then obtain $y = (r, m, 1)\,[E/(r+m)]$. By *consistency*, $(y_i, y_j) = R(c'_N, y_i + y_j)$, and since $(c'_N, y_i + y_j) = (c, E)$, we conclude $R_i(c, E) = qR_j(c, E)$, as claimed.

Substep 3: For each $(c, E) \in \mathscr{C}^N$, $R(c, E) = P(c, E)$.

The result follows from Substep 2 and *claims continuity*.

Step 2: For each $N \in \mathscr{N}$ with $|N| \neq 2$, and each $(c, E) \in \mathscr{C}^N$, $R(c, E) = P(c, E)$.

If $|N| = 1$, the claim is obvious. If $|N| \geq 3$, the claim follows from Step 2 by an application of the Elevator Lemma, using the fact that P is *conversely consistent* and R *consistent*. □

We close this section with a comment on the independence of the axioms in our characterizations. It is obvious that large classes of rules would become admissible if either *equal treatment of equal groups*, *consistency*, or *replication invariance* were dropped. From de Frutos (1999) and Ju and Miyagawa (2002), it follows that in Theorems 1 and 3a, *claims monotonicity* can be dispensed with. See Section 5 for further discussion.

4. Claims problems with a continuum of agents

In this section, we consider problems with a large number of claimants whose claims are small. We model such situations by representing the set of claimants as a continuum (Aumann, 1964). We show that in this context, the proportional rule emerges as essentially (that is, up to measure zero), the only one to satisfy an appropriate version of *equal treatment of equal groups* for the model.

The set of claimants is the unit interval [0, 1]. Let $\mathscr{B}$ denote its Borel subsets. Let $\mathscr{L}^1_+$ denote the set of nonnegative $\mathscr{B}$-measurable and integrable functions from [0; 1] into $\mathbb{R}$. A ***problem*** is a pair $(c, E) \in \mathscr{L}^1_+ \times \mathbb{R}_+$ such that $E \leq \int_{[0,1]} c(t)\,dt$. Let $\mathscr{C}$ denote the set of all problems. An ***awards vector*** for $(c, E) \in \mathscr{C}$ is an element x of $\mathscr{L}^1_+$ such that $\int_{[0,1]} x(t)\,dt = E$, and for almost every $t \in [0, 1]$, $x(t) \leq c(t)$. A ***rule*** is a function defined on $\mathscr{C}$ that associates with each problem $(c, E) \in \mathscr{C}$ an awards vector for (c, E). A rule R satisfies ***equal treatment of equal groups*** if for each $(c, E) \in \mathscr{C}$, and for each pair $\{A, B\}$ of elements of $\mathscr{B}$ such that $\int_A c(t)\,dt = \int_B c(t)\,dt$, we have $\int_A R(c, E)(t)\,dt = \int_B R(c, E)(t)\,dt$. The ***proportional rule*** P is defined by

$$P(c,E)=\frac{E}{\int_{[0,1]} c(t)\,\mathrm{d}t}\,c,$$

if $\int_{[0,1]} c(t)\,\mathrm{d}t>0$, and $P(c,E)=0$ otherwise.

Theorem 6. *A rule R satisfies equal treatment of equal groups if and only if for each* $(c, E)\in\mathscr{C}$, *and for almost every* $t\in[0,1]$, $R(c,E)(t)=P(c,E)(t)$.[14]

Proof. It is trivial to verify that any rule as described in the theorem does satisfy *equal treatment of equal groups*. Conversely, let R be a rule satisfying *equal treatment of equal groups*. The claim is obvious if $\int_{[0,1]} c(t)\,\mathrm{d}t=0$. Otherwise, let $(c,E)\in\mathscr{C}$ and C be the measure on $\mathscr{B}$ whose density is c. Let r be the measure on $\mathscr{B}$ whose density is $R(c, E)$. Clearly, both C and r are countably additive and nonatomic. By Lyapunov's theorem (Aliprantis and Border, 1999, p. 444) $\{(C(A), r(A)): A\in\mathscr{B}\}\subset\mathbb{R}^2$ is convex. Let $A^*\in\mathscr{B}$ satisfy $0<C(A^*)<C([0,1])$. Then, by *equal treatment of equal groups*, if $C(A)=C(A^*)$, then $r(A)=r(A^*)$. Hence, there exists $\alpha\in\mathbb{R}$ such that $r=\alpha C$. By the definition of a rule, $r([0,1])=E$, so that

$$\alpha=\frac{E}{\int_{[0,1]} c(t)\,\mathrm{d}t}.$$

Thus,

$$r=\frac{E}{\int_{[0,1]} c(t)\,\mathrm{d}t}\,C,$$

so that for almost every $t\in[0,1]$,

$$R(c,E)(t)=\frac{E}{\int_{[0,1]} c(t')\,\mathrm{d}t'}\,c(t),$$

where the last equality follows from the Radon–Nikodym theorem (Aliprantis and Border, 1999; p. 437). □

5. Related literature and open questions

It is known that for each N with $|N|\geq 3$, the proportional rule is the only rule on $\mathscr{C}^N$ to satisfy *continuity* and ***no advantageous transfer***, the requirement that no group of

[14] It is of interest that the result holds for any rule defined on a domain consisting of one problem $(c, E)\in\mathscr{C}$.

agents should benefit by transferring claims among themselves (Moulin, 1988; Chun, 1988; Ju and Miyagawa, 2002).[15] Therefore, *no advantageous transfer* implies *group order preservation* whenever $|N| \geq 3$. This is of course not the case for the two-person case, since *no advantageous transfer* is vacuously satisfied, even by rules violating *order preservation*, to which *group order preservation* reduces then.

Another characterization of the proportional rule in the variable population framework is that, when the number of potential claimants is at least three, it is the only rule for which no two agents ever benefit from consolidating their claims into one and appearing as a single claimant, and no agent ever benefits from splitting his claim into two parts and being represented by two claimants whose claims are these two parts (O'Neill, 1982; Chun, 1988; de Frutos, 1999; Ju and Miyagawa, 2002). This is the property of ***no advantageous merging or splitting***.[16] Let us refer to it as Theorem A. Theorem 1 can be obtained as a corollary of Theorem A and the fact that *group order preservation* and *consistency* together imply *no advantageous merging or splitting*. We give the proof of this implication in Appendix A.

Finally we comment on a possible relation between Theorem 1 and Young's (1987) theorem according to which a rule satisfies *joint continuity* (with respect to all variables), *equal treatment of equals*, and *consistency* if and only if it admits a parametric representation.[17] *Equal treatment of equal groups* implies *equal treatment of equals*, but because *claims continuity* is weaker than *joint continuity*, the axioms of Theorem 1 are not stronger than those of Young's theorem. As argued earlier, *joint continuity* is a very natural property, so the weaker version of Theorem 1 obtained by replacing *claims continuity* by *joint continuity*—let us refer to it as Theorem B—is an interesting result. One possible way to prove Theorem B is to use Theorem 1 and the fact that the only parametric rule satisfying *group order preservation* is the proportional rule. Proving this fact turns out to be essentially the same thing as proving Theorem 1 directly, however. Moreover, there are significant benefits to an elementary and independent proof since Young's theorem is one of the deepest theorems of the theory of resolution of conflicting claims.

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[15] The reason for this multiple attribution is that Moulin (1988) considers a related but different model, and that Chun (1988) works with a more general notion of rules, by not requiring *non-negativity* or *claims boundedness*.

[16] This property is most often called *non-manipulability by merging or splitting*. O'Neill (1982) imposed axioms that were later shown to be redundant, such as *anonymity* and *agent-by-agent claims* continuity at at least one point. Chun (1988) considers a more general notion of rules (see footnote 15). de Frutos (1999) imposes *non-negativity* but not *claims boundedness* and establishes uniqueness of the proportional rule on the basis of *no advantageous merging or splitting* alone. Ju and Miyagawa (2002) establish uniqueness for the notion of a rule examined here, also on the basis of *no advantageous merging or splitting* alone. They derive it as a consequence of their characterization of the rule on the basis of *no advantageous transfer*.

[17] A parametric representation of a rule R is a function f defined over some product $[a, b] \times \mathbb{R}$, where $[a,b] \subseteq]-\infty, \infty L$ such that for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, there is $\lambda \in [a, b]$ such that for each $i \in N$, $R_i(c, E) = f(c_i, \lambda)$. The function f is required to be continuous, nowhere decreasing, and to be such that for each $c_i \in \mathbb{R}_+$, $f(c_i, a) = 0$ and $f(c_i, b) = c_i$.

Acknowledgements

William Thomson acknowledges support from NSF under grant SBR-9731431. An early version of this paper was presented at the International Meeting of the Social Choice and Welfare Society, Alicante, Spain, in July 2000. We thank Stephen Ching and the referees for very helpful suggestions.

Appendix A

Proof. (of Lemma 1, the Elevator Lemma) Let R be a *bilaterally consistent* rule, $\bar{R}$ a *conversely consistent* rule, and suppose that for each $N \in \mathcal{N}$ with $|N| = 2$, $R_{|\mathcal{C}^N} = \bar{R}_{|\mathcal{C}^N}$. Let $N \in \mathcal{N}$ with $|N| > 2$, $(c, E) \in \mathcal{C}^N$, and $x = R(c, E)$. We need to show that $x = \bar{R}(c, E)$. Since R is *bilaterally consistent*, then for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = R(c_{N'}, \Sigma_{N'} x_i)$. Since $R = \bar{R}$ on the subdomain of two-claimant problems, then for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = \bar{R}(c_{N'}, \Sigma_{N'} x_i)$. Since $\bar{R}$ is *conversely consistent*, $x = \bar{R}(c, E)$. □

Finally, we include the proof of a claim made in the concluding section.

Lemma 3. *If a rule satisfies equal treatment of equals and consistency, then it satisfies anonymity.*

Proof. Let R be a rule satisfying the hypotheses of the lemma. First, let $N, N' \in \mathcal{N}$ be such that $|N| = |N'|$, and π: $N \to N'$ be a bijection. Let $(c, E) \in \mathcal{C}^N$. We show that for each $i \in N$, $R_{\pi(i)}((c_{\pi(i)})_{i \in N}, E) = R_i(c, E)$. Suppose first that $N \cap N' = \emptyset$. Let $(c', 2E) \in \mathcal{C}^{N \cup N'}$ be defined by $c'_N = c$, and for each $i \in N'$, $c'_{\pi(i)} \equiv c_i$. By *equal treatment of equals*, for each $i \in N$, $R_i(c', 2E) = R_{\pi(i)}(c', 2E)$. Thus, $\Sigma_N R_j(c', 2E) = \Sigma_{N'} R_j(c', 2E) = E$. Applying *consistency* twice, for each $i \in N$, $R_{\pi(i)}((c_{\pi(i)})_{i \in N}, E) = R_{\pi(i)}(c', 2E) = R_i(c', 2E) = R_i(c, E)$.

Next, suppose that $N \cap N' \neq \emptyset$. Let $N'' \subset I \backslash (N \cup N')$ such that $|N''| = |N|$, let π': $N \to N''$ and π'': $N'' \to N'$ be bijections such that $\pi = \pi'' \circ \pi'$, and let $((c_{\pi'(i)})_{i \in N}, E) \in \mathcal{C}^{N''}$. Then, by the previous paragraph, $R_{\pi(i)}((c_{\pi(i)})_{i \in N}, E) = R_{\pi''(\pi'(i))}(((c_{\pi''(\pi''(i))})_{i \in N}), E) = R_{\pi'(i)}((c_{\pi'(i)})_{i \in N}, E) = R_i(c, E)$. □

Theorem 7. *Suppose $|I| = \infty$. If a rule satisfies equal treatment of equal groups and consistency, it satisfies no advantageous merging or splitting.*

Proof. Let R be a rule satisfying the hypotheses of the lemma. Let $N, N' \in \mathcal{N}$ such that $N \cap N' = \emptyset$. Let $(c, E) \in \mathcal{C}^N$ and $(c', E') \in \mathcal{C}^{N'}$ be such that $E = E'$ and Π be a partition of N such that there exists a bijection π: $\Pi \to N'$ such that for each $i \in N'$, $c'_i = \Sigma_{j \in \pi^{-1}(i)} c_j$. We will show that for each $i \in N'$, $R_i(c', E) = \Sigma_{j \in \pi^{-1}(i)} R_j(c, E)$.

Let $c'' \in \mathbb{R}_+^{N \cup N'}$ be defined by $c''_N \equiv c$ and $c''_{N'} \equiv c'$. Note that $(c'', 2E) \in \mathcal{C}^{N \cup N'}$. By *equal treatment of equal groups*, for each $i \in N'$, $R_i(c'', 2E) = \Sigma_{j \in \pi^{-1}(i)} R_j(c'', 2E)$. Therefore, $\Sigma_N R_j(c'', 2E) = \Sigma_{N'} R_j(c'', 2E) = E$. Thus, by *consistency* applied twice, for each $i \in N'$, $R_i(c', E) = R_i(c'', 2E) = \Sigma_{j \in \pi^{-1}(i)} R_j(c'', 2E) = \Sigma_{j \in \pi^{-1}(i)} R_j(c, E)$.

Now, appealing to Lemma 3, we conclude that R satisfies *no advantageous merging or splitting*. □

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