Group Signals

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Abstract

We present a model of information aggregation in which agents' information is represented through partitions over states of the world. We discuss three axioms, meet separability, upper unanimity, and non-imposition, and show that these three axioms characterize the class of oligarchic rules, which combine all of the information held by a pre-specified set of individuals.

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1 Introduction

The difficulty of appropriately defining rules for sharing information in differential information economies is well-documented. Efficiency and core concepts are necessarily information dependent. Privacy of information opens the door to strategic manipulation of information

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revelation on the part of agents. Early works on efficiency concepts in differential information economies include Wilson (1978) and Holmstrom and Myerson (1983).

Here we point out a different difficulty with information sharing. Imagine two agents, each of whom face two distinct signals about the true state of the world. Agent 1 observes whether or not the sun comes out, and Agent 2 observes whether or not the temperature falls below freezing. These are "weather" signals. Likewise, Agent 1 observes the demand for oranges in New York, whereas Agent 2 observes the demand for oranges in New England. These are "demand" signals. The two agents agree ahead of time on a prespecified rule that is to be used in aggregating signals, resulting in a group signal. The rule need not recommend sharing of all information between the agents. The agents can apply their rule to the weather signals; likewise they can apply their rule to the demand signals. The resulting aggregates are a group weather signal and a group demand signal.

On the other hand, each agent has a large signal about "weather and demand," which results from combining her private signal about weather with her private signal about demand. This agent uses this combined "weather and demand" signal to predict the price of orange futures. When aggregating the "weather and demand" signals across agents, the rule produces a group "weather and demand" signal. In general, it need not be the case that the group "weather and demand" signal is the same signal that results from combining the group weather and group demand signal. This fact opens the door to strategic manipulation of the rule—how signals are specified becomes relevant for determining the final group signal. In order to rule out the possibility of such manipulation, we require that these group signals always coincide: a property we call *meet separability.*

Formally, all informational content of a signal in our model is captured by a *partition* of the states of the world. Therefore, a signal is identified with a partition over states of the world.

We consider two other requirements that a rule should satisfy. The first, *upper unanimity*, says that when all agents face an identical signal, the group signal should be a coarsening of this one. (In general, we might require that the group signal is a coarsening of the coarsest common refinement; this turns out to be implied by our axioms.) The second, *non-imposition*, simply states that the rule is non-imposed; that is, any conceivable signal might be the realized signal of the group for some profile of individual signals.

These three axioms, taken together, characterize the *oligarchic* rules. A rule is oligarchic if there is a prespecified group of agents (the oligarchy) who determine the outcome of the rule. In our case, we take the group signal of an oligarchic rule to be the totality of information contained in the signals of the oligarchy (the coarsest common refinement of their signals). Our main result states that a rule satisfies meet separability, upper unanimity, and non-imposition if and only if it is oligarchic. Together with a basic symmetry axiom, this states that the only information sharing rule satisfying the three properties is the rule which always aggregates *all* information.

1.1 Related Literature

The oldest result on the aggregation of partitions is an Arrovian-style impossibility theorem discovered by Mirkin (1975), and refined by Barthélemy et al. (1986) and Fishburn and Rubinstein (1986).[1] This theorem characterized oligarchic rules as the class of rules satisfying an Arrovian-style independence condition and a unanimity condition. The independence

[1] See also Barthélemy (1988). More recently, Nehring (2006) provides a proof of Mirkin's theorem using tools created to study judgment aggregation.

axiom requires that individual signals about demand in New England be ignored when determining the collective signal about demand in New York. The unanimity condition applies when each individual receives the same signal. In that case, the commonly-received signal becomes the collective signal.

Miller (2008) investigates the concept of meet separability in the context of group identification. Vannucci (2008) studies the property in a more general model of group identification in which opinions may be in the form of partitions of the society.

Recently, others have independently proven a result similar to our own. Dimitrov et al. (2009) characterize oligarchic rules as the class of rules satisfying meet separability and two other axioms, *pareto-plus* and *non-triviality.* Pareto-plus requires that, if none of the individual signals distinguish between two states ("rain" and "clouds"), then the group signal also does not distinguish between those two states. Pareto-plus implies the upper unanimity axiom. The non-triviality axiom requires that the group signal must be informative for some profile of individual signals. Non-triviality is implied by the non-imposition axiom.

2 The Model

2.1 Notation and the Model

The set of **agents** is a finite set $N \equiv \{1, ..., n\}$ of individuals. **States of the world** are elements of a finite set Ω. A **signal** is a partition π over Ω. Formally, a signal can be identified with an equivalence relation $p \subseteq \Omega \times \Omega$. Two states stand in the relation if they cannot be distinguished according to the signal. Let $\mathscr{P}$ denote the set of equivalence relations. A **profile** is an N-dimensional vector of equivalence relations $P \equiv (P_1, ..., P_n) \in$

$\mathscr{P}^N$, one for each agent. An **aggregator** is a function $f : \mathscr{P}^N \to \mathscr{P}$ mapping from signal profiles into the aggregate signal.

For two K-dimensional vectors P and Q in $\mathscr{P}^K$, we denote by $\leq$ coordinatewise set inclusion, so that $P \leq Q$ if $P_i \subseteq Q_i$ for all $i \in K$. We denote by $\wedge$ coordinatewise intersection, so that $P \wedge Q \equiv (P_1 \cap Q_1, ..., P_k \cap Q_k)$. Viewing an equivalence relation as a partition, $P \leq Q$ if P **refines** Q. $P \vee Q$ is the finest common coarsening of P and Q, while $P \wedge Q$ is the coarsest common refinement.

2.2 The axioms and the main result

Our first axiom eliminates strategic manipulation by framing of signals. It was described at length in the introduction.

Meet Separability: For all $P, Q \in \mathscr{P}^N$, $f(P) \wedge f(Q) = f(P \wedge Q)$.

Our next axiom is a very weak unanimity property.

Upper Unanimity: If $P_i = P_j$ for all $i, j \in N$, then $f(P) \geq P_1 = ... = P_n$.

Lastly, we require that, for any signal, there is a signal profile realizing that signal.

Non-Imposition: For all $p \in \mathscr{P}$, there exists an $Q \in \mathscr{P}^N$ such that $f(Q) = p$.

The next definition defines a class of rules. An oligarchy is a maximal set of agents whose signals are combined to make the group signal.

Oligarchic Aggregator: There exists $S \subseteq N \setminus \varnothing$ such that $f(P) \equiv \wedge_{i \in S} P_i$.

Theorem 1. *An aggregator satisfies meet separability, upper unanimity, and non-imposition if and only if it is oligarchic. Moreover, all three axioms are independent.*

The proof of the theorem can be found in the appendix.

2.3 Anonymity

The principle of anonymity requires that each agent's signal be given the same weight. Formally, let individuals trade signals amongst themselves according to a permutation $\pi : N \rightarrow N$ of the set of agents. Anonymity requires that the collective signal should not change because agents have traded their signals.

Anonymity: For all π and $P \in \mathscr{P}$, $f(P) = f\left(P_{\pi(1)}, ..., P_{\pi(n)}\right)$.

The only oligarchic aggregator which satisfies anonymity is the one which includes all of the agents' information. The proof of the following proposition is trivial and is left for the readers.

Proposition 2. *An aggregator f satisfies meet separability, upper unanimity, non-imposition, and anonymity if and only if $f(P) \equiv \wedge_{i \in N} P_i$.*

2.4 Relationship to other results

The oldest result on the aggregation of partitions is an Arrovian-style impossibility theorem discovered by Mirkin (1975). Here we discuss the version of the Mirkin theorem found in Fishburn and Rubinstein (1986).

Fishburn and Rubinstein (1986) use the following unanimity property, which implies both the upper unanimity and non-imposition axioms.

Unanimity: If $P_i = P_j$ for all $i, j \in N$, then $f(P) = P_1 = ... = P_n$.

Corollary 3. *An aggregator satisfies meet separability and unanimity if and only if it is oligarchic.*

Fishburn and Rubinstein (1986) also use an Arrovian-style independence axiom, which requires that whether two states of the world are differentiable must not depend on the information about other states of the world. Whether we can collectively differentiate between snow and sleet should only depend on our individual abilities to differentiate between snow and sleet, and must not be affected by whether anyone can differentiate between either of the two and rain. For a signal or profile of signals R and a subset of events $X \subseteq \Omega$, let $R|_X$ denote the informational content of the signal(s) that pertains only to the events in X.

Independence: If $P|_{\{a,b\}} = Q|_{\{a,b\}}$ for some $a, b \in \Omega$, then $f(P)|_{\{a,b\}} = f(Q)|_{\{a,b\}}$.

Independence and unanimity together characterize oligarchic rules. For a proof, see Fishburn and Rubinstein (1986).

Theorem 4. *(Fishburn and Rubinstein, 1986) An aggregator satisfies independence and unanimity if and only if it is oligarchic. Moreover, both axioms are independent.*

3 Meet Homomorphisms and Inverse Functions

To prove Theorem 1, we introduce a more general result about finite lattices.

Let A, Z be finite lattices. Without loss of generality we denote the orders corresponding to the lattices by $\leq$. Any finite lattice has a maximal and minimal element. Without loss of generality we denote these by $\mathbf{1}$ and $\mathbf{0}$ respectively.

We say that a function $f : A \to Z$ is a meet-homomorphism if for all $a, b \in A$, $f(a \wedge b) =$

$f(a) \wedge f(b)$. Likewise, a function is a join-homomorphism if for all $a, b \in A$, $f(a \vee b) = f(a) \vee f(b)$.

A function $f : A \to Z$ is monotonic if for all $a, b \in A$, $a \leq b$ implies $f(a) \leq f(b)$. Defining $x < y$ to mean $x \leq y$ and $x \neq y$, say f is strictly monotonic if for all $a, b \in A$, $a < b$ implies $f(a) < f(b)$. We say that f is surjective if for all $z \in Z$, there exists $a \in A$ such that $f(a) = z$. We say that f is injective if for all a, b, $f(a) = f(b)$ implies $a = b$.

Theorem 5. *A function $f : A \to Z$ is a surjective meet-homomorphism if and only if there exists an injective join-homomorphism $g : Z \to A$ satisfying $g(\mathbf{0}) = \mathbf{0}$ which is a left inverse of f, and for which*

$$f(a) = \bigvee \{z \in Z : g(z) \leq a\}.$$

The proof of the theorem can be found in the appendix.

4 Conclusion

We have presented a model of information aggregation and have characterized oligarchic aggregation rules as the unique class of rules satisfying meet separability, upper unanimity, and non-imposition. Oligarchic aggregation rules combine all of the information received by a pre-selected set of agents. Other rules will necessarily violate one of the three axioms. Rules which violate the meet separability axiom allow the possibility of strategic manipulation—the group signals may be affected by the questions asked. Independent questions about the weather and about demand for oranges may lead to different conclusions about the price of orange futures than would follow from a more direct question. Rules which violate upper unanimity and non-imposition have very undesirable properties—the resulting group signals are derived from something other than the agents' information.

Of interest for future research is the investigation of the aggregation of modal operators. Instead of working with partitions directly, one would work with the corresponding knowledge operators, and investigate axioms placed directly on these.

Appendices

A Proofs

We prove the theorems in reverse order.

A.1 Proof of Theorem 5

Proof. Let the function f be a surjective meet-homomorphism. Because f is a surjection, for all $z \in Z$, the set $u_z \equiv \{a \in A : f(a) \geq z\}$ is nonempty.

Define $g(z) = \bigwedge u_z$. Because f is a meet homomorphism and A is finite,

$$f(g(z)) \quad = \quad f\left(\bigwedge \{a \in A : f(a) \geq z\}\right) = \bigwedge_{f(a) \geq z} f(a) \quad \geq \quad z.$$

Let $z \in Z$. Because f is a surjection, there exists $a_z \in A$ for which $f(a_z) = z$. It follows from the definition of g that $a \geq g(f(a))$. Therefore $a_z \geq g(f(a_z)) = g(z)$. Because f is a meet-homomorphism, it follows trivially that f is monotonic, and therefore that $z = f(a_z) \geq f(g(z)) \geq z$. Thus $f(g(z)) = z$.

To show that g is an injection, let $y, z \in Z$ and suppose that $g(y) = g(z)$. Then

$f(g(y)) = f(g(z))$ and therefore, $y = z$.

To show that g is monotonic, let $y, z \in Z$ such that $y \leq z$. For all $a \in A$, $f(a) \geq z$ implies $f(a) \geq y$. Consequently $\{a : f(a) \geq z\} \subseteq \{a : f(a) \geq y\}$ which implies that $g(z) \geq g(y)$.

Next, we show that g is a join-homomorphism; that is, for all $y, z \in Z$, $g(y \vee z) = g(y) \vee g(z)$. Monotonicity of g implies that $g(y \vee z) \geq g(y) \vee g(z)$. Monotonicity of f implies that $f(g(y) \vee g(z)) \geq f(g(y)) = y$. Similarly, $f(g(y) \vee g(z)) \geq z$, so that $f(g(y) \vee g(z)) \geq y \vee z$. It follows from the definition of g that $g(y) \vee g(z) \geq g(y \vee z)$. Therefore $g(y \vee z) = g(y) \vee g(z)$.

For all $a \in A$, $f(a) \geq \mathbf{0}$ which implies that $g(\mathbf{0}) = \mathbf{0}$.

Last, we show that for all $a \in A$, $f(a) = \bigvee \{z \in Z : g(z) \leq a\}$. Note that $\{z \in Z : g(z) \leq a\}$ is nonempty as $g(\mathbf{0}) \leq a$ for all $a \in A$. Let $a \in A$. Because $g(f(a)) \leq a$ it follows that $f(a) \leq \bigvee \{z \in Z : g(z) \leq a\}$. Next, let $z \in Z$ such that $g(z) \leq a$. By monotonicity of f, $z = f(g(z)) \leq f(a)$. It follows that $f(a) \geq \bigvee \{z \in Z : g(z) \leq a\}$.

To prove the converse, let $g : Z \to A$ be an injective join-homomorphism satisfying $g(\mathbf{0}) = \mathbf{0}$, and define for all $a \in A$, $f(a) = \bigvee \{z \in Z : g(z) \leq a\}$. Note that f is well-defined as $g(\mathbf{0}) \leq a$ for all $a \in A$.

First, we claim that $f(g(z)) = z$ for all $z \in Z$. To prove this, let $x, z \in Z$ such that $g(x) \geq g(z)$. Then because g is a join-homomorphism, $g(x) = g(x) \vee g(z) = g(x \vee z)$. Because g is an injection, it follows that $x = x \vee z$ and therefore $x \geq z$. Next, let $y \in Z$. Then

$$f(g(y)) = \bigvee \{z \in Z : g(z) \leq g(y)\} = \bigvee \{z \in Z : z \leq y\} = y.$$

This shows that f is a surjection.

To show that f is monotonic, let $a, b \in A$ such that $a \leq b$. Then for all $z \in Z$, $g(z) \leq a$ implies that $g(z) \leq b$, so that $f(a) = \bigvee \{z \in Z : g(z) \leq a\} \leq \bigvee \{z \in Z : g(z) \leq b\} = f(b)$.

Further, for all $a \in A$, $g(f(a)) \leq a$. This follows as $g(f(a)) = g(\bigvee \{z \in Z : g(z) \leq a\}) = \bigvee_{g(x) \leq a} g(x) \leq a$, where the second to last inequality follows from the fact that g is a join-homomorphism.

To see that f satisfies meet-separability, let $a, b \in A$. It follows from the monotonicity of f that $f(a \wedge b) \leq f(a) \wedge f(b)$. We have shown that $g(f(a)) \leq a$ and $g(f(b)) \leq b$. It follows from the fact that g is a join-homomorphism that $g(f(a) \wedge f(b)) \leq a \wedge b$. The monotonicity of f implies that $f(a) \wedge f(b) \leq f(a \wedge b)$. Therefore, $f(a \wedge b) = f(a) \wedge f(b)$.

□

A.2 Proof of Theorem 1

Proof. **Step 1**. The set $\mathscr{P}$ forms a lattice under set inclusion, and the set $\mathscr{P}^N$ thus forms a lattice under the product order. The meet separability and non-imposition axioms imply that f is a meet-homomorphism and is a surjection, respectively. As a consequence, Theorem 5 implies that there is an injective join-homomorphism $g : \mathscr{P} \to \mathscr{P}^N$ satisfying $g(\mathbf{0}) = \mathbf{0}$, for which

$$f(P) = \bigvee \{p \in \mathscr{P} : g(p) \leq P\}.$$

Let $J \equiv \{p \in \mathscr{P} : p > \mathbf{0}, p \geq r > \mathbf{0} \text{ implies that } p = r\}$.

Step 2. We show that there is a non-empty subset $S \subseteq N \setminus \varnothing$ such that, for all $p \in J$, $g(p)_i = p$ if $i \in S$, and $g(p)_i = \mathbf{0}$ if $i \notin S$.

Let $p \in J$. Because f is monotone and surjective it follows that $f(\mathbf{0}) = \mathbf{0}$. By upper unanimity, $f(p,...,p) \geq p$. By definition of g and the fact that g is a join homomorphism on a finite lattice, conclude $g(f(p,...,p)) \leq (p,...,p)$. From the monotonicity of g it follows that $(p,...,p) \geq g(f(p,...,p)) \geq g(p)$. Therefore, because $p > \mathbf{0}$ it follows that $(p,...,p) \geq g(p) > \mathbf{0}$. This implies that there is a subset $S \subseteq N \setminus \varnothing$ such that $g_i(p) = p$ if $i \in S$ and $g_i(p) = \mathbf{0}$ if $i \notin S$. If $|X| < 3$, we may proceed to the last step. Otherwise, define $S^p \equiv \{i \in N : g(p)_i = p\}$. Let $q \in J \setminus \{p\}$, and define S^q accordingly. To complete this step we must show that $S^p = S^q$.

Let $r, t \in J \setminus \{p\}$ such that $r \vee t \geq p$ and define S^r and S^t accordingly.[2] Because g is a join-homomorphism, it follows that $g(r) \vee g(t) = g(r \vee t) \geq g(p)$. Now $(g(r) \vee g(t))_i = g_i(p) \geq p$ if and only if $i \in S^r \cap S^t$, which implies that $S^p \subseteq S^r \cap S^t$. Next, because $t \vee r \geq p$ it follows that $p \vee r \geq t$ and $p \vee t \geq r$. By repeating this argument, it follows that $S^r \subseteq S^p \cap S^t$ and $S^t \subseteq S^p \cap S^r$. Therefore $S^p = S^t = S^r$.

If there exists $s \in J \setminus \{p, q\}$ such that $p \vee q \geq s$, this concludes the step. If not, we can always find $r, s, t \in J \setminus \{p, q\}$ such that $p \vee r \geq s$ and $q \vee r \geq t$ and therefore $S^p = S^r = S^q$.[3]

Step 3. We show that there is a non-empty subset $S \subseteq N \setminus \varnothing$ such that, for all $p \in \mathcal{P}$, $g_i(p) = p$ if $i \in S$, and $g_i(p) = \mathbf{0}$ if $i \notin S$.

We have shown that there exists $S \subseteq N$ such that, for all $q \in J$, $g_i(q) = q$ if $i \in S$ and $g_i(q) = \mathbf{0}$ if $i \notin S$. Let $P \in \mathscr{P}^N$. The result now follows as every nonzero element $P \in \mathscr{P}$ can be expressed as a finite join of elements of J, and as g is a join homomorphism.

Step 4. Completing the proof.

Let $p \in \mathscr{P}$. Recall $f(P) = \bigvee \{p \in \mathscr{P} : g(p) \leq P\}$. The statement $g(p) \leq P$ is true if

[2]The existence of such r, t follows as $|X| \geq 3$.

[3]Existence of r, s, t follows in this case as the nonexistence of $s \in J \setminus \{p, q\}$ such that $p \vee q \geq s$ implies that $|X| \geq 4$

and only if for all $i \in S$, $p \leq P_i$, which is true if and only if $p \leq \bigwedge_{i \in S} P_i$. Consequently, $f(P) = \bigvee \left\{ p \in \mathscr{P} : p \leq \bigwedge_{i \in S} P_i \right\} = \bigwedge_{i \in S} P_i$.

Independence of the Axioms. We present three rules; each satisfies two of the axioms but violates the third.

Rule 1. The degenerate rule, $f(P) = \mathbf{1}$ for all $P \in \mathscr{P}^N$. This rule trivially satisfies meet separability and upper unanimity but violates non-imposition.

Rule 2. The rule $f(P) = P_2$, if $P_1 = \mathbf{1}; f(P) = \mathbf{0}$, otherwise. This rule satisfies meet separability and non-imposition but violates upper unanimity.

Rule 3. The lattice polynomial, or majority rule: Let $G \equiv \left\{ S \subseteq N : |S| > \frac{N}{2} \right\}$. $f(P) = \bigvee_{S \in G} \bigwedge_{i \in S} P_i,$. This rule satisfies upper unanimity and non-imposition but violates meet separability.

This completes the proof.

□

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