

On Regret and Options - A Game Theoretic Approach for Option Pricing[†]

Peter M. DeMarzo, Ilan Kremer and Yishay Mansour

Stanford Graduate School of Business and Tel Aviv University

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ABSTRACT.

We study the link between the game theoretic notion of 'regret minimization' and robust option pricing. We demonstrate how trading strategies that minimize regret also imply robust upper bounds for the prices of European call options. These bounds are based on 'no-arbitrage' and are robust in that they require only minimal assumptions regarding the stock price process. We then focus on the optimal bounds and demonstrate that they can be expressed as a value of a zero sum game. We solve for the optimal volatility-based bounds in closed-form, which in turn implies the optimal regret minimizing trading strategy.

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1. Introduction

There is a growing literature in game theory on the strategic concept of regret minimization for games under uncertainty.¹ Regret is defined as the difference between the outcome of a strategy and that of the ex-post optimal strategy. This literature is based on earlier work by Hannan (1957) and Blackwell (1956) who studied dynamic robust optimization, and is related to more recent work on calibration and the dynamic foundations of correlated equilibria.² In this paper we consider a financial interpretation of regret minimization and demonstrate a link to the robust pricing of financial assets. In particular we focus on financial options, which we can think of as contracts that allow investors to minimize their regret when choosing an investment portfolio.

Using the link between regret minimization and option pricing, we then derive robust pricing bounds for financial options. The classic, structural approach to option pricing developed by Black and Scholes (1973) and Merton (1973), posits a specific stock price process (geometric Brownian motion), and then shows that the payoff of an option can be replicated using a dynamic trading strategy for the stock and a risk-free bond. No arbitrage then implies that price of the option must equal the cost of this trading strategy. But because empirical stock prices do not follow the process assumed by Black-Scholes-Merton, their argument is not a true arbitrage: the replication is perfect only for a very restricted set of price paths. While our results are weaker (we provide bounds, rather than exact prices), they are robust in that we do not assume a specific price process.

In sum, the goal of this paper is two-fold. First, we develop a finance-based interpretation for the notion of regret minimization by showing the link to robust (distribution-free) bounds for the value of financial options. Second, we look for the optimal such bounds.

¹This fast growing literature examines the statistical notion of calibration as well as dynamic foundation for correlated or Nash equilibrium. See Hart (2005), Foster Levine and Vohra (1999), and Fudenberg and Levine (1998) for excellent surveys.

²Regret minimization is equivalent to the concept of competitive analysis that is used to evaluate the performance of an algorithm in computer science. The competitive ratio of an algorithm is the maximum, over all realizations, of the ratio of the performance of the best ex-post algorithm to that of the given algorithm (see, e.g., Sleator and Tarjan (1985)).

The roots of regret minimization in game theory can be traced to Hannan (1957) and Blackwell's (1956) work on dynamic optimization when the decision maker has very little information about the environment. They considered a repeated decision problem in which in each period the agent chooses an action from some fixed finite set. Although the set of actions is fixed, the payoffs to these actions vary in a potentially non-stationary manner, so that no learning is possible. They show that in the limit, there is a dynamic strategy that guarantees the agent an average payoff that is at least as high as that from the ex-post optimal static strategy in which the same action is taken repeatedly. Thus, in terms of the long run average payoff, the agent suffers no regret with respect to any static strategy.

Hannan (1957) and Blackwell's (1956) provide foundations to later work in engineering, especially in computer science. Computer scientists are interested in dynamic optimization methods (referred to as "on-line algorithms") for environments in which a specific distribution of the uncertain variables is unknown. They have followed the view that in such environment one should maximize a relative objective function rather than an absolute one. In particular, they evaluate the worst-case loss relative to the optimal strategy if the uncertain variables were known in advance. This differs from the more traditional approach in economics that considers an absolute objective (e.g. Gilboa and Schmeidler (1989)) in such an environment. It is important to note that in this paper we do not take a stand on which is the right approach. Our results hold regardless of what one believes is the right way to optimize or what best describes behavior observed in practice.

We explore the link to financial markets by examining investment decisions in an uncertain environment. Here we can define regret as the difference between the investor's wealth and the wealth he could have obtained had he followed alternative investment strategies. By comparing the investor's payoff to that which could be attained from a buy and hold investment of a the stock or a bond, we can interpret regret as the difference in payoff between a dynamic trading strategy and a call option, allowing us to link regret minimization to no arbitrage upper bounds for option prices. We begin by adapting the Hannan-Blackwell results to an investment setting. To do so, we need to adjust for the fact that in their setting, per period payoffs are additive and drawn from a finite set. In an

investment context, payoffs are multiplicative, and bounds on the per-period returns will be required. While Hannan-Blackwell focus on limiting results (similar to the traditional work on regret minimization), to be useful in an investment context we consider minimizing regret over a finite horizon.

The option price bounds we derive using the Hanan-Blackwell approach do not depend on specific distributional assumptions for the stock price path, and so is in that sense robust. The most straightforward extension of the Hannan-Blackwell approach requires restricting the magnitude of the stock's return each period. A more natural restriction, and one which allows a direct comparison to the Black-Scholes-Merton framework, is to impose restrictions on the realized volatility, or quadratic variation, of the stock price path. In subsection 3.2, we show that simple momentum strategies (in which we invest more in the stock when its return is positive) is effective at limiting regret when the stock's quadratic variation is bounded. These strategies thus lead to bounds for option prices based on the stock's volatility.³

It is important to note that the strategies mentioned above are not necessarily optimal. There might be a lower upper bound corresponding to a hedging strategy that is cheaper. Hence, a natural question is what is the optimal bound/strategy? in section 4, we address this question. We show that it can be viewed as a solution to a finite horizon zero-sum game. Using this approach we compute the bound using dynamic programming and derive a simple closed-form solution. We also derive the optimal robust trading strategy, which is the lowest cost strategy with a payoff that exceeds the option payoff for any stock price path with a quadratic variation below a given bound. These returns also provide the optimal strategies for minimizing regret in our setting.

Finally we compare our price bounds to the Black-Scholes-Merton (BSM) model. Our optimal bounds exceed the BSM price – we can interpret the bounds as the BSM price corresponding to a higher ‘implied volatility.’ Indeed, the pattern of implied volatility determined by our bound resembles the ‘volatility smile’ that has been documented empirically in options markets. We also compare our trading strategy to the delta hedging

³Our results in this regard are related to work by Cover (19xx, 19xx) on the “universal portfolio,” a dynamic trading strategy designed to perform well compared to any alternative fixed-weight portfolio. We discuss the relationship of our results to Cover's in Section XX.

strategy of Black and Scholes. We show that it is similar in nature but that the agent's stock position is less sensitive to movements in the underlying stock price. This insures him against jumps that are not considered by Black and Scholes.

1.1. Illustration

Before turning to the more technical part of the paper it is useful to demonstrate some of the insights in this paper using two simple examples. We begin with an example that demonstrates the equivalence between regret minimization and robust bounds for option prices:

EXAMPLE 1 (Regret and option pricing): Suppose the current stock price is \$1 and the risk free interest rate is zero. In a regret framework, we measure the performance of a strategy by its maximal loss relative to the ex-post optimal asset choice. For example, a T -period dynamic trading strategy with a maximal loss of 20% implies that starting with \$1, our payoff at time T will exceed $0.8 \max\{1, S_T\}$, where S_T is the final stock price. To see how such a strategy can be used to bound the value of a call option on the stock, note that by scaling the strategy by $1.25 = 1/0.8$, we conclude that starting with \$1.25 our strategy would have a payoff that exceeds $\max\{1, S_T\}$. If we partly finance our strategy by borrowing \$1 initially, then after paying off our loan, our final payoff exceeds $\max\{0, S_T - 1\}$, which is the payoff of a T -period call option on the stock with a strike price of \$1. Thus, to avoid arbitrage, the value of the call option cannot exceed the upfront cost of \$0.25. The quality of our bound is determined by the maximal regret of our dynamic trading strategy relative to best static decision (which is either to buy bond or the stock). A loss of 20% translated into an upper bound of \$0.25 for the call option price, and a lower loss would provide a tighter bound.

We continue by presenting a simple example that demonstrates the link between the optimal bounds and zero sum games; it is based on the classic one-period binomial model.

EXAMPLE 2 (Optimal bounds and zero sum games): An investor and an adversary engage in a one-period game. The adversary decides on the return of the risky asset whose initial price is normalized to one. We assume that he can only choose a return of $r = \pm\sigma$ for $\sigma \in (0,1)$, or randomize between the two alternatives. The investor starts with zero wealth and decides on how many shares to buy, $\Delta \in R$. He finances this purchase by borrowing at a zero interest rate; hence, his final wealth is given by $(1+r)\Delta - \Delta = r\Delta$. The game is a zero sum game where the adversary's payoff is the expected difference between the payoff of at-the-money call option and the investor's wealth, $E\{\max\{0, r\} - r\Delta\}$.

In section 4, we prove that a version of the minimax theorem holds for this game (as well as the more general dynamic version we will introduce shortly).⁴ Based on the minimax theorem we can define the value of this simultaneous-move game. In equilibrium the adversary randomizes while the investor does not.

In this simple example it is obvious that the adversary must choose r so that $E(r) = 0$. Otherwise the investor can obtain an infinite through an appropriate choice of Δ . In the multi-period framework the conclusion would be that the adversary chooses the stochastic process to be a Martingale. If the adversary sets the expected return on the risky asset to be zero then the investors expected wealth is zero regardless of the number of shares he buys. Hence, the value of this game is simply the expected payoff of the option, which in our case is $\sigma/2$. In the more general setup this will lead us to a simple numerical procedure to compute the value of the game.

Still, the number of shares the investor buys cannot be arbitrary. For the adversary to randomize he must be indifferent between a return of σ and $-\sigma$; for this to happen the investor must buy $\Delta = 0.5$ shares. In the more general setup we argue that this leads to a differential equation. The number of shares matches the derivative of the value with respect to the stock price.

We have characterized the equilibrium of this game, the adversary sets $\Pr(r = x) = \Pr(r = -x) = 0.5$ while the investor buys $\Delta = 0.5$ shares; the value of the

⁴ We will need to take into account the fact that the set of the investor's strategies is not compact.

game is $\sigma/2$. This matches the solution of the classic one-period binomial model where only one price eliminates arbitrage. In this paper we consider a more general setup in which returns in each period can take more than two values. As a result the markets are incomplete and multiple prices are consistent with No-arbitrage. As we shall discuss in subsection 4.1, the value of the game in this case matches the highest price in this set or the optimal (lowest) upper bound. The investor's equilibrium strategy corresponds to the robust hedging strategy that ensures against any possible price path of the risky asset. Based on this, in section 4.3 we present closed form solution for the case when the quadratic variation is bounded.

1.2. Literature Review

While the Black-Scholes formula is one of the most useful formulas developed in economics, in recent years extensive empirical research has identified several anomalies in the data. In general the formula seems to generate prices for stock index options that are too low. Said another way, the implied volatility of the stock index computed based on the Black-Scholes formula is significantly higher on average than the ex-post realized volatility. In addition, this effect is more pronounced for call options whose strike price is low. This effect is often referred to as the volatility “smirk” or “smile.” As a response to these findings, there has been an active research (e.g., Pan (2002), Eraker, Johannes and Polson (2003), and Eraker (2004)) trying to modify the Black and Scholes formula to account for these discrepancies. These papers examine different stochastic processes for the index, with modifications that include jump processes and stochastic volatility models. The result of our study will complement this analysis by offering a new perspective. Rather than focusing on a specific formulation for the stochastic process we rely on a generic trading strategy that works with any evolution for the risky asset as long as it satisfies some bounds on returns and quadratic variation.

As a result of both academic and practical interest there are several papers that study the restrictions one can impose on the price of options. These papers are similar in spirit to our work as the goal is to provide a robust bound by relaxing the specific assumption made by Black and Scholes. Mykland (2000) considers a stochastic process that is more general than Black and Scholes assumption that the stock has a constant volatility. He

models the stock price as a diffusion process, but allows the volatility to be stochastic. In this case the market need not be complete and we might be unable to replicate a option payoff.⁵ Still he shows that one can use the Black-Scholes price as an upper bound if we take the volatility parameter to be the upper bound over all realizations of the average stochastic volatility. The reason for this can be traced to Merton's argument that in such a framework the Black and Scholes formula holds if the average volatility is known. While such bounds generalize Black and Scholes in a significant way they still impose significant restrictions on the stochastic process. For example, the price path is assumed to be continuous so the stochastic price has no jumps; such jumps were shown to be empirically important by Pan (2002), Eraker, Johannes and Polson (2003), and Eraker (2004). For example, the Merton observation fails in a discrete time version of Black and Scholes; the ability to trade continuously is critical. Still, Mykland's result, similar to our methodology, shows that an upper bound over the integral or average volatility can dramatically improve the bounds compared to the case in which one assumes an upper bound over the instantaneous volatility (see for example Shreve, El Karoui, and Jeanblanc-Picque (1998)).

An alternative approach to that taken here is developed by Bernardo and Lediot (2000) and Cochrane and Saa-Requejo (2000), who strengthen the no-arbitrage condition by using an equilibrium argument. Specifically they assume bounds for the risk-reward ratio that should be achievable in the market. Based on these bounds and existing market prices, they can then determine upper and the lower bounds for new securities that may be introduced into the market.

As mentioned earlier, our research is also related to research in Computer Science/Statistics. In particular there is a literature that applies competitive algorithms in the context of investments. Most of the literature follows the seminal work by Cover (1991). We follow the more conventional paradigm in economics of efficient market and hence provide a different interpretation. We argue that one should think of these trading algorithms as a way to super replicate an option under different conditions.

⁵ A complete market is one in which the existing assets allow all possible gambles on future outcomes.

2. Model

We consider a discrete-time n -period model which time is denoted by $j \in \{0, 1, \dots, n\}$. There is a risky asset (e.g., stock) whose value (price) at time n is given by S_n . We normalize the initial value to one, $S_0 = 1$, and assume that the asset does not pay any dividends. We denote by r_j the return between $j-1$ and j so that $S_j = S_{j-1}(1+r_j)$. Throughout this paper we assume that $r_j > -1$ so that the stock price is always positive; we call $r = r_1, \dots, r_n$ the *price path*. In addition to the risky asset we have a risk free asset (e.g., bond). Unless otherwise stated, we assume that the risk free rate is zero.

A dynamic trading strategy starting with $\$c$ in cash has initial value $G_0 = c$. At each period it distributes its current value G_j , between the assets, investing a fraction x_j in the risky asset and $1-x_j$ in the risk free asset. Since we assume zero interest rate, at time $j+1$ its value is $G_{j+1} = (x_j G_j)(1+r_{j+1}) + (1-x_j)G_j = G_j(1+x_j r_{j+1})$; its final value is G_n .

Let $C(K)$ be the value, at time $j=0$, of a European call option whose strike price is K that matures at time n . This is the present value (at $j=0$) of the final payoff at time T that is given by $\max\{0, S_n - K\}$. We consider restrictions on the possible price path that are represented by a subset of possible price paths, $\Psi \subset \mathcal{R}^n$; we assume Ψ to be compact and that $0 \in \Psi$. For example, in some cases we assume bounded quadratic variation so that $\Psi = \{r \mid r_j > -1, \sum_{j=1}^n r_j^2 < q^2\}$, in other cases we assume a bound on a single day return so that $\Psi = \{r \mid r_j > -1, |r_j| < m\}$, and sometimes we assume both restrictions apply.

Conditional on Ψ , we assume that there is *no arbitrage* in prices. Namely, for any two trading strategies (or financial securities) A_1 and A_2 , that start with cash $\$c_1$ and $\$c_2$, if for any price path in Ψ the future payoff of A_1 is always at least that of A_2 , then $c_1 \geq c_2$. If this were not true and $c_1 < c_2$, assuming that one can sell short assets (and strategies), there would be an arbitrage opportunity: Investing in A_1 and shorting A_2 would lead to a

time 0 gain of $c_2 - c_1$ without the possibility of loss in the future. As a result we have the following definition:

DEFINITION 1. We say that $c = C_\Psi(K)$ is an upper bound if there exists a dynamic trading strategy that starts with $\$c$ and for all possible price path in Ψ its final payoff, G_T , satisfies: $G_n \geq \max\{0, S_n - K\}$.

Our goal in this paper is to show how this is related to regret and what bounds can we obtain. Before proceeding to the next section we should discuss the importance of imposing certain restrictions on the price path given by Ψ . The first part of Merton (1973) addresses this question by asking what can be said about the value of a call option without making any additional assumption about the price path. The answer is that with zero interest rate one can only say that for $k > 0$:

$$C(k) \in [\max(0, S_0 - k), S_0)$$

Hence, the option is not more valuable than the underline asset. This is a very weak bound but it is tight as when allowing for arbitrary price paths the value of the call option can be arbitrary close to it.

3. Regret

As mentioned in the introduction, we first examine the concept of ‘regret minimization’. This concept is based on the seminal work by Hanan (1957) and Blackwell(1956). In this section we describe the basic ‘regret’ concept and demonstrate how one can apply the original method developed by Blackwell (1956) in the context of financial market. We also show how this yields a robust upper bound for at the money call option; later we consider different setup and improve on the bounds obtained here.

Consider a setup in which we repeatedly choose a single action among $I = \{1..I\}$ possible alternatives. Let $\pi_{i,j} \in R$ denote the payoff of alternative i at time j . We make almost no assumptions regarding these payoffs apart from assuming that differences in payoffs are uniformly bounded so that $\|\pi_{i,j} - \pi_{i',j}\| < m$ for all i, i', t for some $m > 0$. In particular the payoff may not follow a stationary distribution or any other distribution. We allow the

agent to randomize and describe a random strategy by $\xi_j \in \{1..I\}$ so that $\xi_j = i$ implies that at time j we choose the i 'th alternative. Our payoff at time t is given by $\pi_{\xi_j, j}$; our average payoff up to time n is given by $\frac{1}{n} \sum_{j=1}^n \pi_{\xi_j, j}$

Given the few assumptions we have made, we do not seek to maximize absolute performance. Instead, we consider a relative benchmark, which is the regret measure. For each alternative i , we focus on the average payoff up to time n , which is given by $\frac{1}{n} \sum_{j=1}^n \pi_{i, j}$. The time n regret of a given strategy measures how it compares to the best static strategy ex-post, $\max_i \{\frac{1}{n} \sum_{j=1}^n \pi_{i, j}\}$. A corollary of Blackwell's approachability implies that:

COROLLARY 1 (no asymptotic regret) There exists a randomized strategy so that for any $\delta > 0$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{j=1}^n \pi_{\xi_j, j} - \max_i \left\{ \frac{1}{n} \sum_{j=1}^n \pi_{i, j} \right\} \right] \geq_{a.s.} -\delta$$

In addition one can bound the expected convergence rates. Given any realization of payoffs, the expected distance converges at a rate of m/\sqrt{n} .

PROPOSITION 1. There exists a randomized strategy so that:

$$\frac{1}{n} E \left(\sum_{j=1}^n \pi_{\xi_j, j} \right) \geq \max_i \left\{ \frac{1}{n} \sum_{j=1}^n \pi_{i, j} \right\} - m \sqrt{(I-1)/n} \quad (1)$$

We provide the proof in the appendix only for the case when there are two alternatives, $I = 2$. While this holds also for $I > 2$, the proof is somewhat more involved and for our purpose the case $I = 2$ is sufficient. Specifically consider two alternatives that are based on the two financial assets that we described in the previous section: a risky asset whose net return at time t is given by r_t and a risk free asset with zero net return. We define payoffs by looking at log-returns by letting $\pi_{0,t} = 0$, $\pi_{1,j} = \ln(1+r_j)$, and assume that $\|\ln(1+r_j)\| < m$.

If at time j we choose at random a single alternative, $i = 0.1$ then we can use Proposition 2 to bound our *expected* regret. Instead, consider now a trading strategy based on the randomized strategy described above. We construct a deterministic strategy so that at time j we invest a fraction of $x_j \equiv E_j \xi_j$ in the risky asset and $1 - x_j$ in the risk free asset. Our return at time j is given by $1 + x_j r_j$, and our final payoff is given by $\prod_{j=1}^n (1 + x_j r_j)$. Since $x_j \in [0, 1]$ and $r_j > -1$ we have that:⁶

$$\ln(1 + x_j r_j) \geq x_j \ln(1 + r_j) = E(\pi_{\xi_j, j})$$

Hence we can conclude that:⁷

$$\sum_{j=1}^n \ln(1 + x_j r_j) \geq E\left(\sum_{j=1}^n \pi_{\xi_j, j}\right) \geq \max\{0, \sum_{j=1}^n \ln(1 + r_j)\} - m\sqrt{n}$$

which implies that our payoff *always* satisfies

$$\prod_{j=1}^n (1 + x_j r_j) \geq \exp(-m\sqrt{n}) \max\{1, S_n\} \quad (1)$$

In the limit our geometric payoff converges to the geometric average payoff of the best asset. For a finite horizon we approximate the best asset ex-post to a factor of $\exp(-m\sqrt{n})$ our multiplicative regret is $1 - \exp(-m\sqrt{n})$.

⁶For a given $x \in [0, 1]$ let $f(r) = 1 + xr$ and $g(r) = (1 + r)^x$. Note that

$f(0) = g(0) = 1$, $f'(0) = g'(0) = x$, and $g''(r) < 0$ for any $r > -1$. Since g is convex while f is linear in r we have that $f(r) \geq g(r)$ for $r > -1$.

⁷One needs to be careful here. Consider a two period model in which the stock price doubles itself in both periods with certainty. Suppose that an investor first chooses with equal probabilities whether to invest his entire wealth in the stock or nothing. He does not change his decision in the second period so in each period the expected fraction invested in the stock equals a half. This random strategy yields 1 with probability 0.5 and 4 with probability 0.5 so on average 2.5. Using the procedure outlined in the text we transform this strategy to a deterministic one by investing half of our wealth in the stock in both periods; this strategy yields 2.25 with certainty. However, once we look at log returns the randomized strategy yields on average $0.5 \ln(4) = \ln(2)$ while the deterministic one yields $2 \ln(1.5) = \ln(2.25)$. Hence, only when we look at logs, our deterministic portfolio performs better.

3.1. Regret based Bounds

In the previous section we have demonstrated how to transform the strategy in Blackwell to an investment strategy in financial markets. As we shall demonstrate this translates to an upper bound for at-the-money call option. More generally, to obtain bounds for different strike prices we consider a modified regret guarantee; we put different weights on the assets:

DEFINITION 2 A dynamic trading strategy has an (α, β) guarantee, if for any price paths in Ψ its final payoff, G_n , satisfies $G_n \geq \max\{\alpha, \beta S_n\}$.

To gain some intuition it is better to first examine a very simple trading strategy. Suppose we decide to use a buy and hold strategy in which we invest a fraction β in the risky asset and $\alpha = 1 - \beta$ in the risk free asset. The future payoff of this fixed portfolio, G_n , is

$$G_n = \alpha + \beta S_n \geq \max\{\alpha, \beta S_n\}$$

This implies that we implemented an (α, β) guarantee for $\beta = 1 - \alpha$. Compare the above to the payoff of a fixed portfolio of β call options each with a strike price of $K = \frac{\alpha}{\beta}$ combined with α invested in the risk free asset. Such a fixed portfolio yields at time n a payoff of exactly $\alpha + \beta \max\{0, S_n - (\alpha/\beta)\} = \max\{\alpha, \beta S_n\}$. By definition, the current price of this fixed portfolio is $\beta C(k = \frac{\alpha}{\beta}, T) + \alpha$. Since $G_n \geq \max\{\alpha, \beta S_n\}$, by the no arbitrage assumption, we have,

$$\beta C\left(\frac{\alpha}{\beta}, n\right) + \alpha \leq 1 \Rightarrow C\left(\frac{\alpha}{\beta}, n\right) \leq \frac{1 - \alpha}{\beta} = 1 = S_0$$

As mentioned before, S_0 is a simple known upper bound on the option price. Our goal is to construct a dynamic trading strategy that starts with \$1 and yields a future payoff that exceeds:

$$\max\{\alpha, \beta S_n\}$$

for some $\alpha + \beta > 1$. Such a strategy yields a non-trivial bound, as stated in the following claim,

PROPOSITION 2. Assume that all price paths are (q^2, m) price paths. A dynamic trading strategy with an (α, β) guarantee ensures that for a call option with strike price $K = \frac{\alpha}{\beta}$, $C^u(K, q^2, m, n) \leq \frac{1-\alpha}{\beta} = \frac{1}{\beta} - K$.

Based on the above claim and proposition 2 we can derive an upper bound for the value of at-the-money call option that is based on Blackwell:

$$\exp(m\sqrt{n}) - 1$$

This bound is quite high; in fact if $m\sqrt{n} > \ln(2)$ then our upper bound is higher than one which is the initial share price; hence to get a meaningful bound $m\sqrt{n}$ cannot be too high. In section 4.3 the optimal bound for this setup which will enable us to compare it to the above bound. A bound that depends on restricting the absolute per period return suffers from the fact that even for high frequency a stock return may be quite high. As a result trying to impose a uniform bound on the absolute return is likely to result in a bound which is too high. The other restriction that we consider that is based on bounding the quadratic variation is much more useful as it relies on a bound of a global property over the entire price path.

3.2. Bounds Based on Quadratic Variation- Universal Portfolios

Following the discussion above we focus our attention to trading strategies that are based on the quadratic variation.⁸ We introduce a momentum strategy that is useful in deriving upper bounds. We first describe a more general version in which we trade I different assets, and its goal is to have its value approximate the value of the best asset. Later we shall see how a simple application indeed yields the desired upper bound on the price of the option.

⁸ These strategies are similar in spirit to the "Universal Portfolios" approach by Cover (1996).

Consider I assets where we denote by $V_{i,j}$ the price of asset i at time j . We normalize the initial value of each asset to be one, i.e., $V_{i,0} = 1$. The value at time j satisfies $V_{i,j} = V_{i,j-1}(1 + r_{i,j})$, where $r_{i,j} \in [-m, m]$ is the immediate return of asset i at time j . Our trading strategy is based on what we refer to as ‘weights’, $\{w_{i,j}\}$. We fix the initial weights so that $\sum_i w_{i,0} = 1$, and then use the update rule $w_{i,j+1} = w_{i,j}(1 + \eta r_{i,j})$, for some parameter $\eta \geq 0$. At time t we form a portfolio where the fraction of investment in asset i is $x_{i,j} = w_{i,j}/W_j$ where $W_j = \sum_i w_{i,j}$. The value of trading strategy is initially, $G_0 = 1$, and $G_j = \sum_{i=1}^I (x_{i,j} G_{j-1})(1 + r_{i,j}) = G_{j-1}(1 + \sum_{i=1}^I x_{i,j} r_{i,j})$.

The following theorem, whose proof appears in the Appendix, summarizes the performance of our online algorithm.

PROPOSITION 3. Given parameters: $\eta \in \left[1, \frac{1}{m} \left(1 - \frac{1}{2(1-m)}\right)\right]$, and $\{w_{i,0}\}$, where

$\sum_i w_{i,0} = 1$, the momentum trading strategy described above, guarantees that for any asset i ,

$$\ln(G_n) \geq \ln(V_{i,n}) - \frac{1}{\eta} \ln \frac{1}{w_{i,0}} - (\eta - 1)q_i^2$$

where $q_i^2 = \sum_{t=1}^n r_{i,t}^2$, and $|r_{i,j}| < m < 1 - \sqrt{2}/2 \approx 0.3$.

Consider now the application to the special case we consider of only two assets. With a slight abuse of notation we let w_0 denote the amount invested in the risky asset and assume that we invest $1 - w_0$ in the risk free asset. Since we assume a zero interest rate we have $q_i^2 = 0$ for the risk free asset and conclude that:

COROLLARY 2 Given parameters: $\eta \in \left[1, \frac{1}{m} \left(1 - \frac{1}{2(1-m)}\right)\right]$, and $w_0 \in (0, 1)$, the momentum trading strategy described above, when applied to a risky asset and a risk free asset, guarantees that

$$\ln(G_n) \geq \max \left\{ \ln(S_n) - \frac{1}{\eta} \ln \frac{1}{w_0} - (\eta-1)q^2, -\frac{1}{\eta} \ln \frac{1}{1-w_0} \right\}$$

where $q^2 = \sum_{j=1}^n r_j^2$, and $|r_j| < m < 1 - \sqrt{2}/2 \approx 0.3$.

From Corollary 1 we have,

$$G_n \geq \max \{ \alpha, \beta S_n \}$$

where

$$\alpha(w_0, \eta) = (1-w_0)^{1/\eta} \quad \text{and} \quad \beta(w_0, \eta) = w_0^{1/\eta} e^{-(\eta-1)q^2}$$

Now consider the bound for a given strike price K . To find the best bound we can optimize over the two parameters η, w_0 , specifically we solve:

$$\begin{aligned} \beta^*(K) &= \max_{\eta, w_0} \beta(w_0, \eta) \quad \text{s.t.} \\ \frac{\alpha(w_0, \eta)}{\beta(w_0, \eta)} &= K \quad \text{and} \quad \eta \in \left[1, \frac{1}{m} \left(1 - \frac{1}{2(1-m)} \right) \right] \end{aligned}$$

One can simplify this problem by using $\frac{\alpha(w_0, \eta)}{\beta(w_0, \eta)} = K$ to solve for w_0 :

$$w_0(\eta, K) = \frac{1}{1 + K^\eta e^{-\eta(\eta-1)q^2}}$$

Hence, we need to solve the following maximization,

$$\beta^*(K) = \max_{\eta} w_0(\eta, K)^{1/\eta} e^{-(\eta-1)q^2} \quad \text{s.t.} \quad \eta \in \left[1, \frac{1}{m} \left(1 - \frac{1}{2(1-m)} \right) \right]$$

Let $\beta^*(K)$ be the solution to the above optimization, our bound is then given by

$$C(K, T) \leq C^u(K, q^2, m, n) \leq \frac{1}{\beta^*(K)} - K$$

4. Optimal Bounds

In prior sections we have looked at particular dynamic trading strategies and the option price bounds that they imply. In this section we consider optimal regret minimizing strategies, and determine the tightest (lowest) possible option price bounds. We focus on the following constraints:

$$\Psi_n(q, m) = \left\{ \sum_{j=1}^n r_j^2 \leq q^2, \max_j |r_j| \leq m \right\}$$

As mentioned before this is the more relevant constraint and it simplifies our exposition; still, most of our results hold for a more general Ψ . Let $V(S, K, q^2, m, n)$ be the minimal cost of a portfolio that super-replicates the option for all n -period stock price paths in $\Psi_n(q, m)$. We can define V recursively as follows. For $n=0$, V is equal to the option payoff:

$$V(S, K, q^2, m, 0) = \max\{0, S - K\}$$

For $n>0$, V is the cost of the cheapest portfolio whose payoff next period, after any allowable return, is sufficient to super-replicate the option from that point onward. Because a portfolio with value V and Δ shares of the stock has payoff $V + r S \Delta$, we have

$$\begin{aligned} V(S, K, q^2, m, n) &= \min_{\Delta, V} V \\ \text{s.t. } V + r S \Delta &\geq V(S(1+r), K, q^2 - r^2, m, n-1) \text{ for all } r \in \Psi_1(q, m) \end{aligned} \quad (2)$$

We first note that since the restrictions we consider are on the returns, we can conclude that

$$V(S, K, q^2, m, n) = K V(S/K, 1, q^2, m, n) \quad (3)$$

We also show in the appendix that:

LEMMA 1. (i) For a given S , $V(S, K, q^2, m, n)$ is convex in K . (ii) For a given K , $V(S, K, q^2, m, n)$ is convex in S .

Given (3), without loss of generality we focus on the case $K=1$; hence we suppress the second argument and write $V(S, q^2, m, n)$. Following the example in the introduction we begin by establishing the connection to zero sum games.

4.1. Optimal bounds as a zero sum game

Consider a zero sum game between an investor and an adversary. The adversary chooses a price path and the investor chooses a trading strategy that starts with $\$W$. The payoff to the adversary is the difference between the final values of the option and the investor's portfolio; the investor's payoff is minus this amount. As we shall demonstrate that the value of this game is the optimal option price bound.

While the game can be described as a one-shot game it is better to consider a dynamic (extensive form) representation. In each period the adversary chooses the next period return, r , or more precisely a random return θ , then the investors decide how many shares to buy, Δ . Formally, we consider the following recursive definition:

$$\text{For } n = 0: f(W, S, q^2, m, 0) = \max\{S - 1, 0\} - W$$

$$\text{For } n \geq 1: f(W, S, q^2, m, n) = \sup_{\theta \in \Sigma(q, m)} \inf_{\Delta} E f(W + \theta \Delta, S(1 + \theta), q^2 - \theta^2, m, n - 1)$$

where θ is the random variable that represents the next return, and $\Sigma(q, m)$ is the set of random variables whose magnitude is bounded by $\min(q, m)$.

The above formulation fits a setup in which the adversary moves first. The investor forms his portfolio after knowing the strategy of the adversary. The investor observes the distribution that the adversary has chosen but not the realized return. Using induction we show in the appendix that:

LEMMA 2. (i) $f(W, S, q^2, m, n) = f(0, S, q^2, m, n) - W$, (ii)

$f(0, S, q^2, m, n) \in [0, S]$, (iii) $E \inf_{\Delta} f(S \theta \Delta, S(1 + \theta), q^2 - \theta^2, m, n) \geq 0$ if and only

if $E \theta \neq 0$, (iv) $f(W, S, q^2, m, n)$ is continuous in q^2 , m and S .

Part (iii) implies that the adversary must use a Martingale measure. It is interesting to note that the Martingale property arises from the fact that otherwise the investor could obtain an infinite payoff. Using part (i) in the above Lemma we can write:

$$f(W, S, q^2, m, n) = \sup_{\theta \in \Sigma(q, m)} \inf_{\Delta} E \{ f(0, S(1 + \theta), q^2 - \theta^2, m, n - 1) - S\theta\Delta \} - W$$

Using part (iii) we can focus on θ that satisfy $E\theta = 0$, and the choice of Δ becomes irrelevant. Since θ is chosen from a compact set (based on C^* topology), continuity implies that

$$f(0, S, q^2, m, n) = \max_{\theta \in \Sigma(q, m)} E f(0, S(1 + \theta), q^2 - \theta^2, m, n - 1) \text{ s.t. } E\theta = 0 \quad (4)$$

Finally we argue that a version of the minimax theorem holds. We need to deal with the fact that Δ is chosen from a non-compact set and hence rely on the version of Sion (1958). Applying the minimax theorem and noting that the optimum is obtained with a finite number, we have

LEMMA 3. From the minimax theorem:

$$f(0, S, q^2, m, n) = \min_{\Delta} \max_{\theta \in \Sigma(q, m)} E \{ f(0, S(1 + \theta), q^2 - \theta^2, m, n - 1) - \theta S \Delta \}$$

Since $f(0, S, q^2, m, 0) = \max\{S - 1, 0\}$, comparing **LEMMA 3** with (2) and using (4) we have proven that:

PROPOSITION 4. The value of the above game matches the optimal upper bound for the value of the option, that is,

$$V(S, q^2, m, n) = f(0, S, q^2, m, n) \quad (5)$$

PROOF OF PROPOSITION 4

The proof immediately follows from the fact that (2) can be written as

$$V(S, q^2, m, n) = \min_{\Delta} \max_{r \in \Psi_1(q, m)} V(S(1 + r), K, q^2 - r^2, m, n - 1) - rS\Delta$$

Due to the fact that in $\min_{\Delta} \max_{\rho \in \Sigma(q,m)} E\{f(0, S(1+\rho), q^2 - \rho^2, m, n-1) - \rho S \Delta\}$ the return r is chosen after Δ is known **LEMMA 3** can be written as:

$$f(0, S, q^2, m, n) = \min_{\Delta} \max_{r \in \Psi_1(q,m)} f(0, S(1+r), q^2 - r^2, m, n-1) - rS\Delta$$

♦

Finally, note that V is bounded and non-decreasing in n . Hence, one can define:

$$\begin{aligned} f(0, S, q^2, m) &\equiv \lim_{n \rightarrow \infty} f(0, S, q^2, m, n) = \\ V(S, q^2, m) &\equiv \lim_{n \rightarrow \infty} V(S, q^2, m, n) \end{aligned}$$

By removing the constraint on the number of stock price movements, this bound equals the maximal value of the option when the stock price evolves continuously with the only constraints on the quadratic variation and the maximal jump size between trading opportunities.

4.2. Numerical Algorithm

We have shown that:

$$\begin{aligned} V(S, q^2, m, n) = & \max_{\rho} E[V(S(1+\rho), q^2 - \rho^2, m, n-1)] \\ \text{s.t.} & E\rho = 0, |\rho| \leq \min(q, m) \end{aligned} \quad (6)$$

The above computation can be simplified by noting that we only need to consider binary random variables, namely:

$$\rho \in \left\{ \begin{array}{l} r_u \text{ with probability } \pi \\ r_d \text{ with probability } 1 - \pi \end{array} \right\}$$

where $r_u > 0, r_d < 0, \pi r_u + (1 - \pi)r_d = 0$. Hence, we can write:

$$\begin{aligned} V(S, q^2, m, n) = & \max_{r_u, r_d} \left\{ \frac{-r_d}{r_u - r_d} V(S(1+r_u), q^2 - r_u^2, m, n-1) + \frac{r_u}{r_u - r_d} V(S(1+r_d), q^2 - r_d^2, m, n-1) \right\} \\ \text{s.t.} & r_u \in [0, \min(q, m)], r_d \in [-\min(q, m), 0] \end{aligned}$$

This problem is straightforward to solve numerically.

4.3. Closed form solution

Case 1: Bounding the per-period return

If $q^2 > n m^2$, the only relevant constraint is the bound m on the magnitude of the stock's return each period. In this case we suppress the argument q , and write $V(S, n, m)$ for the option price bound. The following result shows that in this case, the optimal bound is equivalent to the value of the option computed using an n -period binomial model.

PROPOSITION 5. The option price bound $V(S, n, m)$ is equal to the value of the option when the stock has a binomial distribution with returns $r_t \in \{-m, m\}$ each period.

PROOF: Because V is convex in S , $V(S(1+r), n-1, m)$ is convex in r . Therefore, (6) is solved with r taking on the extreme values of $-m$ and m with equal likelihood. ♦

Case 2: Bounding the quadratic variation

If $q < m \leq 1$, then the bound on the per-period return is not binding. We can suppress the third argument, we also look at the limiting case when $N \rightarrow \infty$ and write $V(S, q^2)$. In this case we have:

$$V(S, q^2) = \min_{\Delta} \max_{r \in [-q, q]} V(S, q^2 - r^2) - rS\Delta \quad (7)$$

and $V(S, 0) = \max\{0, S - k\}$. We argue that:

$$V(S, q^2) = \begin{cases} \frac{1}{2} \left(\frac{q}{1-q} \right) \left((1-q^2)S \right)^{1/q} & \text{for } S \leq s_0 = 1/(1-q^2) \\ \frac{1}{2} \left(\frac{q}{1+q} \right) \left((1-q^2)S \right)^{-1/q} + S - 1 & \text{for } S \geq s_0 = 1/(1-q^2) \end{cases} \quad (8)$$

One can numerically verify the above expression using the numerical procedure we described in subsection 4.2. We are unable to prove it analytically and instead provide few propositions that reveal how we found the above expression.

LEMMA 4. Let V be defined by (7), and let Δ be the number of shares in the optimal portfolio. If V is differentiable with respect to the second argument then $V_1(S, q^2) = \Delta$.

Based on the above proposition we conclude that if V is differentiable with respect to the second argument then (7) implies that:

$$V(S, q^2) + V_1(S, q^2)Sr \geq V(S(1+r), q^2 - r^2) \quad \text{for all } r^2 \leq q^2 \quad (9)$$

The boundary condition is given by:

$$V(S, 0) = \max\{0, S - K\} \quad (10)$$

When $r = \pm q$, (9) combined with the boundary condition yields:

$$V(S, q^2) - V_1(S, q^2)Sq \geq 0 \quad (11)$$

$$V(S, q^2) + V_1(S, q^2)Sq \geq S(1+q) - 1 \quad (12)$$

We derive the expression in (8) by conjecturing that at each point one of the constraints in (11) and (12) binds. Hence,

LEMMA 5. Let V be defined by (9) and (10), then $V(S, q^2) \geq V^*(S, q^2)$, where V^* is defined by (8).

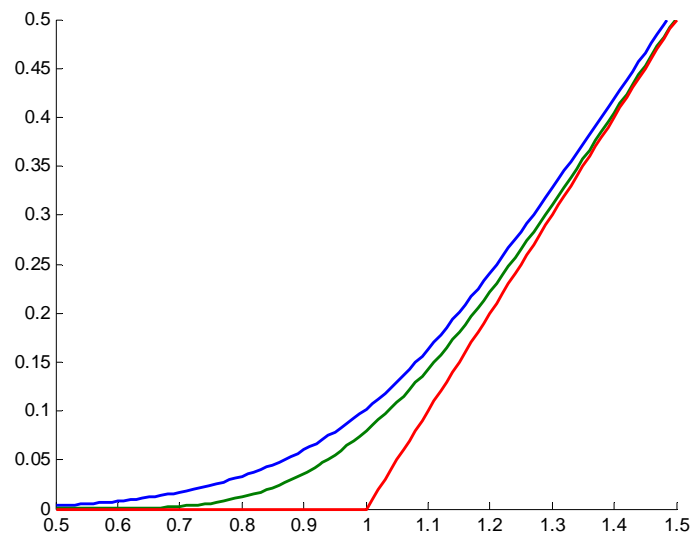
4.4. Comparison to Black and Scholes

Formally the Black and Scholes model is not nested in our model. This is due to the fact that Black and Scholes is a continuous time model while we consider a discrete time model. Nevertheless one can overcome this technical difference using the fact the Black and Scholes model is a limit of discrete time models.

Black and Scholes model with volatility σ^2 can be expressed as the limit of binomial trees where the quadratic variation is $q^2 = \sigma^2$. If we let m denote the daily return and N the number of periods then we keep $nm^2 = \sigma^2$ as N goes to infinity. Since the bounds we derive do not depend on the number of periods we can conclude that the Black and Scholes price

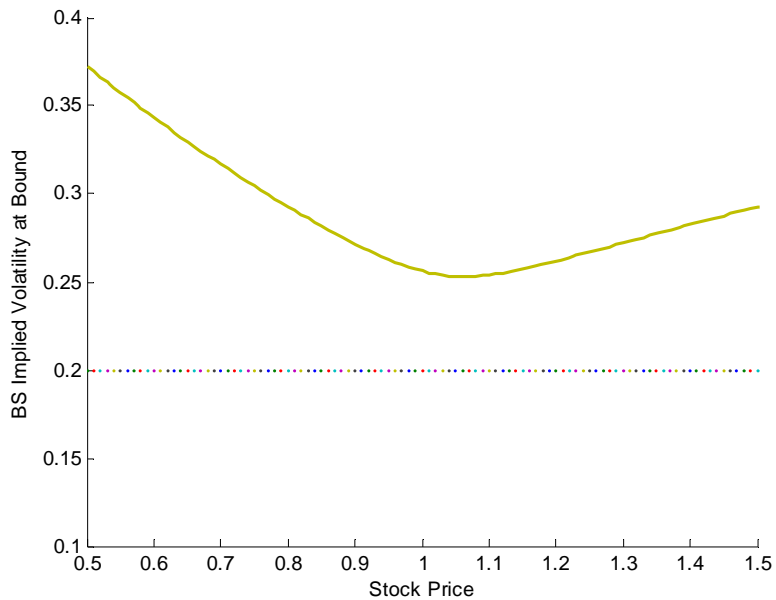
with volatility σ^2 is not higher than our upper bounds when σ^2 is a bound on the quadratic variation.

One may wonder about the restriction of constant quadratic variation in the context of Black and Scholes. When we look at a geometric Brownian motion at discrete intervals, the increments are normally distributed which may suggest that we allow for unbounded quadratic variation. However, in such a case the discrete Black and Scholes trading strategy fails to replicate the option payoff; moreover, the loss is unbounded. A different way of saying this is that while we can define as limit of different sequences of discrete time processes only particular sequences yield the Black and Scholes equation: i.e. binomial trees. At the continuous time limit almost surely a path is continuous and has a fixed quadratic variation.

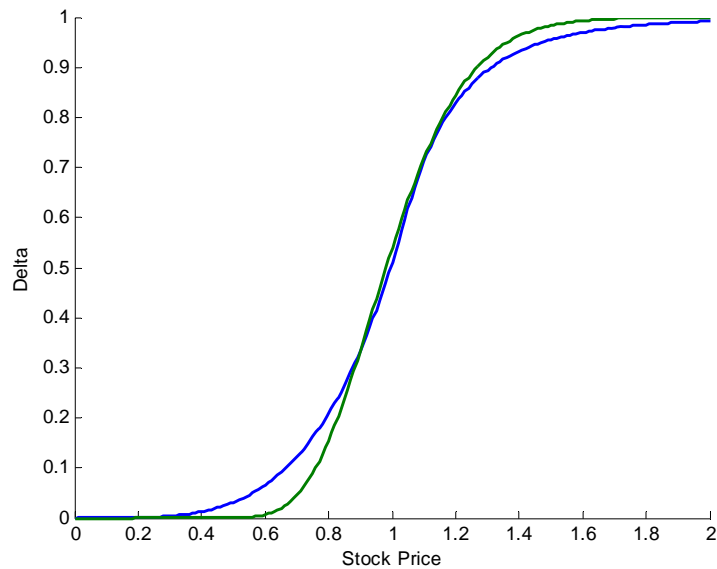


The above graph illustrates the relation between the Black and Scholes price and the expression we derive in (8). The bottom line is the intrinsic value of the option where $K = 1$ as a function of S ; hence, it is $\max\{0, S - 1\}$. The middle line represents the value of an option as according to Black and Scholes when $\sigma^2 = 0.2$, while the top line represent the expression in (8) when $q^2 = 0.2$

An alternative way of representing this relation is by looking at the implied volatility. That is we use the Black and Scholes formula to solve for the volatility:



It is interesting to see that the bound we derived has a smile feature as the implied volatility is higher for options that are deep out of the money or in the money. Finally we plot the number of shares in our hedging strategy, Δ , and compare it to the hedging strategy in Black and Scholes.



5. Conclusion

To be added

6. Appendix

PROOF OF PROPOSITION 1.

To apply Blackwell theorem one needs to describe the setup as a game with vector payoffs. We let $a^{i,j}$ denote the vector payoff of alternative i at time j when we define $a^{i,j} \equiv \pi_{i,j} - \pi_{i',j}$. Our payoff at time j is given by $a^{\xi_j,j}$, which measures our regret relative to the individual strategies. Following Hartt (2003) we let $\bar{a}^n = \frac{1}{n} \sum_{j=1}^n a^{\xi_j,j}$. Using this formulation our goal is to converge as fast as possible to the positive quadrant. In particular if we let $\delta_n = \text{dist}(\bar{a}^n, (R^+)^+)$ then,

$$\delta_n = \max \left\{ 0, \frac{1}{n} E \left[\sum_{j=1}^n \pi_{\xi_n,n} \right] - \max_i \left\{ \frac{1}{n} \sum_{t=1}^n \pi_{i,n} \right\} \right\}$$

Claim 5. When $I = 2$ there exists a randomized strategy so that $\delta_n \leq m/\sqrt{n}$

Proof. Our strategy is given by:

- If $\bar{a}_1^{n-1} \geq 0$ and $\bar{a}_2^{n-1} \geq 0$: we chose the first alternative so that $\xi_n = 1$ with certainty. We are already in the positive quadrant and hence our action is arbitrary.
- If $\bar{a}_1^{n-1} < 0$ and $\bar{a}_2^{n-1} < 0$: we randomize and choose the first action with probability $pr(\xi_n = 1) = \frac{\|\bar{a}_1^{n-1}\|}{\|\bar{a}_1^{n-1}\| + \|\bar{a}_2^{n-1}\|}$.
- If $\bar{a}_1^{n-1} \geq 0$ and $\bar{a}_2^{n-1} < 0$: we chose the risky asset which is the second alternative so that $\xi_n = 2$ with certainty.
- If $\bar{a}_1^{n-1} < 0$ and $\bar{a}_2^{n-1} \geq 0$: we chose the risk free asset which is the first alternative so that $\xi_n = 1$ with certainty.

We argue by that, $E_{n-1}(n^2 \delta_n) \leq (n-1)^2 \delta_{n-1}^2 + m^2$, this implies the result by induction. The claim then noting that $\delta_0 = 0$. We consider the different cases:

1. $\bar{a}_1^{n-1} \geq 0, \bar{a}_2^{n-1} \geq 0$: In this case δ_{n-1} is zero and hence

$$E_{n-1}(n^2 \delta_n) \leq (a_2^{\xi_n,n})^2 = (a_2^{2,n})^2 \leq m^2.$$

2. $\bar{a}_1^{n-1} < 0$ and $\bar{a}_2^{n-1} < 0$ In this case $E_{n-1}a^{\xi_{n,n}} = (\pi_{1,i} - \pi_{2,n})\left(\frac{\|\bar{a}_1^{n-1}\|}{\|\bar{a}_1^{n-1}\| + \|\bar{a}_2^{n-1}\|}, \frac{\|\bar{a}_2^{n-1}\|}{\|\bar{a}_1^{n-1}\| + \|\bar{a}_2^{n-1}\|}\right)$. Note that $\|E_{n-1}a^{\xi_{n,n}}\|^2 \leq m^2$ and that $E_{n-1}a^{\xi_{n,n}}$ is orthogonal to \bar{a}^{n-1} . As a result when we measure our distance from $(0,0)$ we obtain

$$\begin{aligned}\delta_n &\leq E_{n-1}(\|\bar{a}^n\|^2) = \frac{1}{n^2} \left((n-1)^2 \|\bar{a}^{n-1}\|^2 + \|E_{n-1}a^{\xi_{n,n}}\|^2 \right) \\ &\leq \frac{1}{n^2} \left((n-1)^2 \delta_{n-1}^2 + m^2 \right)\end{aligned}$$

3. $\bar{a}_1^{n-1} < 0$ and $\bar{a}_2^{n-1} \geq 0$ or $\bar{a}_1^{n-1} \geq 0$ and $\bar{a}_2^{n-1} < 0$: In this case we are getting closer to the positive quadrant compared to the case where we are already on one of the axis, that is, $\bar{a}_2^{n-1} = 0$ or $\bar{a}_1^{n-1} = 0$; this is essentially covered by the previous case.

♦

PROOF OF PROPOSITION 3

We first establish the following technical lemma.

LEMMA 6. Assuming that $m \in (0,1)$, $r > -m$, and $\eta \in \left[1, \frac{1}{m} \left(1 - \frac{1}{2(1-m)}\right)\right]$ then:

$$\eta \ln(1+r) \geq \ln(1+\eta r) \geq \eta \ln(1+r) - \eta(\eta-1)r^2$$

Proof. For the first inequality define a function

$$f_1(r) = \eta \ln(1+r) - \ln(1+\eta r)$$

We have $f_1(0) = 0$, and

$$f_1'(r) = \frac{\eta}{1+r} - \frac{\eta}{1+\eta r} = \frac{\eta(\eta-1)r}{(1+r)(1+\eta r)}$$

Hence, for $r > 0$ then $f_1'(r) > 0$ and for $r < 0$ we have $f_1'(r) < 0$. Therefore 0 is a minimum point of f_1 .

For the second inequality we have:

$$f_2(r) = \ln(1+\eta r) - \eta \ln(1+r) + \eta(\eta-1)r^2$$

Again, $f_2(0) = 0$ and

$$f_2'(r) = \frac{\eta}{1+\eta r} - \frac{\eta}{1+r} + 2\eta(\eta-1)r = \eta(\eta-1)r \left(2 - \frac{1}{(1+r)(1+\eta r)} \right)$$

We use a similar argument as before and claim that for $r > 0$ then $f_2'(r) > 0$ and for $r < 0$ we have $f_2'(r) < 0$. To show this we only need to verify that:

$$(1+r)(1+\eta r) \geq \frac{1}{2}$$

For $r > 0$ this clearly holds so we focus on $r < 0$. In this case, since the minimum of the expression is when $r = -m$, it is sufficient to guarantee that $(1-m)(1-\eta m) \geq 1/2$. Solving for η we get,

$$\eta \leq \frac{1/2 - m}{m(1-m)} = \frac{1}{m} \left(1 - \frac{1}{2(1-m)} \right)$$

and in addition we need that $m < 1$. ♦

We now can prove the proposition. For each $i = 1, \dots, N$ we get

$$\begin{aligned} \ln \frac{W_{n+1}}{W_1} &\geq \ln \frac{w_{i,n+1}}{W_1} = \ln w_{i,0} + \ln \prod_{j=1}^n (1 + \eta r_{i,j}) \\ &= \ln w_{i,0} + \sum_{t=1}^n \ln(1 + \eta r_{i,j}) \\ &\geq \ln w_{i,0} + \sum_{t=1}^n \left(\eta \ln(1 + r_{i,j}) - \eta(\eta - 1) r_{i,j}^2 \right) \\ &= \ln w_{i,0} + \eta \ln(V_i) - \eta(\eta - 1) Q_i \end{aligned}$$

where $V_{i,n}$ is the value of asset i at time n , and $Q_i = \sum_{j=1}^n r_{i,j}^2$. On the other hand, using $\ln(1 + \eta z) \leq \eta \ln(1 + z)$,

$$\begin{aligned} \ln \frac{W_{n+1}}{W_1} &= \sum_{j=1}^n \ln \frac{W_{j+1}}{W_j} = \sum_{j=1}^n \ln \sum_{i=1}^I (1 + \eta r_{i,j}) x_{i,j} \\ &= \sum_{j=1}^n \ln \left(1 + \eta \sum_{i=1}^I r_{i,j} x_{i,j} \right) = \sum_{j=1}^n \ln(1 + \eta r_{G,j}) \\ &\leq \sum_{j=1}^n \eta \ln(1 + r_{G,j}) = \eta \ln(G_n) \end{aligned}$$

Combining the two inequalities and dividing by $\eta \geq 1$, we get

$$\ln(G_n) \geq \frac{\ln w_{i,0}}{\eta} + \ln(V_{i,n}) - (\eta - 1) Q_i \quad \blacklozenge$$

PROOF OF LEMMA 1

(i) We need to show that:

$$\lambda V(S, K_1, q^2, m, n) + (1 - \lambda)V(S, K_2, q^2, m, n) \geq V(S, \bar{K}, q^2, m, n)$$

For all $\lambda \in [0, 1]$ where $\bar{K} = \lambda K_1 + (1 - \lambda)K_2$

The above holds since a portfolio of λ options with strike K_1 and $(1 - \lambda)$ options with strike K_2 dominates the payoff of a single option with a strike of $\lambda K_1 + (1 - \lambda) K_2$.

(ii) For a fix K we note that $V(1, K/S, q^2, m, n)$ is convex in S as given (i) it is a composition of two convex functions; we also note that it is increasing in S . The proof then follows as if $f(x)$ is increasing and convex in x then so is $xf(x)$ ♦

PROOF OF LEMMA 2.

(i) Follows from a simple argument based on induction that reveals that $f(W, S, q^2, m, n) = f(0, S, q^2, m, n) + W$ (ii) The fact that $f(W, S, q^2, m, n) \leq 0$ follows by induction since we can set $\rho = 0$ The fact that $f(W, S, q^2, m, n) \leq S$ follows by induction using (i) since we can set $\Delta = 1$ (iii) If $E\rho \neq 0$ then since $f(0, S, q^2, m, n - 1) \in [0, S]$ using the fact that Δ is unbounded we can choose Δ so that $Ef(W + \rho\Delta, S(1 + \rho), q^2 - \rho^2, m, n - 1) > 0$ which is a contradiction to (ii). (iv) The proof in both cases follows by induction using the fact that ρ and V are bounded. ♦

PROOF OF LEMMA 3.

Since $E\rho\Delta + Ef(0, S(1 + \rho), q^2 - \rho^2, m, n - 1)$ is linear in the distribution of ρ and Δ and since the space of ρ is compact under an appropriate topology (C^*) enables us to use Sion(1958) and conclude that:

$$f(W, S, q^2, m, n) = W + \sup_{\Delta} \inf_{\rho} E \left\{ \rho\Delta + f(0, S(1 + \rho), q^2 - \rho^2, m, n - 1) \right\}$$

The fact that the space of σ is compact and continuity in q enables to use minimization over q . Since the function is linear in Δ we need to show that we can restrict the domain of Δ to a compact subset of R . As mentioned before when considering $\inf_{\rho} \sup_{\Delta} E\rho\Delta + Ef(0, S(1 + \rho), \sigma^2 - \rho^2, n - 1)$ we can assume Δ to be zero. Hence we focus on $\sup_{\Delta} \inf_{\rho} E \left\{ \Delta\rho + f(0, S(1 + \rho), \sigma^2 - \rho^2, n - 1) \right\}$. Since we know that $f(0, S, \sigma^2, n) > -S$ we can restrict Δ so that $|\Delta| \leq S/\sigma$. ♦

PROOF OF LEMMA 4

Convexity of V in S implies that it is sufficient to show that:

$$\limsup_{\varepsilon \downarrow 0} \frac{V(S(1 + \varepsilon), \sigma^2) - V(S, \sigma^2)}{\varepsilon} \leq \Delta S \tag{13}$$

and

$$\liminf_{\varepsilon \downarrow 0} \frac{V(S, \sigma^2) - V(S(1 - \varepsilon), \sigma^2)}{\varepsilon} \geq \Delta S \quad (14)$$

Using (7) we know that by letting $r = \varepsilon$:

$$V(S, \sigma^2) + \Delta S \varepsilon \geq V(S(1 + \varepsilon), \sigma^2 - \varepsilon^2)$$

Thus for $\varepsilon > 0$:

$$\frac{V(S(1 + \varepsilon), \sigma^2 - \varepsilon^2) - V(S, \sigma^2)}{\varepsilon} \leq \Delta S$$

Hence for (13) one needs to show that

$$\lim_{\varepsilon \downarrow 0} \frac{V(S(1 + \varepsilon), \sigma^2 - \varepsilon^2) - V(S(1 + \varepsilon), \sigma^2)}{\varepsilon} = 0$$

which holds under our differentiability assumption. The proof for (14) is similar when we take $r = -\varepsilon$. ♦

PROOF OF LEMMA 5.

Note that (11) is equivalent to

$$V_l(S, q^2) \leq \frac{V(S, q^2)}{Sq}$$

which limits the rate of decline of V to the left of (s_0, v_0) . The steepest decline occurs if (11) holds with equality. The resulting differential equation has the solution

$$V^l(S, q^2) = c_l S^{1/q}$$

where c_l is chosen so that $V(s_0, q^2) = v_0$; that is, $c_l = v_0 s_0^{-1/q}$.

Similarly, to the right of (s_0, v_0) , (12) determines the minimal rate of increase of V . The slowest rate of increase occurs when (12) binds, or

$$V^r(S, q^2) = c_r S^{-1/q} + S - 1$$

with $c_r = (v_0 - s_0 + 1) s_0^{1/q}$.

Both V^r and V^l are increasing in c_r and c_l , and so are increasing in v_0 . For a given s_0 , what is the lowest possible value of v_0 ? Adding (11) and (12), we find that $V(S, q^2) \geq \frac{1}{2}(S(1+q) - 1)$. Therefore, if we set

$$v_0 = \frac{1}{2}(s_0(1+q) - 1)$$

then the true V exceeds V^r to the right of s_0 and V^l to the left of s_0 . We then find V^* by choosing s_0 to maximize V^r and V^l by maximizing c_r and c_l . In both cases, this occurs with

$$s_0 = 1/(1-q^2)$$

The result then follows by solving for c_r and c_l given (s_0, v_0) . ♦

7. References

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