California Institute of Technology
Division of Humanities and Social Sciences

Professor Bruno Gerard — BEM 104
Fall 2004 — Investments

Midterm Exam
Solutions

1. (10 points each, 20 points total) Are the following statements True, False or Ambiguous? Provide a short justification for your answer. (You are evaluated on your justification.)

   a. (10 points) Because the typical investor in the government bond market has a short-term horizon, the term structure should be upward sloping.
   Solution: Uncertain. The yield to maturity on a long-term zero coupon bond is an average of forward rates up to the maturity of the bond. Forward rates are equal to expected future interest rates plus a risk premium. If expected future interest rates are lower than current rates, then the yield to maturity on a long-term zero coupon bond could be lower than current rates. Hence even with positive risk premia, the term structure could slope down.

   b. (10 points) Protecting a liability by duration matching could be expensive because of all of the trading that is necessary.
   Solution: Uncertain. This of course depends on trading costs and the amount of trading really necessary. The point, however, is that duration matching requires rebalancing as time progresses and as interest rates move. (This assumes that the liability is not funding by zero coupon bonds that exactly match the maturity of the obligation.)

2. (25 points)
   You work for a pension fund. The board of the pension fund is worried about an obligation of $500 million that must be paid out of the pension fund in 5 years. They would like you to design an investment strategy today to finance that obligation. Suppose that the term structure is flat at 8%. (Remember this means that the yields to maturity on all pure discount bonds are 8%).
   a. How much do you need to invest today to fund the obligation?
   b. The board has advised you to invest this money in a combination of 3 year and 10 year zero coupon bonds. They would like you to set this up so that the pension fund is shielded from interest rate fluctuations as much as possible. (You are to protect against

parallel shifts in the yield curve.) How much would you invest in each of the bonds today?
   c. Suppose that immediately after you make your investment interest rates to increase to 9% at all maturities. Without doing any calculations would you expect the change in the value of your investment in the two bonds to be smaller or larger than any change in the value of your liability? Why? (Just words. No calculations!)

Solution:
   a. You need to invest:
$$\frac{500}{1.08^5} = 340.29$$
   b. The duration of the obligation is 5. Let *x* be the proportion invested in the 3 year bond. *x* must satisfy:
$$3 \times x + 10 \times (1 - x) = 5$$
   This implies that *x = 0.71*. Hence 0.71 × 340.29 = 243.06 million should be invested in the 3-year bond and 0.29 × 340.29 = 97.23 million in the 10-year bond.
   c. As interest rates rise the duration of the liability does not change. The duration of the assets falls however since the relative value of the 3-year bond increases. Hence there is convexity in the asset position. The value of the assets will fall by less than the value of the liability.

3. (25 points)
   You work on the bond trading desk of a large bank. You know the following market prices of zero coupon bonds with face value of $100:

Years to Maturity	1	2
Price	98	95

   a. What is the forward rate from year 1 to year 2?
   b. You also find that a 2-year coupon bond with annual coupons of $100 and face value of $1,000 is trading for $1,000. Is this consistent with the prices of the zero coupon bonds given above? If not, is there any way in which you could exploit the bond prices? How would you do it? Be very specific.

Solution:

   a. The forward rate is given by:
$$f_2 = \frac{B_1}{B_2} - 1 = \frac{98}{95} - 1 = 3.2\%$$
   b. The price of the coupon bond should be:

$$P = 100\times\frac{98}{100} + 1100\times\frac{95}{100} = 1143$$

Hence the market price is too low. I would buy this bond and sell one 1-year zero coupon bond and sell 11 2-year zero coupon bonds. The resulting cash flows would be:

	Year 0	Year 1	Year 2
Purchase of coupon bond	-1,000	100	1100
Sale of 1-year zero	98	-100	--
Sale of 2-year zero	11×95=1045	--	-1100
Net CF	+143	0	0

I would do this as many times as possible!

4. (30 points)
Consider two stocks "B" and "C" with the following properties:

$$E(\tilde{r}_B) = 0.10 \quad \sigma_B = 0.15 \quad \rho_{B,C} = 0$$

$$E(\tilde{r}_C) = 0.15 \quad \sigma_C = 0.30$$

The risk-free rate is 5%.
You are advising a client who has $1 million invested. Currently 50% of this money is in stock B and 50% is in stock C.
   a. What is the expected return and standard deviation of the return on your client's portfolio?
   b. What is the Sharpe Ratio of this portfolio?
   c. Your job is to come up with a better portfolio than the one currently chosen by your client. What combination of stocks B and C has the highest Sharpe Ratio possible? (Remember that the portfolio with the highest Sharpe Ratio is the "OP" portfolio.)
   d. If you wish to match the expected return of your client's current portfolio using the portfolio from part c, what would you do?

Solution:

a. The expected return is given by:

$$E(\tilde{r}_p) = w_B \times E(\tilde{r}_B) + w_C \times E(\tilde{r}_C) =$$

$$= 0.5\times 10\% + 0.5\times 15\% = 12.5\%$$

The variance of the portfolio return is given by:

$$\sigma_B^2 = (w_B \times \sigma_B)^2 + (w_C \times \sigma_C)^2 + 2\times w_B \times w_C \times \rho_{B,C} \times \sigma_B \times \sigma_C$$

$$= (0.5\times 0.15)^2 + (0.5\times 0.3)^2 + 2\times 0.5\times 0.5\times 0\times 0.15\times 0.3 = 0.028125$$

Hence the standard deviation is given by: $\sigma_p = \sqrt{0.02815} = 0.168$ or 16.8%.

b. The Sharpe ratio is given by:

$$SR_p = \frac{E(r_p) - r_f}{\sigma_p} = \frac{0.125 - 0.05}{0.168} = 0.45$$

c. The weight on stock B in the OP portfolio is given by:

$$w_B^p = \frac{E(\tilde{r}_B^e)\sigma_C^2 - E(\tilde{r}_C^e)\rho_{BC}\sigma_B\sigma_C}{E(\tilde{r}_B^e)\sigma_C^2 + E(\tilde{r}_C^e)\sigma_B^2 - \left[E(\tilde{r}_B^e) + E(\tilde{r}_C^e)\right]\rho_{BC}\sigma_B\sigma_C}$$

In this case:

$$w_B^p = \frac{0.05\times 0.3^2 - 0.10\times 0}{0.05\times 0.3^2 + 0.10\times 0.15^2 - [0.05 + 0.10]\times 0} = 0.667$$

d. The expected return on this new portfolio is:

$$E(\tilde{r}_{OP}) = 0.667\times 0.10 + 0.333\times 0.15 = 11.67\%$$

Let *w* be the weight on this portfolio and *(1 - w)* on the risk-free security. The *w* must satisfy:

$$(1-w)\times 5\% + w\times 11.67\% = 12.5\%$$

This gives *w = 1.125*. Hence I need to invest 112.5% in the portfolio "OP" and borrow 12.5% of my net asset value at the risk free rate the portfolio. This means I borrow $1 million ×0.125 = $125,000 and invest $1 million × 1.125 × 0.667 = $750,000 in stock B and $1 million × 1.125 × 0.333 = $375,000 in stock C.