California Institute of Technology
Division of Humanities and Social Sciences

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BEM 104
Investments

Example Midterm Exam Questions
Version with Solutions

1. Are the following statements True, False or Ambiguous? Provide a short justification for your answer. (You are evaluated on your justification.)

(a) (5 points) In order for investors to be willing to invest their money for the long run, the yield to maturity on a 15 year bond must always be greater than the yield to maturity on a 5 year bond.
Solution:
False. The yield on a 15 year bond could be lower if investors expect lower interest rates between year 5 and year 15. This could result in forward rates between years 5 and 15 that are much lower than the yield to maturity on a 5 year bond. For example suppose that both bonds are pure discount bonds. Suppose that $YTM_5 = 10\%$ and $f_6 = f_7 = \quad = f_{15} = 5\%$. Then the price of a 15 year bond with \$100 face value is:

$$B_{15} = \frac{100}{(1.10)^5 \times (1.05)^{10}} = 38.12$$

This yields:

$$YTM_{15} = \left(\frac{100}{38.12}\right)^{1/15} - 1 = 6.6\%$$

(b) (5 points) If two assets are negatively correlated, then there is a portfolio of the two assets with a variance of zero.
Solution:
False in general. The variance of a portfolio is given by:
$\sigma_p^2 = w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\rho_{AB}\sigma_A\sigma_A$
where *w* is the percent of investment in asset A. If $\rho_{A,B} = -1$ then $\sigma_p = 0$ if $w = \frac{\sigma_B}{\sigma_A + \sigma_B}$. Otherwise σ_p^2 is larger than zero since:

$$\begin{aligned}\sigma_p^2 &= w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\rho_{AB}\sigma_A\sigma_B \\ &> w^2\sigma_A^2 + (1-w)^2\sigma_B^2 - 2w(1-w)\sigma_A\sigma_B \\ &= 0 if w = \sigma_B/(\sigma_A + \sigma_B)\end{aligned}$$

(c) (5 points) Consider an investor who currently has substantial savings to devote to buying a house, but she would like to buy the house 5 years from now. A sequence of short term investments would mean that the investor

would have to periodically reinvest her money. Therefore the investor should invest in any bond with a maturity of at least 5 years. For example, she could invest in a 10 year bond.
Solution:
Uncertain. If she invests in a 10 year bond she faces price risk. She should match the duration of her obligation (house purchase) with the duration of the bond investment. If a 5 year zero coupon bond is available that would work perfectly. Otherwise she should invest in several bonds so that the duration of the resulting portfolio is 5. If the 10 year bond is a coupon bond it is possible that the duration is 5 in which case it would acceptable

(d) (5 points) Because of the potential for diversification if two stocks are not perfectly correlated a portfolio that invests 50% in each stock must be less risky that a portfolio invested just in one of the stocks.
Solution:
False in general. Although there is a benefit to diversification, if one of the stocks is very risky then the portfolio could be riskier than an investment in the low risk stock. For example suppose that $\sigma_1 = 0.1$, $\sigma_2 = 0.3$ and $\rho_{1,2} = 0$.
The variance of the 50-50 portfolio is:
$\sigma_p^2 = (0.5\times0.1)^2 + (0.5\times0.3)^2 + 2\times0.5\times0.5\times0\times0.1\times0.3 = 0.025$
Hence $\sigma_p = \sqrt{0.025} = 0.1581$. This has higher risk than investing entirely in stock #1.

2. (25 points:)
You are in charge of the bond trading and forward loan department of a large investment bank. Market prices for 1, 2 and 3 year pure discount bonds (zero coupon bonds) with face value of \$100 are displayed on your computer terminal as follows:

Years to Maturity	1	2	3
Price	98	96	90

(a) A new summer intern from Harvard has just told you that he thinks that 3 year treasury notes with annual coupons of \$30 and face value of \$1,000 are trading for \$1,000. Would you ask the intern to recheck the price of this coupon bond? If so, why?

(b) A customer approaches you looking for a quote on a loan of \$20 million dollars to be received by the customer one year from now. The customer will repay the loan two years from now. What forward interest rate would you quote for your customer?

(c) Suppose that your customer is willing to enter into the loan agreement of part 2b. How would you structure your holdings of pure discount bonds so that you can exactly match the future cash flows of this loan?

Solution:

(a) This bond should trade at the price:

$$\text{Price} = \frac{98}{100}\times 30 + \frac{96}{100}\times 30 + \frac{90}{100}\times 1{,}030 = 985.20$$

Since this is different than the price the intern thinks it should be, I would ask him to recheck his numbers.

(b)

$$f_2 = \frac{B_1}{B_2} - 1 = \frac{98}{96} - 1 = 2\%$$

I would quote a rate of 2%.

(c) I would suggest the following transactions:

- Buy 200,000 one year pure discount bonds at a price of 200,000 × 98 = 19,600,000.
- Finance this by selling 19,600,000/96 = 204,167 two year pure discount bonds.

3. (25 points)
Suppose that the yield to maturity on all bonds is 5% (in other words, the yield curve is flat at 5%). You have an obligation to pay $10M in 4 years. You have only two investment opportunities: a 1 year Treasury strip with face value of $100 and a perpetuity that pays an annual coupon of $10 each year.

(a) How much money do you have to invest today in the bond market to entirely fund your obligation?

(b) How would you structure your holdings of the 1 year Treasury strip and the perpetuity so that you are protected against the risk of interest rate fluctuations? How many 1 year Treasury strips and how many perpetuities would you buy?

Solution:

(a) I would have to invest the present value of this obligation, which is:

$$\frac{10M}{1.05^4} = 8{,}227{,}024.75$$

(b) The duration of my obligation is 4. I would like a bond portfolio with this same duration. The duration of the 1 year strip is 1. The duration of the perpetuity is:

$$D_{\text{perpetuity}} = \frac{Y}{Y-1} = \frac{1.05}{.05} = 21$$

If x is the percentage of my investment in the 1 year strip, then I need:

$$x \times 1 + (1-x)\times 21 = 4$$

This implies that x is 0.85. Hence 85% of my money should be in the 1 year strip and 15% of my money should be in the perpetuity. I should invest 0.85 × 8,227,024.75 = 6,992,971.04 in the 1 year strip and 0.15 × 8,227,024.75 = 1,234,053.71 in the perpetuity.
The price of the one year strip is:

$$B_1 = \frac{100}{(1.05)} = 95.24$$

The price of the perpetuity is:

$$P_{perpetuity} = \frac{10}{.05} = 200$$

Hence I should buy 6,992,971.04/95.24 = 73,426.20 1 year strips and 1,234,053.71/200 = 6,170.27 perpetuities.

4. (30 points)
Suppose that you are considering investing in two stocks. After analyzing the two stocks you think that there are two possible states for the economy over the next year: “Good" and “Bad." Each state is equally likely (probability 0.5). The returns of the two securities in each state are as follows:

State	Stock 1 Return	Stock 2 Return
Good	30%	5%
Bad	10%	10%

(a) What is the expected return and standard deviation of each stock return?

(b) What is the covariance and the correlation between the two stock returns?

(c) Draw a picture to illustrate the tradeoff between risk and return that is available by investing in these two stocks.

(d) Suppose that a risk-free investment of 5% is also available. Does this present a profit opportunity to you? Why or why not?

Solution:

a) The expected return on stock #1 and stock #2 are:

$$E(\tilde{r}_1) = 0.5\times 0.30 + 0.5\times 0.10 = 0.20 \text{ or } 20\%$$

$$E(\tilde{r}_2) = 0.5\times 0.05 + 0.5\times 0.10 = 0.075 \text{ or } 7.5\%$$

The variances of the stock returns are:

$$\sigma_1^2 = 0.5\times(0.3-0.2)^2 + 0.5\times(0.1-0.2)^2 = 0.01$$

$$\sigma_2^2 = 0.5\times(0.05-0.075)^2 + 0.5\times(0.1-0.075)^2 = 0.000625$$

The standard deviations of the stock returns are:

$$\sigma_1 = \sqrt{0.01} = 0.1 \text{ or } 10\%$$

$$\sigma_2 = \sqrt{0.000625} = 0.025 \text{ or } 2.5\%$$

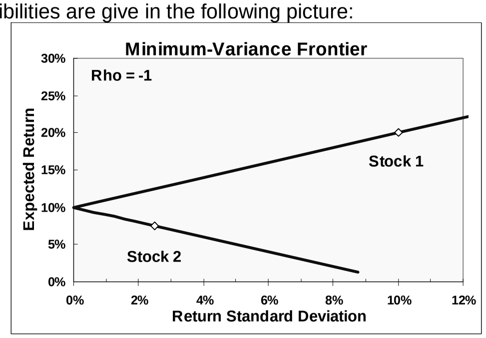
b) The covariance between the two stocks is:

$\sigma_{1,2} = 0.5\times(0.3-0.2)\times(0.05-0.075)+0.5\times(0.1-0.2)\times(0.1-0.075) = -0.0025$

The correlation between the two stocks is

$$\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_2\sigma_2} = \frac{-0.0025}{0.1\times0.025} = -1$$

c) The possibilities are give in the following picture:



d) It is possible to create a risk-free return in the two stocks by holding 20% in stock #1 and 80% in stock #2. This has a return of 10%. As a result there is a significant profit opportunity (an arbitrage opportunity). I would borrow at the risk-free rate and invest in a stock portfolio with 20% in stock #1 and 80% in stock #2.

5. (30 points)
You work in the bond trading group of a large investment bank. You are told that 1, 2 and 3 year treasury strips (zero coupon bonds) are currently trading at the following implied yields to maturity

Time to Maturity (*j*)	1	2	3
YTM_j	5%	4%	8%

(a) The commercial loan department of your firm has asked for your help in constructing a forward loan for one of their customers. This customer would like to enter into a contract to borrow $100 million from your firm a year from now to be repaid 2 years from now. The customer is willing to pay 7% interest on the loan. Should your firm enter into this contract? If so, how would you structure your holdings of Treasury strips so that your firm can exactly match the cash flows required by the loan.

(b) You are told that a 3 year U.S. Treasury note is to be issued today with an annual coupon of $50 and a face (principal) value of $1,000. What is the highest price that your group should pay for this bond?

(c) Suppose that you purchased the bond in part 5b at the price that you calculated. It is now one year later and you just received the first coupon payment on the bond. At this time the yields to maturity on 1, 2 and 3 year Treasury strips are:

Time to Maturity (*j*)	1	2	3
YTM_j	6%	7%	8%

If you were to sell the bond now, what rate of return would you realize on your investment in the bond?

Solution:

a) Assuming that the face value of Treasury strips is 100, the prices of the 1, 2 and 3 year strips are given by:

$$B_1 = \frac{100}{1.05} = 95.24$$
$$B_2 = \frac{100}{1.04^2} = 92.46$$
$$B_3 = \frac{100}{1.08^3} = 79.38$$

The implied forward rate between years 1 and 2 is given by:

$$f_2 = \frac{B_1}{B_2} - 1 = \frac{95.24}{92.46} - 1 = 3\%$$

Since the customer is willing to pay 7% the firm should agree to the contract. To match the cash flows the firm could do the following:

a. Buy 1 million 1 year strips yielding $100 million in year 1 to be delivered to the customer. This costs $95.24 million.
b. Finance the purchase of the 1 year strips by selling 95.24/92.46 = 1.03 million 2 year strips. This creates a liability in year 2 of $103 million. Since the customer willing to pay $107 million in year 2, this liability is more than covered.

b) The most that you should pay for this bond is:

$$P = \frac{50}{1.05} + \frac{50}{1.04^2} + \frac{1050}{1.08^3} = 927.37$$

c) The price of the bond should now be:

$$P_{New} = \frac{50}{1.06} + \frac{1050}{1.07^2} = 964.28$$

The realized rate of return is:

$$rr = \frac{50+964.28}{927.37} - 1 = 9\%$$

6. (25 points)
You have been hired by a small pension fund to help them design a bond portfolio to fund a $10 million obligation that will come due in 4 years. The managers of the fund would like to use a 2 year zero coupon bond along with an 8 year zero coupon bond to fund this obligation. Suppose that the yield curve is flat so that the yields to maturity on all zero coupon bonds are 5%.

(a) Design a portfolio of the two bonds that will protect the pension fund from fluctuations in interest rates.

(b) Suppose that right after you create this portfolio, the yield curve shifts to 6% at all maturities. Calculate what you expect the future value of the investment in the two bonds to be in year 4. Do you meet the obligation of the fund? Explain any difference.

Solution:

(a) The present value of the obligation is:

$$PV = \frac{10,000,000}{1.05^4} = 8,227,024.74$$

Therefore the fund needs to invest $8,227,024.74.
The duration of the obligation is 4. The duration of the two-year bond is 2 and the duration of the eight-year bond is 8. Let x be the percentage of the investment in the 2-year. Then:

$$4 = 2 \times x + 8 \times (1 - x)$$

This implies that *x = 2/3*. Hence the fund should invest *2/3 × 8,227,024.74 = 5,484,683.17* in the two-year zero coupon bond. And *1/3 × 8,227,024.74 = 2,742,341.58* in the eight-year zero coupon bond.

(b) The above investments result in a cash flow of *5,484,683.17 × 1:05*^{2} *= 6,046,863.19* in year two and a cash flow of *2,742,341.58 × 1.05*^{8} *= 4,051,687.50* in year eight. When interest rates are 6% the future value at year 4 of reinvesting the cash received in year 2 is:

$$FV_4(CF_2) = 6,046,863.19 \times 1.06^2 = 6,794,255.48$$

The value of the cash flow from the eight-year zero when sold in year 4 when interest rates are 6% is:

$$\frac{4,051,687.50}{1.06^4} = 3,209,315.99$$

The total received would be *$10,003,571.48 > 10,000,000*. It is larger because this strategy has convexity. When interest rates rise the duration of the portfolio shifts towards the two-year bond. As a result the portfolio is exposed to greater reinvestment risk. Since interest rates have risen this is an additional benefit to the portfolio.

7. (25 points)

You have $1 million currently invested entirely in mutual fund A. You are considering switching into a combination of T-bills and mutual fund B for the next year. Mutual fund B is invested 50% in the stock of ABC and 50% in the stock of XYZ. A one-year T-bill with face value $10,000 is currently selling for $9,523. You have come up with the following assessments of the return to mutual fund A along with the returns to ABC and XYZ for the next year.

$$E(\tilde{r}_A) = 0.10 \quad E(\tilde{r}_{ABC}) = 0.10 \quad E(\tilde{r}_{XYZ}) = 0.18$$
$$\sigma_A = 0.20 \quad \sigma_{ABC} = 0.20 \quad \sigma_{xyz} = 0.30$$
$$\rho_{ABC,XYZ} = 0.5$$

Using just the T-bill and mutual fund B show how you can construct an investment of your $1 million such that the portfolio is better than your current investment in mutual fund A. You must show clearly why the new portfolio is better and indicate the dollar investment in T-bills and in mutual fund B.

Solution:

To solve this problem we need the expected return and standard deviation of the return to mutual fund B. The expected return is given by:

$$E(\tilde{r}_B) = 0.5 \times E(\tilde{r}_{ABC}) + 0.5 \times E(\tilde{r}_{XYZ}) =$$
$$= 0.5 \times 10\% + 0.5 \times 18\% = 14\%$$

The variance of the return to mutual fund B is given by:

$$\sigma_B^2 = (0.5 \times \sigma_{ABC})^2 + (0.5 \times \sigma_{XYZ})^2 + 2 \times 0.5 \times 0.5 \times \rho_{ABC,XYZ} \times \sigma_{ABC} \times \sigma_{XYZ}$$
$$= (0.5 \times 0.2)^2 + (0.5 \times 0.3)^2 + 2 \times 0.5 \times 0.5 \times 0.5 \times 0.2 \times 0.3 = 0.0475$$

Hence *σ_B = 0.218*.
We also need the one-year risk-free rate. This is given by the price of the one year T-Bill:

$$1 + r_f = \frac{10,000}{9,523} = 1.05$$

Hence *r_f = 5%:*
Let's find a portfolio with the same return as mutual fund A. Let *w* be the weight in this portfolio invested in mutual fund B. Then

$$(1 - w) \times 5\% + w \times 14\% = 10\%$$

This implies that *w = 5/9*. The standard deviation of this portfolio's return is *σ_p = 5/9 × 0.218 = 12.1%*. The portfolio has the same expected return as mutual A, but lower standard deviation.
Alternatively we could find a portfolio with the same return standard deviation as that of mutual fund A. In this case we set:

$$w \times 0.218 = 0.20$$

This implies that *w = 0.917*. The expected return of this portfolio is

$$(1 - 0.917) \times 5\% + 0.917 \times 14\% = 13.25\%$$

The expected return on this portfolio is larger than that of mutual fund A.