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Cloning manipulation of the Borda rule

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Abstract

In an election contest, a losing candidate a can manipulate the election outcome in his favor by introducing his clone b in the choice set, the clone b being defined as an alternative which is ranked immediatly below a in the individual preferences. We characterise the voting situations where this manipulation is efficient for the Borda rule and express its vulnerability for a 3 alternative election.

Keywords: Borda rule, Manipulation, Clone.

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1 Introduction

From the theorem of Gibbard-Satterthwaite (1973, 1975) asserting that all reasonable voting procedures are sensitive to manipulation by strategic voters, an important litterature has been developped on the possible strategies of voters and their efficiencies. Recently, a different way of acting upon the election result has araised. By changing the choice set rather than the preferences, the final outcome might be radically different and favour the manipulators. Saari proved (1989, 1990) that adding or removing a candidate may affect seriously the ranking of scoring rules. Inspired by the theorem of Gibbard-Satterthwaite, Dutta, Jackson and Le Breton (2001) assert that all reasonable voting procedures are sensitive to strategic candidacy, i.e., a candidate can alter the election outcome by deciding whether or not to enter the choice set.

In this paper, we will focus on a specific way of changing the choice set suggested by Dummett (1998) that he called *agenda manipulation*. We will refer to this manipulation as *cloning manipulation* in order not to be confused with the agenda used in parliament to choose amendment. A losing candidate may try to manipulate the election outcome by promoting the candidature of a clone, i.e., another candidate who is considered similar to him, by all the voters. Dummett noticed that the Borda rule may suffer from this manipulation

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and gave some examples as the one displayed in Example 1.
The Borda rule (see Borda 1781) belongs to the class of point ranking rules where points are given to each candidate acccording to his rank in the preference of the voters. When n voters have to choose among m alternatives, each one assigns $m-r$ points to the candidate with the rank r in his individual preference. The Borda score of a candidate is the amount of points that he collects and the chosen candidate is the one with the highest Borda score. In Example 1, eleven voters have to choose among four alternatives a, b, c, and d. Each column represents a preference ordering, with the number of individuals with this specific preference, and the last column represents the number of points that a voter gives to a candidate according to his rank.

Example 1

2	3	2	1	2	1	*points*
b	c	c	a	b	b	3
d	d	a	b	a	a	2
a	a	d	d	d	c	1
c	b	b	c	c	d	0

In Example 1, a is chosen with a Borda score of 18 (17 for b, 16 for c and 15 for d). Dummett assumes that before the vote is taken, a fifth alternative, e, is introduced by d, whom every voter rank immediately below d. The chosen candidate among this new choice set is d with a Borda score of 26 (24 for a, 23 for b, 22 for c, and 15 for e).
As the reader can notice, this manipulation is sufficiently powerfull to make d chosen while he has the lowest Borda score with the initial voting situation. In this example, cloning manipulation can radically change the oucome in favour of the manipulator and it is natural to raise the question of the theoretical frequency of occurence of this manipulation. The procedure suggested by Gerhlein and Fishburn (1976) to obtain analytical representation of the probability of an event is of great interest for our purpose. From a system of linear inequalities characterizing an event, their procedure is able to provide polynomials expressing the likelihood of occurence of this event. The voting situations where cloning manipulation occurs can be characterized, with some conditions, by a system of linear inequalities. Unfortunately, when the coefficient of the variables of the linear inequalities are fraction of integer, which is the case for cloning manipulation, it becomes extremely difficult to give an analytical solution. Huang and Chua (2000)[1] provide a useful algorithm derived from the method of Gehrlein-Fishburn. We use the Huang-Chua technique which permits us to obtain polynomials expressing the likelihood of cloning manipulation for the Borda rule.
Basic notions and assumptions are introduced in Section 2. We characterize the voting situations where cloning manipulation can be successful, present the method of Huang and Chua and give the vulnerability of the Borda rule to cloning manipulation in a three candidate election in Section 3. We discuss our results in Section 4.

[1] see also Gehrlein, 2002.

2 Notations

Let X be the set of possible alternatives and A a finite subset of X, with $|A| = m$. [2] The set of individuals who choose among the candidates is $I = \{1, \ldots, i, \ldots, n\}$ with $|I| = n$. We assume that the individuals are able to rank all the alternatives without ties in X. The preferences on a finite subset A are the restriction of P_i on A. The six possible preference orderings over A will be numbered as follows when $A = \{a, b, c\}$:

Table 1

n_1	n_2	n_3	n_4	n_5	n_6
a	a	c	c	b	b
b	c	a	b	c	a
c	b	b	a	a	c

A *voting situation* is a vector $s = (n_1, n_2, n_3, n_4, n_5, n_6)$, with n_t the number of type t voters and $\sum_{t=1}^{6} n_t = n$, that gives the distribution of the n voters over the six possible preference types. S^n is the set of all possible voting situations. For a context A, a *social choice function* (SCF) $g : \cup_{i=1}^{n} S^n \to A$, assigns to each voting situation a nonempty subset of A, $g(A, s)$. We shall assume throughout the paper that the social decision for the context A only depends upon the restriction of the preferences in the profile on A.[3] N_{xy} is the number of voters who prefer x to y, $S^{x}_{B,s}$ is the Borda score of x for the voting situation s and $S^{xy}_{B,s}$ is the difference of Borda score between x and y for the voting situation s. In case of a tie, we use the lexicographic order to choose the winner.
The definition of a clone we give express the idea of a candidate creating another candidate, always ranked after him.[4]

Definition 1 *A candidate y is a clone of x for a voting situation s if and only if:*
$\forall z \in X \setminus \{x, y\},\ \forall i \in I,\ xP_iz \Longleftrightarrow yP_iz$ *and* $\forall i \in I,\ xP_iy$

Given a voting situation s, we say that a SCF is vulnerable to cloning manipulation at s if the outcome of a vote is better for a candidate when his clone is introduced in the context. The vulnerability of the Borda rule will be the number of voting situations where this rule is vulnerable to cloning manipulation at s when compared with the number of all possible voting situations.

3 Vulnerability to cloning manipulation

We characterise the voting situations at which the Borda rule is senstive to cloning manipulation. Lemma 1 describes the case where only one losing candidate manipulate while the other losing candidate doesn't react.

[2]The cardinality of A will vary when an attempt of cloning manipulation takes place.

[3]This is not the condition of Independance of Irrelevant Alternatives of Arrow(1963): in a context A, the social preference between x and y may depend upon all the preferences on A, not only on the pair $\{x, y\}$. But the preference on $\{x, y\}$ is not influenced by z, which is not in the menu A.

[4]For diffferent definitions of a clone, see Laslier (1999), Tideman (1987) and Zavist and Tideman (1989).

Lemma 1 *Suppose $m = 3$ and consider a voting situation $s = (n_1, n_2, n_3, n_4, n_5, n_6)$. The Borda rule is sensitive to single cloning manipulation at s if and only if*
$-S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$ *and* $((N_{ba} > S^{ab}_{B,s}$ *and* $N_{bc} \geq S^{cb}_{B,s})$ *or* $(N_{ca} > S^{ac}_{B,s}$ *and* $N_{cb} > S^{bc}_{B,s}))$
$-S^{ba}_{B,s} > 0$ *and* $S^{bc}_{B,s} \geq 0$ *and* $((N_{ab} \geq S^{ba}_{B,s}$ *and* $N_{ac} \geq S^{ca}_{B,s})$ *or* $(N_{cb} > S^{bc}_{B,s}$ *and* $N_{ca} > S^{ac}_{B,s}))$
$-S^{ca}_{B,s} > 0$ *and* $S^{cb}_{B,s} > 0$ *and* $((N_{ac} \geq S^{ca}_{B,s}$ *and* $N_{ab} \geq S^{ba}_{B,s})$ *or* $(N_{bc} \geq S^{cb}_{B,s}$ *and* $N_{ba} > S^{ab}_{B,s}))$

Proof.
• We assume that the Borda rule is sensitive to single cloning manipulation and a chosen initially: $S^{ab}_{B,s} \geq 0$ and $S^{ac}_{B,s} \geq 0$.
The candidate b creates d. We obtain the following voting situation s':

n_1	n_2	n_3	n_4	n_5	n_6	points
a	a	c	c	b	b	3
b	c	a	b	d	d	2
d	b	b	d	c	a	1
c	d	d	a	a	c	0

The new Borda scores are :
$S^a_{B,s'} = S^a_{B,s} + N_{ab}$, $S^b_{B,s'} = S^b_{B,s} + n$, $S^c_{B,s'} = S^c_{B,s} + N_{cb}$ and $S^d_{B,s'} = S^b_{B,s}$
b is chosen if he beats the other candidates. As d is a clone of b, d is always beaten by b.
b beats a if the difference between their new score is strictly positive which is equivalent to write $N_{ba} > S^{ab}_{B,s}$ and b beats c if the difference between their new score is positive which is equivalent to write $N_{bc} \geq S^{cb}_{B,s}$.
• We assume $S^{ab}_{B,s} \geq 0$, $S^{ac}_{B,s} \geq 0$, $N_{ba} > S^{ab}_{B,s}$ *and* $N_{bc} \geq S^{cb}_{B,s}$.
$S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$ implies a chosen.
$N_{ba} > S^{ab}_{B,s} \iff S^b_{B,s} + n > S^a_{B,s} + N_{ab}$
$\iff 3(n_5 + n_6) + 2(n_1 + n_4) + n_2 + n_3 > 3(n_1 + n_2) + 2n_3 + n_6$ (1)
$N_{bc} \geq S^{cb}_{B,s} \iff S^b_{B,s} + n \geq S^c_{B,s} + N_{cb}$
$\iff 3(n_5 + n_6) + 2(n_1 + n_4) + n_2 + n_3 \geq 3(n_3 + n_4) + 2n_2 + n_5$ (2).
The conditions (1) and (2) correspond to the fact of having yet 6 preference types and 4 candidates as Borda gives now 3 points for a top candidate. One property of the Borda rule is that each voter of type n_t gives $m(m-1)$ points to the candidates. As we know that there is 4 candidates and the number of points given by each voter of type n_t for a, b and c we find that the new candidate d is always ranked after b in the individual preferences. We conclude that d is a clone of b. From the conditions (1) and (2) we know that b is chosen: the Borda rule is sensitive to single cloning manipulation. The other cases are similar and we omitt them. Q.E.D.

Lemma 2 describes the situations where the two losing candidates create simultaneously a clone. They act upon the choice set at the same time but with a different aim.

Lemma 2 *Suppose $m = 3$ and consider a voting situation $s = (n_1, n_2, n_3, n_4, n_5, n_6)$ The Borda rule is sensitive to simultaneous cloning manipulation at s if and only if*
$-S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$
and $((S^b_{B,s} > S^{ab}_{B,s} + N_{ac}$ *and* $S^{bc}_{B,s} \geq N_{cb} - N_{bc})$ *or* $(S^c_{B,s} > S^{ac}_{B,s} + N_{ab}$ *and* $S^{cb}_{B,s} > N_{bc} - N_{cb}))$
$-S^{ba}_{B,s} > 0$ *and* $S^{bc}_{B,s} \geq 0$

$and\, ((S^{a}_{B,s} \geq S^{ba}_{B,s} + N_{bc} \,and\, S^{ac}_{B,s} \geq N_{ca} - N_{ac}) \,or\, (S^{c}_{B,s} > S^{bc}_{B,s} + N_{ba} \,and\, S^{ca}_{B,s} > N_{ac} - N_{ca}))$
$-S^{ca}_{B,s} > 0 \,and\, S^{cb}_{B,s} > 0$
$and\, ((S^{a}_{B,s} \geq S^{ca}_{B,s} + N_{cb} \,and\, S^{ab}_{B,s} \geq N_{ba} - N_{ab}) \,or\, (S^{b}_{B,s} \geq S^{cb}_{B,s} + N_{ca} \,and\, S^{ba}_{B,s} > N_{ab} - N_{ba}))$

Proof.
See Annex.

The lemmas 1 and 2 give the complete characterization of the voting situations where the manipulators are able to win after cloning manipulation (single or simultaneous). Let $CM(n)$ be the set of these voting situations. The conditions that characterize $CM(n)$ can be translated in terms of the n_t's. We assume each voting situation to be equally likely to occur (this is the Impartial Anonymous Culture condition used by Gehrlein and Fishburn (1976)). Under this assumption, it is possible to obtain an analytical representation for the number of elements in $CM(n)$ as a function of n and write the polynomial as following: $|CM(n)| = x_5n^5+x_4n^4+x_3n^3+x_2n^2+x_1n+x_0$ for a sequence of $n = r+ej$, $j = 0, 1, 2, \ldots, 5$, where the term e is the periodicity of the integer sequence for which the representation is valid. The procedure of Huang and Chua permits to find the values of the coefficients of $CM(n)$ [5]. We use computer enumeration to evaluate exact values for the number of the situations characterized by Lemma 1 and 2 for each $n = r + ej$, $j = 0, 1, \ldots, 5$ (see Table 2). We set up 6 equations with 6 unknowns $(x_0, x_1, ..., x_5)$ and solve the 6 simultaneous equations. The Huang-Chua algorithm permits us to find that the periodicity e is 24 (resp.15) for single (resp. simultaneous) cloning manipulation. Finally, we divide the cardinality of $CM(n)$ by the total number of situations, and obtain the following representations for the vulnerability of the Borda rule to cloning manipulation.
V(B^{cl},r(e)) (resp. V(B^{simcl},r(e))) is the vulnerability of Borda rule to single (resp. simultaneous) cloning manipulation with n $\equiv$ r(mod e). We express only the first polynomial.

Proposition 1

$$V(B^{cl}, 1(24)) = \frac{1909n^5 + 24420n^4 + 111750n^3 + 208580n^2 + 94221n - 440880}{3072(n+1)(n+2)(n+3)(n+4)(n+5)}$$

$$V(B^{simcl}, 1(15)) = \frac{1023n^5 + 12900n^4 + 58745n^3 + 109260n^2 + 32280n - 214208}{1875(n+1)(n+2)(n+3)(n+4)(n+5)}$$

4 Concluding remarks

The algoritm of Huang and Chua is of great usefull when we want to look at the likelihood of an event characterized by a system of linear inequalities. They provide a usefull tool which make easier the study of the likelihood of success of the manipulation by cloning candidate. Our results (see Table 2) show the great vulnerability of the Borda rule to this manipulation. We can compare it to the classical manipulation attempted by voters misrepresenting their individual preference. In an election contest of three candidates when n is large the vulnerability of the Borda rule to manipulation of coalition of strategic voters

[5]See Huang and Chua (2000) for a general proof.

is of about 50% (see Favardin, Lepelley, Serais 2001) whereas it is of 62% (resp. 55%) for single (resp. simultaneous) cloning manipulation. These results reinforce the intuition of Dummett about the importance of this manipulation. However, this manipulation is a very specific case of manipulation by changing the choice set. If one doesn't impose specific condition over the choice set, the Borda rule may be robuster than other positional rule. Gehrlein and Fisburn (1980) looked at the likelihood of a positional rule to choose the same candidate when one modified the choice set by eliminating one (resp. one or two) losing candidate of the choice set in the case of a three (resp. four) candidate election. They proved that, in each case, the Borda rule was the positional rule which maximise the probability of electing the same candidate after this modification of the choice set.

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Table 2: Vulnerability of the Borda rule to cloning manipulation (three candidate election)

n	*Single*	n	*Simultaneous*
1	0	1	0
2	0.4286	2	0.2857
3	0.3571	3	0.3571
4	0.4127	4	0.3333
5	0.4524	5	0.4087
6	0.4675	6	0.4026
7	0.4735	7	0.4090
8	0.5012	8	0.4312
9	0.5055	9	0.4380
10	0.5115	10	0.4469
11	0.5240	11	0.4560
12	0.5294	12	0.4584
13	0.5346	13	0.4628
14	0.5399	14	0.4711
15	0.5445	15	0.4753
16	0.5490	16	0.4776
17	0.5530	17	0.4807
18	0.5552	18	0.4836
19	0.5584	19	0.4868
20	0.5622	20	0.4905
⋮	⋮	⋮	⋮
⋮	⋮	91	0.5318
145	0.6121	⋮	⋮
⋮	⋮	⋮	⋮
∞	0.6214	∞	0.5456

5 Annex

Proof Lemma 2

• We assume that the Borda rule is sensitive to simultaneous cloning manipulation and a chosen initially. We have $S^{ab}_{B,s} \geq 0$ and $S^{ac}_{B,s} \geq 0$

We assume that b is chosen after the partisans of b and c have acted upon the choice set by creating clone d and e. We have the following profile s'

n_1	n_2	n_3	n_4	n_5	n_6	points
a	a	c	c	b	b	4
b	c	e	e	d	d	3
d	e	a	b	c	a	2
c	b	b	d	e	c	1
e	d	d	a	a	e	0

The new Borda scores are :

$S^a_{B,s'} = S^a_{B,s} + N_{ab} + N_{ac}$, $S^b_{B,s'} = S^b_{B,s} + n + N_{bc}$, $S^c_{B,s'} = S^c_{B,s} + n + N_{cb}$, $S^d_{B,s'} = S^b_{B,s} + N_{bc}$ and $S^e_{B,s'} = S^c_{B,s} + N_{cb}$.

b is chosen if he beats the other candidates. As d is the clone of b, d is always beaten by b: b beats a if the difference between their new score is strictly positive: $S^{ba}_{B,s'} > 0$, which is equivalent to write $S^b_{B,s} > S^{ab}_{B,s} + N_{ac}$.

b beats c if the difference between their new score is positive: $S^{bc}_{B,s'} \geq 0$, which is equivalent to write $S^{bc}_{B,s} \geq N_{cb} - N_{bc}$

e is the clone of c so we have always $S^{ce}_{B,s'} \geq 0$, as $S^{bc}_{B,s'} \geq 0$ it implies $S^{be}_{B,s'} \geq 0$.

• We assume $S^{ab}_{B,s} \geq 0$, $S^{ac}_{B,s} \geq 0$, $S^b_{B,s} > S^{ab}_{B,s} + N_{ac}$ *and* $S^{bc}_{B,s} \geq N_{cb} - N_{bc}$.

$S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$ implies a chosen.

$S^b_{B,s} > S^{ab}_{B,s} + N_{ac} \iff 2S^b_{B,s} > S^a_{B,s} + N_{ac}$

$\iff S^b_{B,s} + N_{ba} + N_{bc} > S^a_{B,s} + N_{ac}$

$\iff S^b_{B,s} + N_{ba} + N_{bc} + N_{ab} > S^a_{B,s} + N_{ac} + N_{ab}$

$\iff S^b_{B,s} + n + N_{bc} > S^a_{B,s} + N_{ab} + N_{ac}$

$\iff 4(n_5 + n_6) + 3n_1 + 2n_4 + n_2 + n_3 > 4(n_1 + n_2) + 2(n_3 + n_6)$ (1)

$S^{bc}_{B,s} \geq N_{cb} - N_{bc} \iff S^b_{B,s} + n + N_{bc} \geq S^c_{B,s} + n + N_{cb}$

$\iff 4(n_5 + n_6) + 3n_1 + 2n_4 + n_2 + n_3 > 4(n_3 + n_4) + 3n_2 + 2n_5 + n_1 + n_6$ (2)

The conditions (1) and (2) correspond to the fact of having yet 6 preference type but with 5 candidates as Borda gives now 4 points for a top candidate. We know that the number of candidates and the number of points given by each voter of type n_t for a, b and c so we find that the two new candidates d and e are always ranked respectively after b and c in the individual preferences. We conclude that d and e are respectively the clone of b and c. Borda rule is sensitive to simultaneous cloning manipulation. The other cases are similar and we omitt them. Q.E.D.