Optimal Contracting with Moral Hazard and Behavioral Preferences*

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Abstract. We consider a continuous-time principal-agent model in which the agent's effort cannot be contracted upon, and both the principal and the agent may have non-standard, cumulative prospect theory type preferences. We find that the optimal contracts are likely to be "more nonlinear" than in the standard case with concave utility preferences. In the special case when the principal is risk-neutral, we show that she will offer a contract which effectively makes the agent less risk averse in the gain domain and less risk seeking in the loss domain, in order to align the agent's risk preference better with the principal's. We also find that, for specific parameter values, the shape of our optimal contract fits empirical data well.

Keywords: principal–agent problem, cumulative prospect theory, contracts, moral hazard, control, backward stochastic differential equation.

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1 Introduction

In this paper we consider optimal contracting between two parties – the principal ("she") and the agent ("he") – in continuous time, when the effort of the agent cannot be contracted upon. Cvitanić, Wan and Zhang (2009) develop a theory for general concave utility functions for the two parties. Motivated by behavioral criteria, specifically that of the (cumulative) prospect theory (CPT; Kahneman and Tversky (1979), Tversky and Kahneman (1992)), in the present paper we go a step further and allow the principal and the agent to have non-concave, CPT type preference functions.

Our model studies the case of "hidden actions" or "moral hazard", in which the agent's control (effort) of the drift of the output process cannot be contracted upon, either because it is unobserved by the principal, and/or because it is not legally enforceable. Hence, the contract is a function of only terminal values of the underlying output process.

The seminal paper on the continuous-time principal-agent problems is Holmstrom and Milgrom (1987). In that paper the principal and the agent have exponential utility functions and the optimal contract is linear. Their work was generalized and extended by Schättler and Sung (1993, 1997), Sung (1995, 1997), Müller (1998, 2000), and Hellwig and Schmidt (2002). The papers by Williams (2009) and Cvitanić, Wan and Zhang (2009) (henceforth CWZ 2009), use the stochastic maximum principle and forward-backward stochastic differential equations (FBSDEs) to characterize the optimal compensation for more general utility functions, under moral hazard. Cvitanić and Zhang (2007) and Carlier, Ekeland and Touzi (2007) consider also the adverse selection case of "hidden type", in which the principal does not observe the "intrinsic type" of the agent. Sannikov (2008) re-awakens the interest in the continuous-time principal–agent problem by finding a tractable model for solving the problem with a random time of retiring the agent and with continuous payments to the agent.

Optimal contracting with cumulative prospect theory (CPT) preferences, and, in particular, with the agent being loss averse, has already been studied in Dittmann, Maug and Spalt (2010). They calibrate the model to CEO compensation data and find that it explains better the observed compensation contracts than the standard utility preferences (risk aversion) model. This shows the usefulness of studying such models.

There are two main contributions of our paper. First, we show that the optimal payoff depends in a nonlinear way on the value of the output at the time of payment, and may be "more nonlinear" than with the standard, concave preferences. Second, we prove that, with a risk-neutral principal, and under some technical conditions, the optimal contract convexifies the agent's preference function if it is a classical concave utility function; and the optimal contract convexifies the agent's preference function in the gain part and concavifies it in the loss part if the agent has an S-shaped behavioral utility function.

We then study in details examples with a risk-neutral principal, and an agent who has piecewise logarithmic or power objective functions respectively. Notably, we find that the CPT preferences "increase" the nonlinearity of the optimal contract, thus providing an additional rationale for the existence of option-like contracts in practice.

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 presents the general approach to solving the agent's and the principal's optimization problems. Section 4 studies the case of a risk-neutral principal. Section 5 provides detailed examples. Finally, we conclude with Section 6.

2 The Model

We would like to have a model for the output process X of the form

$$dX_t = u_t v_t dt + v_t dW_t \tag{1}$$

for a Brownian motion process W, where u_t represents the effort of the agent, and v_t is the volatility process. As is usually noted in contract theory, choosing u is equivalent to choosing a probability measure over the underlying probability space. Thus, we proceed by developing the following weak formulation of the model (see Cvitanić and Zhang 2012 for more on the weak formulation).

Let B be a standard Brownian motion under some probability space with probability measure Q, and $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{0\le t\le T}$ be the filtration on $[0,T]$ generated by B and augmented by Q-null sets. For any $\mathbf{F}^B$-adapted square integrable process v, let

$$X_t \triangleq x_0 + \int_0^t v_s dB_s \tag{2}$$

be the output process (e.g., in the case of a fund manager managing wealth for a client, X_t is the wealth process). Then, the filtration $\mathbf{F}^X$ generated by X, is contained in the filtration $\mathbf{F}^B$.

We now introduce the agent's effort process u, assumed to be a functional of B. Given u, we define

$$B_t^u \triangleq B_t - \int_0^t u_s ds, \quad M_t^u \triangleq \exp\left(\int_0^t u_s dB_s - \frac{1}{2}\int_0^t |u_s|^2 ds\right), \tag{3}$$

along with a new probability measure Q^u by $dQ^u/dQ \triangleq M_T^u$. By Girsanov's Theorem, under technical conditions, B^u is a Brownian motion under Q^u. A well known sufficient condition is the so-called Novikov condition. We will allow only those control processes u for which B^u is, in fact, a Brownian motion. We have, then

$$dX_t = v_t dB_t = u_t v_t dt + v_t dB_t^u. \tag{4}$$

That is, the triple (X, B^u, Q^u) is a weak solution to the SDE (1). The choice of u corresponds to the choice of the probability measure Q^u, thus to the distribution of the process X. It is well known in the literature that this is the only way to change probability measures in Brownian models, while keeping them equivalent.

Remark 1.

(i) Even though we start with X as a martingale process under Q, it is in general *not* a martingale process under Q^u, the relevant measures under which we will be taking expected values in our optimizations problems below.

(ii) It is also possible to work with a multi-dimensional version of the model, by starting with a d-dimensional Brownian motion $B = (B_1, \ldots, B_d)$ under Q, and then, for a vector process $u = (u_1, \ldots, u_d)$, defining d-dimensional Brownian motion $B^u = (B_1^u, \ldots, B_d)$ under appropriate measure Q^u by $B_{i,t}^u \triangleq B_{i,t} - \int_0^t u_{i,s} ds$. We choose not to pursue this since there would be no essential difference other than notational complexity.

At time T, the principal gives the agent compensation in the form of a payoff $C_T = I(X_\cdot)$, where $I : C[0, T] \to \mathbb{R}$ is a (deterministic) mapping. Clearly, C_T is $\mathcal{F}_T^X$-measurable. On the other hand, for any $\mathcal{F}_T^X$-measurable contract payoff C_T, there exists some functional I such that $C_T = I(X_\cdot)$. Thus, choosing C_T is equivalent to choosing I. Note that the absolute value of v can also be observed by the principal through the quadratic variation. In this sense, if we allow v to be controlled, it would be chosen by the principal. In light of this, from now on we fix v and concentrate on the effort choice. We call C_T a *contract*.

Given a contract C_T and a function $U_A(\cdot, \cdot)$ representing the agent's preferences, the agent's problem is to choose the optimal effort (control) u in order to maximize his objective

$$V_A(C_T) \triangleq \sup_u V_A(u; C_T) \triangleq \sup_u E^u[U_A(C_T, G_T^u)], \tag{5}$$

where

$$G_t^u \triangleq \int_0^t g(s, X_s, u_s, v_s) ds \tag{6}$$

is the accumulated cost of the agent and $E^u \triangleq E^{Q^u}$ denotes the expectation under Q^u.

A contract C_T is *implementable* if there exists an effort process u^{C_T} which maximizes the agent's utility given the contract. i.e.

$$V_A(u^{C_T}; C_T) = V_A(C_T). \tag{7}$$

The typical cases studied in the literature are the *separable* case with $U_A(x, y) = U_A(x) - y$, and the *non-separable* case with $U_A(x, y) = U_A(x - y)$, where, with a slight abuse of notation, $U_A(\cdot)$ denotes the function of one argument only.

Given a function $U_P(\cdot)$, the principal chooses the optimal C_T^* to maximize her objective

$$V_P(C_T^*) = V_P \triangleq \max_{C_T, u^{C_T}} V_P(u^{C_T}; C_T), \tag{8}$$

where

$$V_P(u; C_T) \triangleq E^u[U_P(X_T - C_T)] = E[M_T^u U_P(X_T - C_T)] \tag{9}$$

under the *participation constraint* or *individual rationality (IR) constraint*

$$V_A(C_T) \geq R. \tag{10}$$

Following the convention in the principal-agent literature, we assume that if the agent is indifferent between two actions, he chooses the one that makes the principal better off. This is why u^{C_T} appears as a decision variable in the principal's problem (8). If u^{C_T} is unique for any given C_T, then it can be removed from the set of decision variables.

3 The Agent's Problem: Separable Utility and Quadratic Cost; CWZ (2009)

In this section we consider a particular case when the agent has separable utility $U_A(x, y) = U_A(x) - y$ and the cost is a quadratic function of the effort, i.e. $g(t, x, u, v) = g(u) = ku^2/2$. for some $k > 0$. We also assume that v is given so that the principal just needs to choose C_T. The choice of such a cost function is standard in the contract theory literature, partly for tractability reasons; see, e.g., Bolton and Dewatripont (2005). In the context of the example of a portfolio manager, the fact that cost function g does not depend on x and v means that the cost of effort does not depend on the volatility or the size of the portfolio.

This is the case solved in Cvitanić, Wan and Zhang (2009) for the case of standard utility functions. We recall the method here for the convenience of the reader, and adapt it slightly to be able to apply it to more general preferences.

Under this setting we now define the feasible sets for u and C_T. Roughly speaking, all we need is sufficient integrability so that the weak formulation is meaningful.

Definition 1. *The set $\mathcal{A}_1$ of admissible effort processes u is the space of $\mathbf{F}^B$-adapted processes u such that*

(i) $\exp\left[\int_0^T ku_t^2/2dt\right] < \infty$;

(ii) $E[|M_T^u|^2] < \infty$;

(iii) $E\left[\left(\int_0^T u_t^2 dt\right)^4\right] < \infty$.

Definition 2. *The set $\mathcal{A}_2$ of admissible contracts is the space of $\mathcal{F}_T^X$-measurable random variables C_T such that*

(i) $E[|U_A(C_T)|^4 + e^{2U_A(C_T)/k}] < \infty$;

(ii) $\hat{u}$ *from* (12) *below satisfies* $E\left[\left(\int_0^T \hat{u}_t^2 dt\right)^4\right] < \infty$;

(ii) $E[|U_P(X_T - C_T)|^2 + e^{U_A(C_T)/k}|U_P(X_T - C_T)|] < \infty$.

Definition 2-(ii) is a natural condition: it requires that the agent's optimal effort process in response to a given contract is admissible. In particular, we show in Remark 2 below that any bounded $\mathcal{F}_T^X$-measurable C_T satisfies this condition.

We first state a useful known result for linear backward stochastic differential equations (BSDEs); see, for example, CWZ (2009).

Lemma 1. *Assume that u is $\mathbf{F}^B$-adapted, Girsanov's Theorem holds true for (B^u, Q^u), and $E[|M_T^u|^2] < +\infty$. Then for any $\mathcal{F}_T^B$-measurable ξ with $E^u[|\xi|^2] < \infty$, there exists a unique Q^u-square integrable, $\mathbf{F}^B$-adapted pair (Y, Z) that solves the following BSDE*

$$Y_t = \xi - \int_t^T Z_s dB_s^u. \tag{11}$$

The following theorem is a variation on a result from CWZ (2009). In that paper $U_A(\cdot)$ is assumed to be concave, which we do not need to assume here. Moreover, our admissible sets $\mathcal{A}_1$ and $\mathcal{A}_2$ are somewhat different. For these reasons, we provide a proof here.

We use notation E_t for conditional expectation given σ-algebra $\mathcal{F}_t^B$.

Theorem 1. *For any contract $C_T \in \mathcal{A}_2$, the agent's optimal effort $\hat{u} \in \mathcal{A}_1$ is obtained from solving the following BSDE*

$$\bar{Y}_t = E_t[e^{U_A(C_T)/k}] = e^{U_A(C_T)/k} - \int_t^T \hat{u}_s \bar{Y}_s dB_s, \tag{12}$$

which has a unique $\mathbf{F}^B$-adapted solution $(\bar{Y}, \hat{u})$. Moreover, the optimal value for the agent's problem is

$$V_A(C_T) = k \log \bar{Y}_0 = k \log E[e^{U_A(C_T)/k}]. \tag{13}$$

Proof We first show that the BSDE (12) admits a unique solution and $\hat{u} \in \mathcal{A}_1$. By Definition 2-(i), the following linear BSDE

$$\bar{Y}_t = e^{U_A(C_T)/k} - \int_t^T \bar{Z}_s dB_s.$$

has a unique solution $(\bar{Y}, \bar{Z})$. Note that $\bar{Y}_t > 0$. Define $\hat{u}_t \triangleq \bar{Z}_t / \bar{Y}_t$. Then $(\bar{Y}, \hat{u})$ satisfies (12).

Since $\bar{Y}_t > 0$ is continuous and $E\left[\int_0^T |\bar{Z}_t|^2 dt\right] < \infty$, we have $\int_0^T \hat{u}_t^2 dt < \infty$ a.s. Moreover, direct calculation yields that

$$M_t^{\hat{u}} = \bar{Y}_t / \bar{Y}_0,$$

which means that $M^{\hat{u}}$ is an $\mathbf{F}^B$-martingale under Q. So Girsanov's theorem holds for $(B^{\hat{u}}, Q^{\hat{u}})$. In addition, by Definition 2-(i), $E[|M_T^{\hat{u}}|^2] = E[e^{2U_A(C_T)/k}]/\bar{Y}_0^2 < \infty$. Finally, the definition of $\mathcal{A}_2$ implies that $\hat{u}$ satisfies Definition 1-(iii). Therefore $\hat{u} \in \mathcal{A}_1$.

We now show that $\hat{u}$ is optimal for the agent's problem. For any $u \in \mathcal{A}_1$, consider the remaining utility of the agent at time t

$$W_t^{A,u} = E_t^u \left[U_A(C_T) - \frac{k}{2} \int_t^T u_s^2 ds \right].$$

Then $W_t^{A,u}/k - \frac{1}{2}\int_0^t u_s^2 ds$ is a Q^u-martingale with the terminal value $U_A(C_T)/k - \frac{1}{2}\int_0^T u_t^2 dt$, which is Q^u-square integrable by the definitions of $\mathcal{A}_1$ and $\mathcal{A}_2$. So it follows from Lemma 1 that there exists an $\mathbf{F}^B$-adapted Q^u-square integrable process $Z^{A,u}$ such that

$$W_t^{A,u}/k - \frac{1}{2}\int_0^t u_s^2 ds = U_A(C_T)/k - \frac{1}{2}\int_0^T u_s^2 ds - \int_t^T Z_s^{A,u} dB_s^u.$$

Switching from B^u to B, we have

$$W_t^{A,u}/k = U_A(C_T)/k + \int_t^T \left[u_s Z^{A,u} - \frac{1}{2} u_s^2 \right] ds - \int_t^T Z_s^{A,u} dB_s. \tag{14}$$

On the other hand, from (12), it follows

$$\log \bar{Y}_t = U_A(C_T)/k - \frac{1}{2}\int_t^T \hat{u}_s^2 ds - \int_t^T \hat{u}_s dB_s^u. \tag{15}$$

Note that $W_0^{A,u}/k = E^u[U_A(C_T)/k - \frac{1}{2}\int_0^T u_s^2 ds]$. Thus $\log \bar{Y}_0 = W_0^{A,\hat{u}}$.

Notice that[1]

$$\begin{aligned}
W_0^{A,\hat{u}}/k - W_0^{A,u}/k &= \int_0^T \left[\frac{1}{2}\left(\hat{u}_t^2 + u_t^2\right) - u_t Z_t^{A,u} \right] dt + \int_0^T \left(Z_t^{A,u} - \hat{u}_t \right) dB_t \\
&\geq \int_0^T \left(\hat{u}_t u_t - u_t Z_t^{A,u} \right) dt + \int_0^T \left(Z_t^{A,u} - \hat{u}_t \right) dB_t \\
&= \int_0^T \left(Z_t^{A,u} - \hat{u}_t \right) dB_t^u.
\end{aligned}$$

[1]Due to the lack of Lipschitz continuity caused by the quadratic term $\hat{u}_t^2$, we cannot directly apply the Comparison Theorem for BSDEs here.

The equality holds if and only if $u = \hat{u}$. Since $E^u\left[\int_0^T |Z_t^{A,u}|^2 dt\right] < \infty$ and

$$E^u\left[\int_0^T \hat{u}_t^2 dt\right] = E\left[M_T^u \int_0^T \hat{u}_t^2 dt\right] \le E\left[|M_T^u|^2 + \left(\int_0^T \hat{u}_t^2 dt\right)^2\right] < \infty,$$

we have $W_0^{A,\hat{u}} \ge W_0^{A,u}$, where the equality holds if and only if $u = \hat{u}$. ∎

Remark 2. If C_T is bounded then Definition 2-(ii) is satisfied. Indeed, if $(\bar{Y}, \hat{u})$ solves (12), then there is a constant $\epsilon > 0$ such that $\epsilon < \sup_{0 \le t \le T} \bar{Y}_t < 1/\epsilon$ and $E[\langle \bar{Y} \rangle_T^4] < 1/\epsilon$. Since $\int_0^T \hat{u}_t^2 \bar{Y}_t^2 dt = \langle \bar{Y} \rangle_T$, we have $\int_0^T \hat{u}_t^2 dt < \langle \bar{Y} \rangle_T / \epsilon^2$. As a result,

$$E\left[\left(\int_0^T \hat{u}_t^2 dt\right)^4\right] < \frac{1}{\epsilon^8} E[\langle \bar{Y} \rangle_T^4] < \frac{1}{\epsilon^9} < \infty.$$

From (12) we have $\bar{Y}_t = \bar{Y}_0 M_t^{\hat{u}}$, which, together with (13), yields

$$M_T^{\hat{u}} = e^{-V_A(C_T)/k} \cdot e^{U_A(C_T)/k}, \quad e^{V_A(C_T)/k} = E[e^{U_A(C_T)/k}]. \tag{16}$$

Equivalently, by taking into account the participation constraint,

$$M_T^{\hat{u}} = e^{-\tilde{R}/k} \cdot e^{U_A(C_T)/k}, \quad e^{\tilde{R}/k} = E[e^{U_A(C_T)/k}], \tag{17}$$

for some $\tilde{R} \ge R$. So the principal's problem becomes

$$\begin{aligned} &\max_{C_T, \tilde{R}} \quad e^{-\tilde{R}/k} E\left[e^{U_A(C_T)/k} U_P(X_T - C_T)\right] \\ &\text{subject to} \quad \tilde{R} \ge R, \quad E[e^{U_A(C_T)/k}] = e^{\tilde{R}/k}. \end{aligned}$$

Proposition 1. *For every pair $(C_T, \tilde{R})$ in which $\tilde{R} > R$, there is a pair $(\tilde{C}_T, R)$ which attains a strictly higher value in the above problem.*

Proof Fix a pair $(C_T, \tilde{R})$ such that $\delta \triangleq \tilde{R} - R > 0$ and $E[e^{U_A(C_T)/k}] = e^{\tilde{R}/k}$.

Define $f(x) \triangleq U_A^{-1}(U_A(x) - \delta)$. Then $f(C_T) < C_T, U_A(f(C_T)) = U_A(C_T) - \delta$ and $E\left[e^{U_A(f(C_T))/k}\right] = e^{R/k}$. We have

$$
\begin{aligned}
& e^{-R/k} E\left[e^{U_A(f(C_T))/k} U_P(X_T - f(C_T))\right] \\
& = e^{-\tilde{R}/k} E\left[e^{U_A(C_T)/k} U_P(X_T - f(C_T))\right] \\
& > e^{-\tilde{R}/k} E\left[e^{U_A(C_T)/k} U_P(X_T - C_T)\right].
\end{aligned}
$$

Thus, $(f(C_T), R)$ is strictly better than $(C_T, \tilde{R})$ in the above problem. ∎

In the light of the above proposition, we can set $\tilde{R} = R$. Hence, the principal's problem simplifies to

$$
\begin{aligned}
\max_{C_T} \quad & e^{-R} E\left[e^{U_A(C_T)/k} U_P(X_T - C_T)\right] \\
\text{subject to} \quad & E[e^{U_A(C_T)/k}] = e^{R/k}.
\end{aligned}
\tag{18}
$$

Thus, in view of the Lagrange approach, we get the following

Theorem 2. *If there exists C_T^λ that maximizes*

$$
e^{U_A(C_T)/k}[U_P(X_T - C_T) + \lambda] \tag{19}
$$

where $\lambda \in \mathbb{R}$ and there exists λ_0 such that

$$
E\left[e^{U_A(C_T^{\lambda_0})/k}\right] = e^{R/k}, \tag{20}
$$

then $C_T^{\lambda_0}$ is the optimal contract.

4 Optimal Contracts with Risk-Neutral Principal

In the above setting (the agent has a separable utility and a quadratic cost), we now investigate the optimal contracts with a risk-neutral principal, that is, $U_P(x) = x$.. The latter assumption is standard in contract theory, especially in applications to corporate finance, in which it is assumed that all the shareholders of the firm hiring the agant represent the principal, and those shareholders are well-diversified in the market, hence risk-neutral with respect to one particular firm. For simplicity of notation, we set $k = 1$ in the rest of the paper. We study two cases for the agent's preference, one with a neoclassical (concave) utility function, and the other dictated by the cumulative prospect theory (CPT).

A main issue we address is how an optimal contract effectively changes the agent's risk preference. For this purpose we now introduce the notion of concavifying and convexifying utility function.

Let A^U denote the coefficient of absolute risk aversion of a strictly increasing function U:

$$A^U(x) \triangleq -\frac{U''(x)}{U'(x)}, \tag{21}$$

whereever $U''(x)$ exists and $U'(x) \neq 0$.

Note that $A^U(x) \geq 0 (\leq 0)$ if and only if $U(\cdot)$ is concave (convex) at x.

Following Ross (2004), we introduce the following definition.

Definition 3. *A contract $f(\cdot)$ concavifies (convexifies) a utility function $U(\cdot)$, if there exists an increasing concave (convex) function, $T(\cdot)$ such that $U(f(x)) = T(U(x))$.*

Lemma 2. *(Ross 2004) A smooth contract $f(\cdot)$ concavifies (convexifies) $U(\cdot)$ if and* *only if*

$$\frac{f''}{f'} \leq (\geq) A^U(f)f' - A^U. \tag{22}$$

Remark 3. Note that

$$A^{U\circ f} - A^U = -\frac{U''(f)(f')^2 + U'(f)f''}{U'(f)f'} - A^U = A^U(f)f' - A^U - \frac{f''}{f'}.$$

Thus, we deduce by Lemma 2 that f concavifies (convexifies) U if and only if the coefficient of absolute risk aversion of $U \circ f$ is larger (smaller) than that of U, i.e.

$$f \text{ concavifies (convexifies) } U \Leftrightarrow A^{U\circ f} \geq (\leq) A^U. \tag{23}$$

We now want to solve the problem of maximizing (19) with $U_P(x) = x$. Define

$$F(c, y) \triangleq e^{U_A(c)}(y - c), \tag{24}$$

where we think of y as $X_T + \lambda$. Then the target is to maximize $F(\cdot, y)$ for every y. Clearly, the first order condition is $U_A'(c)(y - c) = 1$.

4.1 Agent with Concave Utility

We first consider the simpler case with a classical utility function $U_A(\cdot)$, i.e. $U_A(\cdot)$ is overall concave. Denote $c_{\min} \triangleq \inf\{c : U_A(c) > -\infty\}$. We assume that $U_A(\cdot)$ is smooth on $(c_{\min}, +\infty)$ and let $A(\cdot)$ be the coefficient of absolute risk aversion of $U_A(\cdot)$. We introduce the following condition.

Condition 1. *$A'(x) \leq 0$, $c_{\min} > -\infty$, and there exists $\beta \leq c_{\min}$ such that $A(x)(x-\beta)$ is a convex function with $-A'(x)(x - \beta) \leq A(x)$ on $x > c_{\min}$.*

Example 1. Take $U_A(c) = \log(c - c_{\min})$. Then $A(x) = 1/(x - c_{\min}),\ \ x > c_{\min}$, which implies $A'(x) = -1/(x - c_{\min})^2 < 0$. Choose $\beta = c_{\min}$. Then we have $A(x)(x - \beta) = 1$ and $-A'(x)(x - \beta) = A(x)$. So $U_A(\cdot)$ satisfies Condition 1.

Example 2. Let $c_{\min} \in (-\infty, +\infty)$ be given and take $U_A(c) = (c - c_{\min})^\gamma/\gamma$ with $\gamma < 1, \gamma \neq 0$. Then

$$
\begin{aligned}
&A(x) = \frac{1-\gamma}{x - c_{\min}}, \quad A'(x) = -\frac{1-\gamma}{(x - c_{\min})^2} < 0, \\
&A(x)(x - c_{\min}) = 1 - \gamma, \quad -A'(x)(x - c_{\min}) = A(x).
\end{aligned}
$$

Again, $U_A(\cdot)$ satisfies Condition 1.

Example 3. Let $c_{\min} \in (-\infty, +\infty)$ be given and take $U_A(c) = -\exp\{-Ac\}$. Then A is a constant and $c_{\min} = -\infty$. So $U_A(\cdot)$ does not satisfy Condition 1.

Theorem 3. *Denote $u(\cdot) = U_A(\cdot)$, which is assumed to be concave and smooth on $(c_{\min}, +\infty)$. Then, the equation*

$$
H(c, y) \triangleq u'(c)(y - c) - 1 = 0, \quad (c, y) \in (c_{\min}, +\infty) \times \mathbb{R}, \tag{25}
$$

admits a unique root $\bar{c}(y) \in (c_{\min}, y),\ \ \forall y > c_{\min} + 1/u'(c_{\min}+)$, and $H(c, y) < 0$ on $(c_{\min}, +\infty) \times (-\infty, c_{\min} + 1/u'(c_{\min}+)]$. Moreover, we have the following results.

(i) The optimal contract, if it exists, is

$$
C_T^*(Y_T) = \begin{cases} \bar{c}(Y_T), & \text{if } \ Y_T > c_{\min} + 1/u'(c_{\min}+); \\ c_{\min}, & \text{if } \ Y_T \leq c_{\min} + 1/u'(c_{\min}+). \end{cases}
$$

(ii) $\bar{c}'(y) \in (0, 1),\ \ \forall y > c_{\min} + 1/u'(c_{\min}+)$.

(iii) If Condition 1 holds, or A is a constant function, then $\bar{c}(\cdot)$ convexifies $u(\cdot)$ on $(c_{\min} + 1/u'(c_{\min}+), +\infty)$.

Proof Note that $H_c(c,y) = u''(c)(y-c) - u'(c) < 0$ for $c < y$, $H(c,y) < 0$ for $c \geq y$. If $y > c_{\min} + 1/u'(c_{\min}+)$, then $H(c_{\min}+, y) > 0$, in which case (25) admits a unique root $\bar{c}(y) > c_{\min}$. If $y \leq c_{\min} + 1/u'(c_{\min}+)$, then $H(c,y) < 0, \forall c > c_{\min}$. This proves (i).

(ii) By $\frac{dH(\bar{c}(y),y)}{dy} \equiv 0,\ \ y > c_{\min} + 1/u'(c_{\min}+)$, we have

$$\bar{c}'(y) = \frac{1}{1 + A(\bar{c}(y))(y - \bar{c}(y))}, \quad y > c_{\min} + 1/u'(c_{\min}+). \tag{26}$$

Since $A(\cdot) > 0$ and $y > \bar{c}(y)$, it follows that $\bar{c}'(y) \in (0,1)$.

(iii) By Lemma 2, it suffices to prove $D(y) \triangleq A(y) + \frac{\bar{c}''(y)}{\bar{c}'(y)} - A(\bar{c}(y))\bar{c}'(y) \geq 0,\ \ \forall y > c_{\min} + 1/u'(c_{\min}+)$. Taking derivative on both sides of (26) with respect to y yields

$$\bar{c}''(y) = -\frac{(y - \bar{c}(y))[A'(\bar{c}(y)) + A(\bar{c}(y))^2]}{[1 + A(\bar{c}(y))(y - \bar{c}(y))]^2}\bar{c}'(y), \quad y > c_{\min} + 1/u'(c_{\min}+). \tag{27}$$

Now suppose that Condition 1 holds. Then there exists $\beta \leq c_{\min}$ such that $A(x)(x - \beta)$ is convex on $x > c_{\min}$. We thus have

$$A(y) \geq \frac{(\bar{c}(y) - \beta)(y - \bar{c}(y))A'(\bar{c}(y))}{y - \beta} + A(\bar{c}(y)), \quad y > c_{\min} + 1/u'(c_{\min}+). \tag{28}$$

Therefore

$$\begin{aligned} D(y) \geq & \frac{A(\bar{c}(y))^3(y - \bar{c}(y))^2}{[1 + A(\bar{c}(y))(y - \bar{c}(y))]^2} \\ & + A'(\bar{c}(y))(y - \bar{c}(y))\left\{\frac{\bar{c}(y) - \beta}{y - \beta} - \frac{1}{[1 + A(\bar{c}(y))(y - \bar{c}(y))]^2}\right\}. \end{aligned}$$

Since $A'(x) \leq 0,\ \ \forall x > c_{\min}$, if $\frac{\bar{c}(y) - \beta}{y - \beta} - \frac{1}{[1 + A(\bar{c}(y))(y - \bar{c}(y))]^2} \leq 0$, we have $D(y) \geq 0$; otherwise, by $-A'(x)(x - \beta) \leq A(x),\ \ \forall x > c_{\min}$ we have

$$D(y) \geq \frac{A(\bar{c}(y))^3(y - \bar{c}(y))^2}{[1 + A(\bar{c}(y))(y - \bar{c}(y))]^2}$$

$$
\begin{aligned}
&-\frac{A(\bar{c}(y))(y-\bar{c}(y))}{\bar{c}(y)-\beta}\left\{\frac{\bar{c}(y)-\beta}{y-\beta}-\frac{1}{[1+A(\bar{c}(y))(y-\bar{c}(y))]^2}\right\}\\
=&\frac{A(\bar{c}(y))(y-\bar{c}(y))^2[1-A(\bar{c}(y))(\bar{c}(y)-\beta)]^2}{(\bar{c}(y)-\beta)(y-\beta)[1+A(\bar{c}(y))(y-\bar{c}(y))]^2}\\
\geq&0, \quad \forall y>c_{\min}+1/u'(c_{\min}+).
\end{aligned}
$$

Finally, if A is a constant, we have, for any $y>c_{\min}+1/u'(c_{\min}+)$,

$$
\bar{c}'(y)=\frac{1}{1+(y-\bar{c}(y))A}, \quad \bar{c}''(y)=-\frac{(y-\bar{c}(y))A^2}{[1+(y-\bar{c}(y))A]^2}\bar{c}'(y).
$$

Therefore

$$
D(y)=\frac{(y-\bar{c}(y))^2A^3}{[1+(y-c(y))A]^2}>0, \quad \forall y>c_{\min}+1/u'(c_{\min}+). \tag{29}
$$

■

The preceding theorem shows that the optimal contract is generally a nonlinear increasing function of the final asset value, and the marginal increasing rate $\bar{c}'(\cdot)\in(0,1)$, which implies that both the agent's and the principal's payoffs are strictly increasing with respect to the final asset value. We will show later that this is not the case if the agent has an S-shaped behavioral utility function. More precisely, in some domains the marginal increasing rate of the optimal contract can be larger than 1; see also Remark 5 below.

4.2 Agent with S-shaped utility

In this subsection we consider the case where the agent has behavioral preferences. More precisely, we use an S-shaped utility $U_A(\cdot)$ for the agent's risk preference, with a reference point K, as defined next.

An S-shaped utility $U_A(\cdot)$ with reference point K is defined by

$$U_A(x) = \begin{cases} u_+(x-K), & x \geq K; \\ -u_-(K-x), & x < K, \end{cases} \tag{30}$$

where $K \geq 0$ is a constant, $u_+(\cdot)$ and $u_-(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$, are strictly increasing, concave, with $u_+(0) = u_-(0)$.

The reference point K separates gains and losses. That is, $(x-K)^+$ is viewed as the gain part and $(K-x)^+$ the loss part. One often requires $u'_+(x) \leq u'_-(x)$ around K to model the behavior of what is called *loss-aversion*, meaning that a marginal loss is greater than a marginal gain of the same size around the reference point. Overall, the S-shaped utility $U_A(\cdot)$ is concave for gains and convex for losses and steeper for losses than for gains. An agent with such a preference evaluates payoffs relative to the reference point, rather than at their absolute values. She is not uniformly risk-averse, but risk-averse on gains and risk-seeking on losses, and more sensitive to losses than to gains.

Let A be the coefficient of absolute risk aversion of U_A, which is defined except at $x = K$. Introduce the following conditions, needed to insure the existence of the optimal solution in the principal's problem:

Condition 2.

A1. $A'(x) \leq 0$ *and there exists* $\beta \leq K$ *such that* $A(x)(x-\beta)$ *is a convex function and* $-A'(x)(x-\beta) \leq A(x)$ *on* $x > K$.

A2. $A'(x) \leq 0$ *and there exists* $\eta \geq K$ *such that* $A(x)(\eta - x)$ *is a concave function and* $-A'(x)(\eta - x) \leq -A(x)$ *on* $x < K$.

A3. $-A(x)(K-x) < 1$ *for* $x < K$.

Example 4. *(Piecewise power, S-shaped)* Take

$$U_A(c) = \begin{cases} \frac{1}{\gamma}(c-K)^\gamma, & c > K; \\ -\frac{\theta}{\gamma}(K-c)^\gamma, & c \le K, \end{cases} \tag{31}$$

where $\theta > 1$ (implying loss aversion), $K > 0$ and $0 < \gamma < 1$ are constants. Then $A(x) = \frac{1-\gamma}{x-K}$ and $A'(x) = -\frac{1-\gamma}{(x-K)^2} < 0 \ \forall x \neq K$. Choosing $\beta = \eta = K$, we have $A(x)(x-\beta) = 1-\gamma$ and $-A'(x)(x-\beta) = A(x)$ on $x > K$; and $A(x)(\eta - x) = \gamma - 1$ and $-A'(x)(\eta - x) = -A(x)$ on $x < K$. Furthermore, $-A(x)(K-x) = 1-\gamma < 1$ for $x < K$. Therefore the S-shaped piecewise power utility defined above satisfies Condition 2.

We will revisit this example in Subsection 5.2.

Remark 4. One can check that an S-shaped piecewise logarithmic utility also satisfies Condition 2. However, an S-shaped piecewise exponential utility, whose $A(\cdot)$ is a negative constant on $(-\infty, K)$, does not satisfy Condition 2-A3.

Define $H(\cdot,\cdot) : (K, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $L(\cdot,\cdot) : (-\infty, K) \times \mathbb{R} \to \mathbb{R}$ through $F_c(\cdot,\cdot)$:

$$H(c,y) \triangleq F_c(c,y)e^{-U_A(c)} = U_A'(c)(y-c) - 1, \tag{32}$$

$$L(c,y) \triangleq F_c(c,y)e^{-U_A(c)} = U_A'(c)(y-c) - 1. \tag{33}$$

That is,

$$F_c(c,y)e^{-U_A(c)} = \begin{cases} H(c,y), & c > K, \\ L(c,y), & c < K. \end{cases} \tag{34}$$

Note that $H(c,y)$ is only defined on $c > K$ and $L(c,y)$ is only defined on $c < K$.

Lemma 3. *Let $u(\cdot)$ be the gain part of $U_A(\cdot)$, i.e. $u(\cdot)$ is defined on $[K, +\infty)$ and*

$u(x) = U_A(x)$ *for* $x \in [K, +\infty)$. *Then, the equation*

$$H(c, y) = u'(c)(y - c) - 1 = 0, \quad (c, y) \in (K, +\infty) \times \mathbb{R}, \tag{35}$$

admits a unique root $\bar{c}(y) \in (K, y)$, $\forall y > K + 1/u'(K+)$, *and* $H(c, y) < 0$ *on* $(K, +\infty) \times (-\infty, K + 1/u'(K+)]$. *Moreover, in the case* $y > K + 1/u'(K+)$, *we further have the following results:*

(i) $\bar{c}(y) = \arg\max_{c \geq K} F(c, y)$.

(ii) $\bar{c}'(y) \in (0, 1)$.

(iii) Under Condition 2-A1, $\bar{c}(\cdot)$ *convexifies* $u(\cdot)$.

Proof It can be proved exactly the same as Theorem 3. ∎

Lemma 4. *Let* $u(\cdot)$ *be the loss part of* $U_A(\cdot)$, *i.e.* $u(\cdot)$ *is defined on* $(-\infty, K]$ *and* $u(x) = U_A(x)$ *for* $x \in (-\infty, K]$. *Assume that* $\lim_{c \to -\infty} L(c, y) > 0$ *for any* $y < K$ *and Condition 2.-A3 holds. Then the equation* $L(c, y) = 0$ *admits a unique root* $\underline{c}(y) \in (-\infty, y)$ *for* $y < K$. *Moreover, we have the following results:*

(i) $\underline{c}(y) = \arg\max_{c \in \mathbb{R}} F(c, y)$, $\forall y < K$.

(ii) $\underline{c}'(y) \in (1, +\infty)$, $\forall y < K$.

(iii) Under Condition 2-A2, $\underline{c}(\cdot)$ *concavifies* $u(\cdot)$ *on* $(K, +\infty)$.

Proof Note that we can write $L_c(c, y) = U'(c)(-A(c)(y - c) - 1)$. By Condition 2-A3., $L_c(c, K) < 0$. But then because $A(c) < 0$, we have $L_c(c, y) < 0$ for $y < K$. Since $L(c, y) < 0$ for $c \geq y$ and $\lim_{c \to -\infty} L(c, y) > 0$, $L(c, y) = 0$ admits a unique root $\underline{c}(y) < y$ for $y < K$. Note that $H(c, y) < 0$ on $(K, +\infty) \times (-\infty, K)$. Thus, for $y < K$, $\underline{c}(y)$ maximizes $F(c, y)$, as stated in (i).

(ii) By $\frac{dL(\underline{c}(y),y)}{dy} = 0, \ \forall y < K$, we obtain

$$\underline{c}'(y) = \frac{1}{1 + A(\underline{c}(y))(y - \underline{c}(y))}, \quad y < K. \tag{36}$$

Since $A(\cdot) < 0$ and $1 + A(\underline{c}(y))(y - \underline{c}(y)) > 0$ (Condition 2-A3), it follows that $\underline{c}'(y) \in (1, +\infty)$.

(iii) In view of Lemma 2, it suffices to prove $D(y) \triangleq A(y) + \frac{\underline{c}''(y)}{\underline{c}'(y)} - A(\underline{c}(y))\underline{c}'(y) \leq 0 \ \forall y < K$. Taking derivative on both sides of (36) in y yields

$$\underline{c}''(y) = -\frac{(y - \underline{c}(y))[A'(\underline{c}(y)) + A(\underline{c}(y))^2]}{[1 + A(\underline{c}(y))(y - \underline{c}(y))]^2}\underline{c}'(y), \quad y < K. \tag{37}$$

Now suppose that Condition 2.-A2 holds. Then there exists $\eta \geq K$ such that $A(x)(\eta - x)$ is concave on $x < K$. We thus have

$$A(y) \leq \frac{(\eta - \underline{c}(y))(y - \underline{c}(y))A'(\underline{c}(y))}{\eta - y} + A(\underline{c}(y)), \quad y < K. \tag{38}$$

Therefore

$$\begin{aligned} D(y) \leq & \frac{A(\underline{c}(y))^3(y - \underline{c}(y))^2}{[1 + A(\underline{c}(y)(y - \underline{c}(y))]^2} \\ & + A'(\underline{c}(y))(y - \underline{c}(y)) \left\{ \frac{\eta - \underline{c}(y)}{\eta - y} - \frac{1}{[1 + A(\underline{c}(y))(y - \underline{c}(y))]^2} \right\}, \quad y < K. \end{aligned}$$

Since $A'(x) \leq 0 \, \forall x < K$, if $\frac{\eta - \underline{c}(y)}{\eta - y} - \frac{1}{[1 + A(\underline{c}(y))(y - \underline{c}(y))]^2} \geq 0$, we have $D(y) < 0$; otherwise, by $-A'(x)(\eta - x) \leq -A(x), \ \forall x < K$, we have

$$\begin{aligned} D(y) \leq & \frac{A(\underline{c}(y))^3(y - \underline{c}(y))^2}{[1 + A(\underline{c}(y))(y - \underline{c}(y))]^2} \\ & + \frac{A(\underline{c}(y))(y - \underline{c}(y))}{\eta - \underline{c}(y)} \left\{ \frac{\eta - \underline{c}(y)}{\eta - y} - \frac{1}{[1 + A(\underline{c}(y))(y - \underline{c}(y))]^2} \right\} \\ = & \frac{A(\underline{c}(y))(y - \underline{c}(y))^2[1 + A(\underline{c}(y))(\eta - \underline{c}(y))]^2}{(\eta - y)(\eta - \underline{c}(y))[1 + A(\underline{c}(y))(y - \underline{c}(y))]^2} < 0, \quad \forall y < K. \end{aligned}$$

■

Now fix λ and denote $Y_T \triangleq X_T + \lambda$. Suppose that there is a deterministic function $C_T^*(\cdot) : \mathbb{R} \to \mathbb{R}$ such that $C_T = C_T^*(Y_T)$ maximizes $F(C_T, Y_T)$.

Lemma 5. *Suppose that* $\lim_{c \to -\infty} L(c, K) > 0$. *Then there exists some* $c_0 < K$ *such that* $L(c, y) > 0$ *on* $(-\infty, c_0) \times [K, \infty)$. *Furthermore, there exists some* $y_1 > K$ *such that* $L(c, y) > 0$ *on* $(-\infty, K) \times (y_1, \infty)$. *In particular,* $C_T^*(y) \geq c_0$ *for* $y \geq K$ *and* $C_T^*(y) \geq K$ *for* $y \geq y_1$.

Proof By continuity, there exists $c_0 < K$ such that $L(c, K) > 0$ for $c < c_0$. Since $L(c, y)$ is increasing with respect to y, we have $L(c, y) > 0$ on $(-\infty, c_0) \times [K, \infty)$.

Let $y_1 = K + 1/U_A'(c_0)$. Then $L(c, y) > 0$ on $[c_0, K) \times (y_1, \infty)$. Therefore $L(c, y) > 0$ on $(-\infty, K) \times (y_1, \infty)$.

Since $L(c, y)$ and $F_c(c, y)$ have the same sign on $(-\infty, K) \times \mathbb{R}$, we conclude that $C_T^*(y) \geq c_0$ for $y \geq K$ and $C_T^*(y) \geq K$ for $y \geq y_1$. ■

We summarize the above results in the following theorem.

Theorem 4. *Suppose that* $\lim_{c \to -\infty} L(c, y) > 0$ *for any* $y \in \mathbb{R}$ *and Condition 2-A3 holds. Then an optimal contract* $C_T^*(\cdot)$ *must satisfy the following:*

(i) *For* $y < K$, $C_T^*(y) = \underline{c}(y)$.

(ii) *For* $y > K + 1/U_A'(K+)$, *if* $C_T^*(y) \geq K$, *then* $C_T^*(y) = \bar{c}(y)$.

(iii) *There exists some* $y_1 > K + 1/U_A'(K+)$ *such that* $C_T^*(y) = \bar{c}(y)$ *for* $y > y_1$.

(iv) *For* $K \leq y \leq K + 1/U_A'(K+)$, $C_T^*(y) \leq K$.

(v) *For* $y \geq K$, *there exists some* $c_0 < K$ *such that* $C_T^*(y) \geq c_0$.

Proof These results follow from Lemma 3, Lemma 4 and Lemma 5. ■

Remark 5. When the final output is sufficiently low so that $y < K$ (and therefore $C_T^*(y) = \underline{c}(y)$), by Lemma 4 the marginal increasing rate of the optimal compensation with respect to the final asset value is greater than 1, which means that the marginal increasing rate of the principal's payoff, $X_T - C_T^*$, is negative. This result is specific to the S-shaped behavioral utilities. Due to the convexity of the loss part of the S-shaped utility, $\underline{c}' > 1$ is the most efficient way to meet the participation constraint. In the case of standard preferences the marginal increasing rate of the compensation is between 0 and 1, and the principal's and the agent's payoffs are both increasing with respect to the asset value, as is the case in Theorem 3 for the classical utilities and in Theorem 4 when $C_T^*(y) = \bar{c}(y)$. Nevertheless, not unlike the standard case, both $\underline{c}(\cdot)$ and $\bar{c}(\cdot)$ are increasing functions, thus providing incentives for the agent to favor higher outcome of the asset value.

Theorem 5. *Suppose that $U_A(\cdot)$ satisfies Condition 2 and $\lim_{c\to-\infty} L(c,y) > 0$ for any $y \in \mathbb{R}$. Then we have the following:*

(i) $C_T^*(\cdot)$ concavifies $U_A(\cdot)$ on the loss part on $(-\infty, K)$.

(ii) There exists $y_1 > K + 1/U_A'(K+)$, such that $C_T^*(\cdot)$ convexifies $U_A(\cdot)$ on the gain part on $(y_1, +\infty)$.

Proof These follow from Lemmas 3 and 4. ∎

Remark 6. By Theorem 5, an optimal contract convexifies the agent's utility on the gain part where the agent is risk averse, and concavifies the utility on the loss part where the agent is risk seeking. Since the principal is risk-neutral, the optimal contract helps to align the agent's risk preference with the principal's.

Denote by C_T^λ the maximizer of (19) and $S(\lambda) \triangleq E\left[e^{U_A(C_T^\lambda)}\right]$. The following proposition addresses the existence of λ_0 that meets the participation constraint (20): $S(\lambda_0) = e^R$.

Proposition 2. *Suppose that $U_A(\pm\infty) = \pm\infty$, $S(\lambda)$ is finite for some λ and the* *conditions from Theorem 4 hold. Then*

$$\lim_{\lambda \to +\infty} S(\lambda) = +\infty, \quad \lim_{\lambda \to -\infty} S(\lambda) = 0.$$

In particular, if we assume that $S(\cdot)$ is continuous[2]*, then there exists λ_0 such that $S(\lambda_0) = e^R$ for any arbitrarily given $R \in \mathbb{R}$.*

Proof Denote $Y_T^\lambda = X_T + \lambda$. Define a subset of $\mathbb{R}$, $I(\lambda) \triangleq \{y > K + 1/U_A'(K+) : C_T^\lambda(y) = \bar{c}(y)\}$ and denote $\bar{I}(\lambda) \triangleq [K, \infty)/I(\lambda)$. By Theorem 4, there exists $y_1 > K + 1/U_A'(K+)$ which is independent of λ (see the proof of Lemma 5) such that $I(\lambda) \supset (y_1, \infty)$ for any λ. Moreover, $C_T^\lambda(Y_T^\lambda)$ consists of three parts: $C_T^\lambda(Y_T^\lambda) = \underline{c}(Y_T^\lambda)$ for $Y_T^\lambda < K$; $C_T^\lambda(Y_T^\lambda) = \bar{c}(Y_T^\lambda)$ for $Y_T^\lambda \in I(\lambda)$ and $C_T^\lambda(Y_T^\lambda) \in (c_0, K]$ for $Y_T^\lambda \in \bar{I}(\lambda)$ where $c_0 < K$ is a constant independent of λ.

Recall that both $\bar{c}(\cdot)$ and $\underline{c}(\cdot)$ are increasing. Since $0 = H(\bar{c}(y), y) = U_A'(\bar{c}(y))(y - \bar{c}(y)) - 1$, it is obvious that $\lim_{y \to +\infty} \bar{c}(y) = +\infty$ which implies that for a fixed ω and hence fixed X_T, $C_T^\lambda(Y_T^\lambda) \to +\infty$ as $\lambda \to +\infty$. Since $\underline{c}(y) < y$, for a fixed ω and hence fixed X_T, $C_T^\lambda(Y_T^\lambda) \to -\infty$ as $\lambda \to -\infty$. The conclusion follows from the facts that $U_A(\pm\infty) = \pm\infty$ and $S(\lambda)$ is finite for some λ. ∎

4.3 Comparison with First-Best Contracts

In the so-called first-best case in a principal–agent problem, the principal can contract upon the agent actions, therefore can optimally choose the action to maximize her objective function. The principal's problem is defined as, upon introducing a Lagrangian multiplier λ,

$$\max_{C_T, u} \{E[X_T^u - C_T] + \lambda E[U_A(C_T) - G_T^u - R]\}$$

[2]Later we show in Section 5.1 that the corresponding $S(\cdot)$ is continuous.

$$= \max_{u} E[X_T^u - \lambda G_T^u] + \max_{C_T} E[\lambda U_A(C_T) - C_T] - \lambda R.$$

Given $\lambda \geq 0$, the first best-optimal contract is determined as

$$C_T^* \in \arg\max_{c} \left[\lambda U_A(c) - c\right],$$

which is a constant (if it exists) and, notably, independent of X_T. This is a standard result, driven by the fact that the risk-neutral principal does not mind taking all the risk, and, because the principal chooses the actions, she does not have to provide incentives (related to the final output) to the agent. In contrast, we have seen in the previous sections that the optimal contracts with moral hazard are (non-constant) functions of X_T. Thus, the value function of the principal is strictly less in the second-best, moral hazard case than in the first-best case. As an extreme example, CWZ (2009) show that if $dX_t = X_t(u_t dt + \sigma_t dB_t)$, $\sigma_t > 0$ is deterministic and there exists $\epsilon > 0$ such that $\epsilon < \sigma_t < \frac{1}{\epsilon}$, then the principal's first-best optimal utility is $+\infty$ (i.e., the principal's problem is ill-posed). This can also be shown here when the agent has an S-shaped utility as in the above examples.

4.4 Risk-Averse Principal

The assumption that the principal is risk-neutral is standard in contract theory, because in the main applications the principal is an institution represented by many members, such as shareholders of a company, who are likely to have well-diversified portfolio of assets in the outside market, and are thus not risk-averse with respect to the performance of this particular institution. Nevertheless, it might be of interest to consider the case of a risk-averse principal; so we briefly discuss the case when U_P is not linear.

When we have enough smoothness, the first order condition for optimal C_T for the optimization problem in Theorem 2 is of the form

$$\frac{U_P'(X_T - C_T)}{U_A'(C_T)} = \frac{1}{k}U_P(X_T - C_T) + \lambda \tag{39}$$

and thus the optimal contract is still a function $c(X_T)$ of the terminal value X_T only. Using this equation we can find the sensitivity $c'(x)$ of the optimal contract with respect to the output, omitting the functions arguments:

$$c' = 1 - \frac{U_A''}{\frac{U_P''}{U_P'}U_A' + U_A'' - \frac{1}{k}(U_A')^2}.$$

We see from here that the difference between the risk-neutral case and the risk-averse case is the term $\frac{U_P''}{U_P'}U_A'$, which is zero in the risk-neutral case. With U_A increasing, if we assume that U_P is a standard utility function, increasing and strictly concave, then that term is negative. Thus, in the concave domain in which $U_A''(x) < 0$, we have $c'(x) < 1$; but $c'(x)$ is likely to be closer to one than when the principal is risk-neutral. [3] This means that, in the region in which the agent is risk-averse, the more risk-averse principal (higher risk aversion $\frac{U_P''}{U_P'}$) is likely to transfer more risk to the agent, which is not surprising. In the agent's risk-seeking domain in which $U_A''(x) > 0$, assuming Condition 2.-A3, Lemma 3 implies $c'(x) > 1$ for the risk neutral principal. This means that the term $U_A'' - \frac{1}{k}(U_A')^2$ in the denominator is negative, and subtracting a negative term from it makes $c'(x)$ smaller, keeping everything else fixed. Thus, under condition Condition 2.-A3 which requires that the agent is only moderately risk-seeking (i.e., $-A(x)$ should not be too large), the more risk-averse principal may actually transfer less risk to the agent. This is because the principal's risk exposure is $|1 - c'(x)|$, so it gets smaller by making $c'(x)$ smaller when $c'(x) > 1$. To summarize, in both cases, the risk-averse principal chooses $c'(x)$ closer to one than

[3] We say "likely" because when we change from risk-neutral to risk-averse principal, the optimal contract $c(x)$, on which all our functions depend, also changes, so we cannot make a direct comparison.

the risk-neutral principal. Taking this to extreme, the infinitely risk-averse principal would simply sell the whole firm to the agent ($c'(x) = 1$), a standard result from the classical contract theory.

4.5 Randomization under S-Shaped Utility

When the agent preference is behavioral with an S-shaped utility function, he is risk-seeking in the loss part of the domain of his utility function. It therefore may be advantageous for the principal to introduce randomized contracts in order to benefit from the agent's willingness to engage in some gambles. In this section we show that, indeed, randomization may strictly Pareto improve the contract. However, we are unable to find the optimal randomized contract, a task which appears to be a very hard mathematical problem, and is thus left for future study.

Consider a σ-field $\mathcal{G}_T$ independent of $\mathcal{F}_T^X$, and the σ-field $\mathcal{H}_T$ generated by $\mathcal{G}_T$ and $\mathcal{F}_T^X$. We consider the same setting as in Section 3 except that now admissible contracts are allowed to be $\mathcal{H}_T$-measurable. In other words, in addition to $\mathcal{F}_T^X$-measurable contracts in the original problem, the principal may randomize such contracts based on an independent random source corresponding to $\mathcal{G}_T$.

Assume that the principal is risk-neutral, i.e., $U_P(x) = x$. Applying the same arguments leading to problem (18), that is, going through the proof of Theorem 1, it can be verified that the problem now becomes

$$\begin{array}{ll} \max\limits_{C_T} & J(C_T) := \max\limits_{C_T} e^{-R} E\left[e^{E(U_A(C_T)|\mathcal{F}_T^X)} (X_T - C_T) \right] \\ \text{subject to} & E[e^{E(U_A(C_T)|\mathcal{F}_T^X)}] = e^R \end{array} \tag{40}$$

where the maximization is taken over contracts C_T that are $\mathcal{H}_T$-measurable. Clearly, when no randomization is applied then (40) reduces to (18) (where $U_P(x) = x$).

Let U_A be an S-shaped utility function such that $U_A(c)$ is strictly concave for $c > K$

and strictly convex for $c \le K$ where K is the reference point. Let C_T be a (non-randomized) contract which is $\mathcal{F}_T^X$-measurable, and $\bar{C}_T$ be a randomization of C_T, namely, a $\mathcal{H}_T$-measurable random variable such that $E(\bar{C}_T|\mathcal{F}_T^X) = C_T$. We then construct another randomization of C_T only on $C_T \le K$:

$$\tilde{C}_T := C_T 1_{C_T > K} + \bar{C}_T 1_{C_T \le K}.$$

We now show that such a randomized contract is Pareto improving if $C_T \le X_T$ (so that the principal gets a non-negative payoff).[4]

Clearly, $\tilde{C}_T$ is $\mathcal{H}_T$-measurable, and

$$E(\tilde{C}_T|\mathcal{F}_T^X) = C_T 1_{C_T > K} + E(\bar{C}_T|\mathcal{F}_T^B) 1_{C_T \le K} = C_T.$$

Thus, we also have $X_T \ge E(\tilde{C}_T|\mathcal{F}_T^X)$ and

$$\begin{aligned}
& e^{-R} E\left[e^{E(U_A(\tilde{C}_T)|\mathcal{F}_T^X)}(X_T - \tilde{C}_T)1_{C_T \le K}\right] \\
\ge \;& e^{-R} E\left\{E\left[e^{U_A(E(\tilde{C}_T|\mathcal{F}_T^X))}(X_T - \tilde{C}_T)1_{C_T \le K}\middle|\mathcal{F}_T^X\right]\right\} \\
= \;& e^{-R} E\left\{E\left[e^{U_A(C_T)}(X_T - \tilde{C}_T)1_{C_T \le K}\middle|\mathcal{F}_T^X\right]\right\} \\
= \;& e^{-R} E\left[e^{U_A(C_T)}(X_T - C_T)1_{C_T \le K}\right],
\end{aligned}$$

where the inequality is due to the convexity of $U_A(c)$ on $c \le K$ and Jensen's inequality. On the other hand,

$$e^{-R} E\left[e^{E(U_A(\tilde{C}_T)|\mathcal{F}_T^X)}(X_T - \tilde{C}_T)1_{C_T > K}\right] \equiv e^{-R} E\left[e^{E(U_A(C_T))}(X_T - C_T)1_{C_T > K}\right].$$

Combining the above, we conclude $J(\tilde{C}_T) \ge J(C_T)$. Moreover, by the same argument

[4] For some parameter configurations the optimal non-randomized contract C_T^* satisfies $C_T^* \le X_T$, so that its randomization $\tilde{C}_T^*$ will be preferable to all non-randomized contracts; see Proposition 3 below for some cases under which $C_T^* \le X_T$.

we derive

$$E\left[e^{E(U_A(\tilde{C}_T)|\mathcal{F}_T^X)}\right] \geq E[e^{U_A(C_T)}] = e^R.$$

Thus, if $P(C_T \leq K) > 0$ then due to the strict convexity of U_A on $c \leq K$ the preceding inequality is equality if and only if $E(\tilde{C}_T|\mathcal{F}_T^X) = \tilde{C}_T$, that is, if $\tilde{C}_T = C_T$. This shows that, for any contract that pays the agent in the loss domain with a positive probability and in that domain his utility is strictly convex, the above randomization will strictly increase the agent's utility while not reducing the principal's utility. This is to be contrasted to the static setting of DMS (2010) in which they assume that there is a lower bound on the contract payoff and the randomization is not optimal[5].

5 Examples

In this section, we apply the general theory established in the previous section to specific examples and obtain more concrete and complete results. As before the principal is risk-neutral. The agent's preferences are taken to be the standard preferences from the behavioral literature, that is, log, power and exponential S-shaped preferences, with a kink at the inflection point.

5.1 Piecewise Logarithmic Utility

Consider the agent's preferences given by

$$U_A(c) = \begin{cases} \log(1+c-K), & c > K; \\ -\theta\log(1+K-c), & c \leq K, \end{cases} \tag{41}$$

[5]Instead, whenever the optimal contracts pays a value in the loss space, it pays the lowest possible amount, the lower bound.

where $\theta > 1$ (representing loss-aversion) and $K > 0$ (reference point) are constants. Here $U_A(\cdot)$ is convex on $c < K$, reflecting a risk-seeking attitude on losses.

By Lemma 3, $\bar{c}(y) = (y+K-1)/2$ for $y > K+1$. Recall $L(\cdot,\cdot)$ defined in (33). Since $L(c,y) \geq 0$ on $(-\infty, K) \times [K+1/\theta, +\infty)$, we have $C_T^*(y) \geq K$ for $y \geq K+1/\theta$. Therefore $C_T^*(y) = \bar{c}(y)$ for $y > K+1$. In addition, recall $H(c,y)$ defined in (32). The fact that $H \leq 0$ on $(K,+\infty) \times (K, K+1]$ yields that $C_T^*(y) = K$ for $K + 1/\theta \leq y \leq K+1$.

For $y < K + 1/\theta$, $L(c,y)$ is strictly decreasing with respect to c and $L(-\infty, y) > 0, L(K-, y) < 0$. Therefore $L(C_T^*(y), y) = 0$, or $C_T^*(y) = (\theta y - K - 1)/(\theta - 1)$.

To summarize, we obtain the optimal C_T^* as follows

$$C_T^*(Y_T) = f(Y_T) \triangleq \begin{cases} \frac{Y_T+K-1}{2}, & Y_T > K+1; \\ K, & K + \frac{1}{\theta} \leq Y_T \leq K+1; \\ \frac{\theta Y_T - K - 1}{\theta - 1}, & Y_T < K + \frac{1}{\theta}, \end{cases} \tag{42}$$

where $Y_T = X_T + \lambda$.

Remark 7. CWZ (2009), Example 3.1, consider the same problem but with a classical logarithmic utility for the agent. In that example, the optimal contract is a linear function $C_T^* = (X_T+\lambda)/2$. Here, with behavioral preferences, the contract is piecewise linear. This provides an additional explanation for the use of option-like contracts – non-linearity may arise because of the behavioral preferences of the agent.

Remark 8. (i) By Lemma 3 and Lemma 4, the optimal contract, $f(\cdot)$, convexifies $U_A(\cdot)$ for $y > K+1$ and concavifies it for $y < K$. Note that for $y > K+1$ and $y < K$,

$$A^{U_A \circ f} = A^{U_A}(f) f' - \frac{f''}{f'} = A^{U_A}(f) f'.$$

Since $f' > 0$, $A^{U_A \circ f}$ and A^{U_A} have the same sign for $y > K+1$ and $y < K$, which means that $U_A \circ f$ is still concave on $y > K+1$ and convex on $y < K$. In other words, $f(\cdot)$ convexifies $U_A(\cdot)$ for $y > K+1$ but keeps its concavity; $f(\cdot)$ concavifies $U_A(\cdot)$ for $y < K$ but keeps its convexity. The contract $f(\cdot)$ makes $U_A(\cdot)$ in parts less concave and in parts less convex to align better with the principal's risk neutrality.

(ii) For $K < y < K + 1/\theta$, U_A is concave while $U_A \circ f$ is convex. Thus, $f(\cdot)$ heavily convexifies $U_A(\cdot)$ in order to keep $U_A \circ f$ better aligned with the risk neutrality. Finally, for $y \in [K + 1/\theta, K+1]$, the optimal compensation is a constant K.

In what follows we consider the optimal contract as a function of X_T rather than $Y_T \equiv X_T + \lambda$.

Denote $\alpha \triangleq K + 1 - \lambda$. Then (42) reads

$$C_T^*(X_T) = g(X_T) \triangleq \begin{cases} \frac{X_T - \alpha}{2} + K, & X_T \geq \alpha; \\ K, & \alpha - \frac{\theta-1}{\theta} \leq X_T < \alpha; \\ \frac{\theta}{\theta-1}(X_T - \alpha) + K + 1, & X_T < \alpha - \frac{\theta-1}{\theta}. \end{cases} \tag{43}$$

Let $\alpha_0 \triangleq K + 1 - \lambda_0$ where λ_0 is the Lagrangian multiplier that satisfies the participation constraint.

Proposition 3. *Suppose $X_T \geq 0$. Then agent has a non-negative payoff if and only if $\alpha_0 \leq \frac{\theta-1}{\theta}(K+1)$. The principal has a non-negative payoff if and only if $\alpha_0 \geq \frac{\theta-1}{\theta} + K$.*

Proof The first claim follows from (43). By (43), we also have

$$X_T - C_T^* = \begin{cases} \frac{X + \alpha_0}{2} - K, & X_T \geq \alpha_0; \\ X_T - K, & \alpha_0 - \frac{\theta-1}{\theta} \leq X_T < \alpha_0; \\ \frac{\theta}{\theta-1}(\alpha_0 - \frac{X_T}{\theta}) - K - 1, & X_T < \alpha_0 - \frac{\theta-1}{\theta}, \end{cases}$$

from which the second claim follows. ∎

Thus, even when the final outcome to be shared is non-negative, $X_T \geq 0$, the principal or the agent may have negative payoffs. However, it is impossible that both the principal and the agent receive negative payoffs simultaneously. The probabilities of receiving negative payoffs depend on the value of α_0.

We now show whether and how the contract $g(\cdot)$ in (43) convexifies or concavifies the agent's utility function. A direct computation yields

$$A^{U_A \circ g} - A^{U_A} = \begin{cases} \frac{1}{2+x-\alpha_0} - \frac{1}{1+x-K}, & x > \alpha_0 \vee K; \\ -\frac{1}{x-\alpha_0} + \frac{1}{1+K-x}, & x < \left(\alpha_0 - \frac{\theta-1}{\theta}\right) \wedge K; \\ \frac{1}{2+x-\alpha_0} + \frac{1}{1+K-x}, & \alpha_0 < x < K; \\ -\frac{1}{x-\alpha_0} - \frac{1}{1+x-K}, & K < x < \alpha_0 - \frac{\theta-1}{\theta}. \end{cases} \tag{44}$$

Proposition 4. *The optimal contract $g(\cdot)$ concavifies $U_A(\cdot)$ on $\{X_T < (\alpha_0 - (\theta - 1)/\theta) \wedge K\}$, which is in the loss domain ($C_T^* < K$). In addition, if $\alpha_0 \leq K + 1$, then the optimal contract convexifies $U_A(\cdot)$ on $\{X_T > \alpha_0 \vee K\}$, which is in the gain domain ($C_T^* > K$).*

Proof When $X_T < \left(\alpha_0 - \frac{\theta-1}{\theta}\right) \wedge K$, we have $-\frac{1}{X_T-\alpha_0} + \frac{1}{1+K-X_T} > 0$; when $\alpha_0 \leq K+1$ and $X_T > \alpha_0 \vee K$, we have $\frac{1}{2+X_T-\alpha_0} - \frac{1}{1+X_T-K} < 0$. By (44) and Lemma 2, the conclusions follow. ∎

Remark 9. From (43) we see that, when $\alpha_0 < X_T < K$, the optimal contract maps the asset value below the reference point ($X_T < K$) to a compensation above K ($C_T^* > K$). Since the utility is S-shaped, the contract concavifies the utility by effectively changing risk-seeking into risk-aversion. For $K < X_T < \alpha_0 - (\theta - 1)/\theta$, the situation is opposite, that is, the contract maps the asset value above the reference point to a compensation below it, convexifying the utility by converting risk-aversion into risk-seeking. On the other hand, if

$$\alpha_0 - \frac{\theta - 1}{\theta} \leq K \leq \alpha_0, \tag{45}$$

or $\alpha_0 \in [K, K + (\theta - 1)/\theta]$, then (43) implies that the principal pays the agent more than K only when the outcome of the managed asset exceeds α_0, which is above K, and pays him less than K only if the outcome falls below a threshold $\alpha_0 - (\theta - 1)/\theta$, which is below K.

5.2 Piecewise Power Utility

In this subsection we study the case where the agent has a piecewise power, S-shaped utility function.

Example 5. *(Piecewise power, S-shaped)* Take

$$U_A(c) = \begin{cases} \frac{1}{\gamma}(c - K)^{\gamma}, & c > K; \\ -\frac{\theta}{\gamma}(K - c)^{\gamma}, & c \leq K, \end{cases} \tag{46}$$

where $\theta > 1$, $K > 0$ and $0 < \gamma < 1$ are constants.

The functions H and L defined in (32) and (33) read

$$H(c, y) = (c - K)^{\gamma - 1}(y - c) - 1, \quad c > K,$$
$$L(c, y) = \theta(K - c)^{\gamma - 1}(y - c) - 1, \quad c < K.$$

Denote

$$\delta \triangleq \gamma(1 - \gamma)^{\frac{1-\gamma}{\gamma}} \theta^{-\frac{1}{\gamma}}. \tag{47}$$

Lemma 6. *For $y \in (K, K + \delta)$, $L(c, y) = 0$ admits two roots $c_A(y) < c_P(y) < K$.*

In addition, $L(c,y) > 0$ *for* $c \in (-\infty, c_A(y)) \cup (c_P(y), K)$ *and* $L(c,y) < 0$ *for* $c \in (c_A(y), c_P(y))$. *Furthermore,* $L(c,y) \geq 0$ *for* $(c,y) \in (-\infty, K) \times [K+\delta, +\infty)$.

Proof For $c < K < y$, denote

$$j(c,y) \triangleq L_c(c,y)(K-c)^{1-\gamma}\frac{1}{\theta(1-\gamma)} = \frac{y-K}{K-c} - \frac{\gamma}{1-\gamma}. \tag{48}$$

Note $j(\cdot, y)$ is strictly increasing, $j(K-, y) = +\infty$ and $j(-\infty, y) < 0$. Therefore $L(\cdot, y)$ attains its minimum at $c_0(y) < K$ such that $j(c_0(y), y) = 0$, i.e.

$$\begin{aligned} c_0(y) =& (K - (1-\gamma)y)\gamma^{-1}, \\ L(c_0(y), y) =& \delta^{-\gamma}(y-K)^{\gamma} - 1. \end{aligned}$$

Therefore, $L(c_0(y), y) < 0$ if and only if $y - K < \delta$. For the case $y - K < \delta$, $L(c,y) = 0$ admits two roots since $L(-\infty, y) = L(K-, y) = +\infty$ and $L_c(c,y)$ has the same sign as $j(c,y)$. ∎

Next, define the function B by

$$B(u_A, u_P, y) \triangleq \arg\max_{u \in \{u_A, u_P\}} F(u,y).$$

Note that $L(c,K) = \theta(K-c)^{\gamma} - 1$ is strictly decreasing for $c < K$. Thus $L(c,K) = 0$ admits a unique root, denoted by $\underline{c}(K)$. Taking into account that $H(c,K) < 0$, we conclude that $C_T^*(K) = \underline{c}(K)$.

Theorem 6. *The optimal contract for an agent with the utility function (46),* $C_T^*(y)$, *is the following*

$$C_T^*(y) = \begin{cases} \bar{c}(y), & y \geq K+\delta; \\ B(\bar{c}(y), c_A(y), y), & K < y < K+\delta; \\ \underline{c}(y), & y \leq K. \end{cases}$$

where $\bar{c}(\cdot)$ *and* $\underline{c}(\cdot)$ *are from Lemma 3 and Lemma 4 with* $U_A(\cdot)$ *defined in* (46).

Proof By Theorem 4, for $y > K$, if $C_T^*(y) \geq K$, then $C_T^*(y) = \bar{c}(y)$; and for $y < K$, $C_T^*(y) = \underline{c}(y)$.

For $y \geq K+\delta$, by Lemma 6, $F_c(c, y) = L(c, y)e^{U_A(c)} \geq 0, \ \forall c < K$. Thus $C_T^*(y) \geq K$. Combining it with Theorem 4, we get $C_T^*(y) = \bar{c}(y)$.

For $K < y < K+\delta$, by Lemma 6, $c_A(y) = \arg\max_{c<K} F(c, y)$. Therefore $C_T^*(y) = B(\bar{c}(y), c_A(y), y)$. ∎

Combining Theorem 6 with Lemma 3 and Lemma 4, we obtain the following result.

Corollary 5. *The optimal contract, as a function of* $Y_T = X_T + \lambda$, *convexifies* $U_A(\cdot)$ *for* $Y_T \geq K+\delta$ *and concavifies it for* $Y_T < K$. ∎

Remark 10. Here, we obtain the same qualitative result as that with the piecewise log utility in the previous example, i.e. the optimal contract induces the agent to be less risk averse in the gain part and less risk seeking in the loss part (see Remark 3). However, unlike in the previous example, the optimal contract here is not piecewise linear; indeed it is "more nonlinear".

Recall from Lemma 3 and Lemma 4 that $0 < \bar{c}'(y) < 1, \ \forall y > K$ and $\underline{c}'(y) > 1, \ \forall y < K$. The following proposition investigates $\bar{c}''(y)$ and $\underline{c}''(y)$.

Proposition 6. *Consider* $\bar{c}(\cdot)$ *and* $\underline{c}(\cdot)$ *from Lemma 3 and Lemma 4 with* $U_A(\cdot)$ *defined in* (46). *We have*

(i) $\bar{c}''(y) > 0, \ \forall y > K$. *In addition, if* $\gamma \in (0, 1/2]$, *then* $U_A \circ \bar{c}$ *is concave on*

$(K, +\infty)$.

(ii) $\underline{c}''(y) > 0, \ \forall y < K$ *and* $U_A \circ \underline{c}$ *is convex on* $(-\infty, K)$.

Proof (i) By definition, $H(\bar{c}(y), y) \equiv 0, \ y > K$. Taking derivative on both sides with respect to y yields

$$\bar{c}'(y) = \frac{\bar{c}(y) - K}{(1-\gamma)(y - \bar{c}(y)) + \bar{c}(y) - K}, \quad \bar{c}''(y) = \frac{\gamma \bar{c}'(y)(1 - \bar{c}'(y))}{(1-\gamma)(y - \bar{c}(y)) + \bar{c}(y) - K}. \quad (49)$$

Since $K < \bar{c}(y) < y$ and $0 < \bar{c}'(y) < 1, \ \forall y > K$, we have $\bar{c}''(y) > 0, \ \forall y > K$. In addition, for $y > K$,

$$A^{U_A \circ \bar{c}}(y) = A^{U_A}(\bar{c}(y)) - \frac{\bar{c}''(y)}{\bar{c}'(y)} = \frac{(1-\gamma)[\bar{c}(y) - K + (1-2\gamma)(y - \bar{c}(y))]}{(1-\gamma)(y - \bar{c}(y)) + \bar{c}(y) - K}.$$

Thus, if $\gamma \in (0, 1/2]$, then $A^{U_A \circ \bar{c}}(y) > 0, \ \forall y > K$.

(ii) Similarly, we have

$$\underline{c}''(y) = \frac{\gamma \underline{c}'(y)(\underline{c}'(y) - 1)}{K - \underline{c}(y) - (1-\gamma)(y - \underline{c}(y))}, \quad \forall y < K. \quad (50)$$

Then $\underline{c}''(y) > 0$ follows by noting $\underline{c}(y) < y < K$ and $\underline{c}'(y) > 1$. Since the composition of two increasing convex function is convex, $U_A \circ \underline{c}$ is convex on $(-\infty, K)$. ∎

Remark 11. Note that the optimal contract is also convex on the loss part. Somewhat surprisingly, the convex contract $\underline{c}(\cdot)$ compounded with the convex function $U_A(\cdot)$ *concavifies* the latter on the loss part. Since the composition of two increasing convex functions, $U_A \circ \underline{c}$, is still convex, the optimal contract keeps the convexity of $U_A(\cdot)$ on the loss part. In contrast, $U_A \circ \bar{c}$ is only concave for $\gamma \in (0, 1/2]$ but may not be overall concave anymore for $\gamma \in (1/2, 1)$. Note that $1 - \gamma$ is the coefficient of the relative risk aversion on the gain part of the agent's utility. Therefore the optimal contract keeps the concavity of the gain part of $U_A(\cdot)$ if the relative risk aversion is

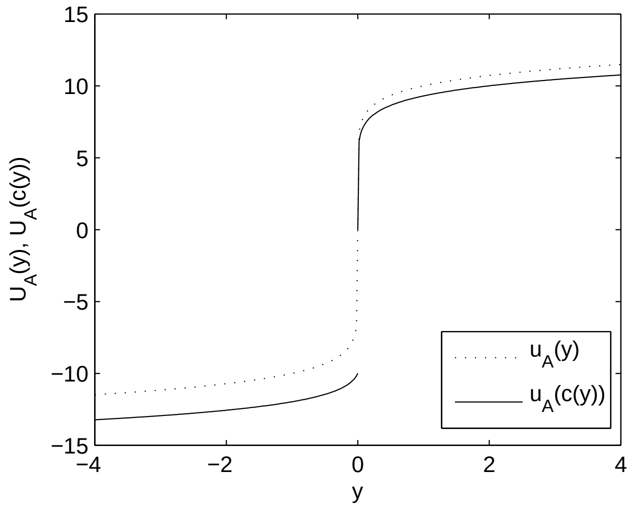

Figure 1: Piecewise power, $\gamma = 0.1, \theta = 1, K$ is normalized to be 0. The dotted line is U_A, the solid line is $U_A \circ c$.

higher and possibly converts the concavity into convexity if the relative risk aversion is lower.

Figure 1 shows the effective concavifying/convexifying of the agent's utility function by $\bar{c}(\cdot)$ and $\underline{c}(\cdot)$. In Figure 2, the contract function is plotted. The dotted line is linear, comparing to which we see that $c(\cdot)$ is convex, while the dashed line is the identity function, comparing to which we see that $c'(\cdot) > 1$ in the loss part.

Suppose a function $T(\cdot)$ satisfies $U_A(c(y)) = T(U_A(y))$ (recall Definition 3). The function $T(\cdot)$ should be convex on the gain part and concave on the loss part. In Figure 3 the dashed lined is linear, compared to which we see the convexity or concavity of T.

6 Conclusions

We find the optimal contracts that the principal should pay to the agent who applies effort to the output process, in the moral hazard framework in which they may have prospect theory type preferences. In particular, we discuss in details the opti-

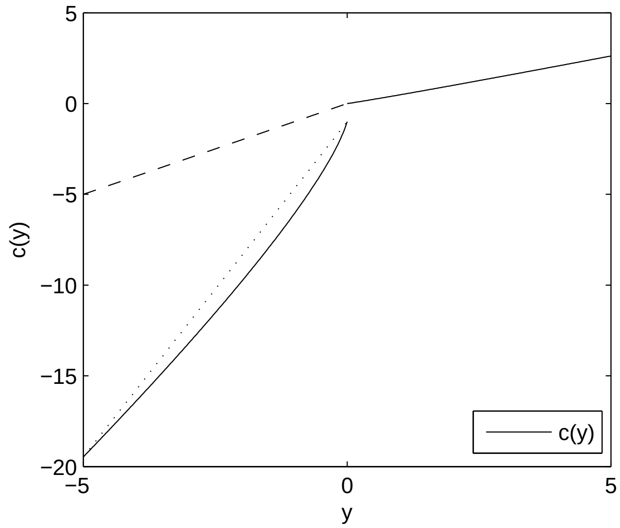

Figure 2: Piecewise power, $\gamma = 0.1, \theta = 1, K$ is normalized to be 0. The solid line is c. The dotted line is linear. The dashed line is the identity function.

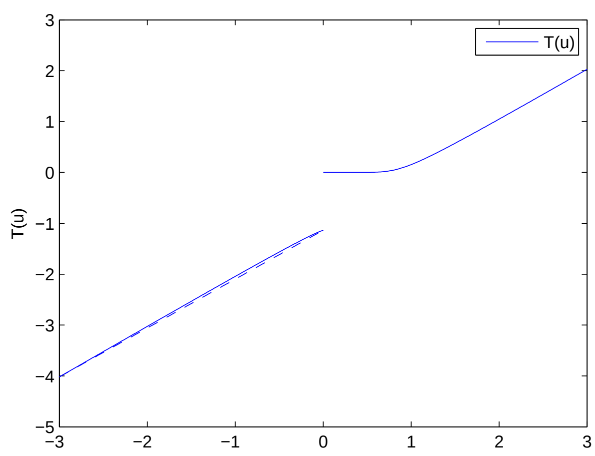

Figure 3: The function $T(\cdot).\gamma = 0.88, \theta = 2.25, K$ is normalized to be zero.

mal contracts when the principal is risk-neutral while the agent has a non-standard, behavioral preference function. We find that the contracts are likely to be more non-linear with those preferences. It would be of interest to extend this analysis to the case of a continuous stream of compensation payments, as is done in Sannikov (2008) for the case of an agent with standard preferences. Moreover, in this paper we only consider non-randomized contracts, and leave the hard problem of finding the optimal randomized contract for future research.

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