# HEDGING AND PORTFOLIO OPTIMIZATION UNDER TRANSACTION COSTS: A MARTINGALE APPROACH*

by


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## Abstract

We derive a formula for the minimal initial wealth needed to hedge an arbitrary contingent claim in a continuous-time model with proportional transaction costs; the expression obtained can be interpreted as the supremum of expected discounted values of the claim, over all (pairs of) probability measures under which the "wealth process" is a supermartingale. Next, we prove the existence of an optimal solution to the portfolio optimization problem of maximizing utility from terminal wealth in the same model; we also characterize this solution via a transformation to a hedging problem: the optimal portfolio is the one that hedges the inverse of marginal utility evaluated at the shadow state-price density solving the corresponding dual problem, if such exists. We can then use the optimal shadow state-price density for pricing contingent claims in this market. The mathematical tools are those of continuous-time martingales, convex analysis, functional analysis and duality theory.


**Abbreviated Title:** Transaction Costs

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## 1. INTRODUCTION.

We obtain a formula for the minimal initial wealth needed to *hedge* an arbitrary contingent claim in a continuous-time model with proportional transaction costs. The expression obtained can be interpreted as the supremum of expected values of the discounted value of the claim, under all feasible "equivalent supermartingale measures", namely the probability measures under which an appropriate "discounted wealth process" is a supermartingale. These results are similar in spirit to some of those obtained by Jouini & Kallal (1991) and, in a discrete-time model, by Kusuoka (1995) and Shirakawa & Konno (1995). Unlike most of the work on hedging under transaction costs (starting with Leland (1985)), we require almost sure rather than just approximate hedge. In other words, the investor has to be able to pay off the claim at the exercise time, no matter what the path of the stock-price has been. This gives an upper bound for the claim price which is typically quite high. For example, in the case of the European call, this mimimal hedging price is equal to the price of the stock at the time the option is purchased (this result was conjectured in Davis & Clark (1994) and was proved by Soner, Shreve & Cvitanić (1995), as well as by Levental & Skorohod (1995)). Nevertheless, we find the result useful in studying the portfolio optimization problem on a finite time-horizon, and also for a preference-based method for pricing claims under transaction costs.

The methodology proceeds as follows. We characterize the solution to the *portfolio optimization problem* of maximizing utility from terminal wealth in the model, via a transformation to a hedging problem; the optimal portfolio is the one that hedges the inverse of marginal utility, evaluated at the shadow state-price density which solves the corresponding dual problem. This hedging-duality approach has been used previously in models of incomplete markets, markets with constraints and markets with nonlinear drifts in the wealth process of the investor (Cvitanić and Karatzas (1992, 1993)), but seems to be new in the context of models with transaction costs. A related approach based on the stochastic maximum principle for singular control problems, as developed in Cadenillas & Haussman (1994), is suggested in Cadenillas & Haussman (1993). The typical approach to utility maximization under transaction costs has been the analytical study of the value function, and the description of the optimal strategy as one with no transactions in a certain region, and with minimal transactions at the boundary in order always to keep the holdings vector inside the region. Such was the spirit of the pioneering work of Magill and Constantinides (1976), and of the more mathematical papers by Taksar, Klass & Assaf (1988), Davis and Norman (1990), Shreve & Soner (1994). Those papers deal with the consumption optimization problem on an infinite-horizon. The finite-horizon problem is studied in Davis, Panas & Zariphopoulou (1993). Our approach gives different insights, can be applied to the case of time-dependent and random market coefficients, but provides no explicit description of optimal strategies, except for the cases in which it is optimal not to trade at all. The latter is the case when the difference between the return rate of the stock and the interest rate is nonnegative but small, and/or the time-horizon is small relative to the transaction costs. This is in contrast with the infinite-horizon case, in which it is always optimal to hold some money in the stock if the return rate is positive, no matter how small it is. It should be of considerable interest, to find additional examples that admit explicit solutions.

We also prove the existence of an optimal trading strategy directly, using standard functional-analytic arguments, without imposing extra assumptions such as the existence of an optimal solution to the dual problem.

Finally, we indicate how to use the optimization result for a utility-based approach to the pricing of contingent claims in such a market. Namely, following the approach of Davis (1994) as applied in Karatzas & Kou (1994), we use the price obtained as the expected value of the claim under the probability measure corresponding to the optimal shadow state-price density (in the dual problem), since with this price the investor becomes neutral between diverting or not diverting a small amount of his funds into the option, resulting in zero marginal rate of substitution.

Our model for proportional transaction costs is the same as that of Davis & Norman (1990). We describe it precisely in Section 2, and define what we mean by hedging in this market. We derive a formula for the minimal hedging price in Section 4, using the auxiliary martingales (shadow state-price densities) introduced in Section 3. In Sections 5 and 6 we describe the utility maximization problem, characterize its solution, and present some simple examples. Section 7 proposes a method for pricing contingent claims in a market with transaction costs, using the optimal state-price density from the dual problem. The long and technical proof of a closedness result for hedging strategies is provided in an Appendix.

Here are some related works on the subject of transaction costs, that the reader may find useful to consult: Avellaneda & Parás (1993), Bensaid, Lesne, Pagès & Scheinkman (1992), Boyle & Tan (1994), Boyle & Vorst (1992), Dewynne, Whalley & Wilmott (1993), Davis & Panas (1994), Davis & Zariphopoulou (1995), Edirisinghe, Naik & Uppal (1993), Figlewski (1989), Flesaker & Hughston (1994), Hodges & Clewlow (1993), Gilster & Lee (1984), Henrotte (1993), Hodges & Neuberger (1989), Hoggard, Whalley & Wilmott (1993), Merton (1989), Morton & Pliska (1993), Panas (1993), Pliska & Selby (1993), Shen (1990), and Toft (1993).

## 2. THE MODEL; DEFINITION OF HEDGING.

We consider a financial market consisting of one riskless asset, called *bank account* (or *bond*) with price $B(\cdot)$ given by

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1; \tag{2.1}$$

and of one risky asset, called *stock*, with price-per-share $S(\cdot)$ governed by the stochastic equation

$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)] \ , \quad S(0) = p \in (0,\infty), \tag{2.2}$$

for $t \in [0,T]$. Here $T \in (0,\infty)$ is the time-horizon, and $W = \{W(t), 0 \le t \le T\}$ a standard, one-dimensional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$; we shall denote by $\mathbf{F} = \{\mathcal{F}(t)\}$ the $\mathbf{P}$-augmentation of the natural filtration $\mathcal{F}^W(t) = \sigma(W(s);\ 0 \le s \le t), 0 \le t \le T$ generated by $W$. The coefficients of the model in (2.1), (2.2) – i.e., the interest rate $r(\cdot)$, the stock-appreciation-rate $b(\cdot)$ and the stock-volatility $\sigma(\cdot) > 0$ – are assumed to be bounded and $\mathbf{F}-$progressively measurable processes; furthermore, $\sigma(\cdot)$ is also assumed to be bounded away from zero (uniformly in $(t,\omega)$).

Now, a *trading strategy* is a pair $(L,M)$ of $\mathbf{F}-$adapted processes on $[0,T]$, with left-continuous, nondecreasing paths and $L(0) = M(0) = 0$; $L(t)$ (respectively, $M(t)$) represents the total amount of funds transferred from bank-account to stock (respectively, from stock to bank-account) by time $t$. Given *proportional transaction costs* $0 < \lambda, \mu < 1$ for such transfers, and initial holdings $x, y$ in bank and stock, respectively, the portfolio holdings $X(\cdot) = X^{x,L,M}(\cdot), Y(\cdot) = Y^{y,L,M}(\cdot)$ corresponding to a given trading strategy $(L,M)$, evolve according to the equations:

$$X(t) = x - (1+\lambda)L(t) + (1-\mu)M(t) + \int_0^t X(u)r(u)du,\ \ 0 \le t \le T \tag{2.3}$$

$$Y(t) = y + L(t) - M(t) + \int_0^t Y(u)[b(u)du + \sigma(u)dW(u)],\ \ 0 \le t \le T. \tag{2.4}$$

**2.1 Definition:** A *contingent claim* is a pair $(C_0, C_1)$ of $\mathcal{F}(T)-$measurable random variables. We say that a trading strategy $(L,M)$ *hedges the claim* $(C_0, C_1)$ *starting with* $(x,y)$ as initial holdings, if $X(\cdot), Y(\cdot)$ of (2.3), (2.4) satisfy

$$X(T) + (1-\mu)Y(T) \ge C_0 + (1-\mu)C_1 \tag{2.5}$$

$$X(T) + (1+\lambda)Y(T) \geq C_0 + (1+\lambda)C_1. \tag{2.6}$$

(Here and in the sequel, comparisons of random variables, in the form of equalities or inequalities, are interpreted "almost surely".)

**Interpretation:** Here $C_0$ (respectively, $C_1$) is understood as a target-position in the bank-account (resp., the stock) at the terminal time $t = T$: for example

$$C_0 = -q\mathbf{1}_{\{S(T)>q\}}, \; C_1 = S(T)\mathbf{1}_{\{S(T)>q\}} \tag{2.7}$$

in the case of a European call-option; and

$$C_0 = q\mathbf{1}_{\{S(T)<q\}}, \; C_1 = -S(T)\mathbf{1}_{\{S(T)<q\}} \tag{2.8}$$

for a European put-option (both with exercise price $q \geq 0$).

"Hedging", in the sense of (2.5) and (2.6), simply means that "one is able to cover these positions at $t = T$". Indeed, assume that we have both $Y(T) \geq C_1$ and (2.5), in the form

$$X(T) + (1-\mu)[Y(T) - C_1] \geq C_0 \; ; \tag{2.5$'$}$$

then (2.6) holds too, and (2.5)$'$ shows that we can cover the position in the bank-account as well, by transferring the amount $Y(T) - C_1 \geq 0$ to it. Similarly, suppose we have $Y(T) < C_1$ and (2.6), in the form

$$Y(T) + \frac{1}{1+\lambda}[X(T) - C_0] \geq C_1 \; ; \tag{2.6$'$}$$

then (2.5) holds as well, and (2.6)$'$ shows that we can again cover both positions by keeping $C_0$ in the bank-account and transferring the difference $X(T) - C_0$ to the stock.

**2.3 Remark:** The equations (2.3), (2.4) can be written in the equivalent form

$$d\left(\frac{X(t)}{B(t)}\right) = \left(\frac{1}{B(t)}\right)[(1-\mu)dM(t) - (1+\lambda)dL(t)] \, , \quad X(0) = x \tag{2.9}$$

$$d\left(\frac{Y(t)}{S(t)}\right) = \left(\frac{1}{S(t)}\right)[dL(t) - dM(t)]\,, \quad Y(0) = y \tag{2.10}$$

in terms of "number-of-shares" (rather than amounts) held.

## 3. AUXILIARY MARTINGALES.

Consider the class $\mathcal{D}$ of pairs of strictly positive $\mathbf{F}$−martingales $(Z_0(\cdot), Z_1(\cdot))$ with

$$Z_0(0) = 1\,, \quad z := Z_1(0) \in [p(1-\mu), p(1+\lambda)] \tag{3.1}$$

and

$$1 - \mu \le R(t) := \frac{Z_1(t)}{Z_0(t)P(t)} \le 1 + \lambda, \quad \forall\ 0 \le t \le T, \tag{3.2}$$

where

$$P(t) := \frac{S(t)}{B(t)} = p + \int_0^t P(u)[(b(u) - r(u))du + \sigma(u)dW(u)]\,, \quad 0 \le t \le T \tag{3.3}$$

is the *discounted stock price.*

The martingales $Z_0(\cdot), Z_1(\cdot)$ are the feasible *state-price densities* for holdings in bank and stock, respectively, in this market with transaction costs; as such, they reflect the "constraints" or "frictions" inherent in this market, in the form of condition (3.2). From the martingale representation theorem (e.g. Karatzas & Shreve (1991), §3.4) there exist $\mathbf{F}$−progressively measurable processes $\theta_0(\cdot), \theta_1(\cdot)$ with $\int_0^T (\theta_0^2(t) + \theta_1^2(t))dt < \infty$ a.s. and

$$Z_i(t) = Z_i(0) \exp\left\{\int_0^t \theta_i(s)dW(s) - \frac{1}{2}\int_0^t \theta_i^2(s)ds\right\}\,, \quad i = 0, 1; \tag{3.4}$$

thus, the process $R(\cdot)$ of (3.2) has the dynamics

$$\begin{aligned} dR(t) =& R(t)[\sigma^2(t) + r(t) - b(t) - (\theta_1(t) - \theta_0(t))(\sigma(t) + \theta_0(t))]dt \\ &+ R(t)(\theta_1(t) - \sigma(t) - \theta_0(t))dW(t), \quad R(0) = z/p. \end{aligned} \tag{3.5}$$

**3.1 Remark:** A rather "special" pair $(Z_0^*(\cdot), Z_1^*(\cdot)) \in \mathcal{D}$ is obtained, if we take in (3.4) the processes $(\theta_0(\cdot), \theta_1(\cdot))$ to be given as

$$\theta_0^*(t) := \frac{r(t) - b(t)}{\sigma(t)} \ , \quad \theta_1^*(t) := \sigma(t) + \theta_0^*(t) \ , \quad 0 \le t \le T, \tag{3.6}$$

and let $Z_0^*(0) = 1, \ p(1-\mu) \le Z_1^*(0) = z \le p(1+\lambda)$. Because then, from (3.5), $R^*(\cdot) := \frac{Z_1^*(\cdot)}{Z_0^*(\cdot)P(\cdot)} \equiv \frac{z}{p}$; in fact, the pair of (3.6) and $z = p$ provide the *only* member $(Z_0^*(\cdot), Z_1^*(\cdot))$ of $\mathcal{D}$, if $\lambda = \mu = 0$. Notice that the processes $\theta_0^*(\cdot)$, $\theta_1^*(\cdot)$ of (3.6) are *bounded.*

**3.2 Remark:** Let us observe also that the martingales $Z_0(\cdot), Z_1(\cdot)$ play the role of *adjoint processes* to the "number-of-share holdings" processes $X(\cdot)/B(\cdot)$, $Y(\cdot)/S(\cdot)$, respectively, in the sense that

(3.7)
$$\begin{aligned} Z_0(t)\frac{X(t)}{B(t)} + Z_1(t)\frac{Y(t)}{S(t)} + \int_0^t \frac{Z_0(s)}{B(s)}[(1+\lambda) - R(s)]dL(s) + \int_0^t \frac{Z_0(s)}{B(s)}[R(s) - (1-\mu)]dM(s) \\ = x + \frac{yz}{p} + \int_0^t \frac{Z_0(s)}{B(s)}[X(s)\theta_0(s) + R(s)Y(s)\theta_1(s)]dW(s) \ , \quad 0 \le t \le T \end{aligned}$$

is a $\mathbf{P}-$local martingale, for any $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ and any trading strategy $(L, M)$; this follows directly from (2.9), (2.10), (3.4) and the product rule. Equivalently, (3.7) can be re-written as

$$\begin{aligned} &\frac{X(t) + R(t)Y(t)}{B(t)} + \int_0^t \frac{(1+\lambda) - R(s)}{B(s)}dL(s) + \int_0^t \frac{R(s) - (1-\mu)}{B(s)}dM(s) \\ &= x + \frac{yz}{p} + \int_0^t \frac{R(s)Y(s)}{B(s)}(\theta_1(s) - \theta_0(s))dW_0(s) = \mathbf{P_0} - \text{local martingale,} \end{aligned} \tag{3.8}$$

where

$$W_0(t) := W(t) - \int_0^t \theta_0(s)ds, \quad 0 \le t \le T \tag{3.9}$$

is (by Girsanov's theorem, e.g. Karatzas & Shreve (1991), §3.5) a Brownian motion under the equivalent probability measure

$$\mathbf{P_0}(A) := E[Z_0(T)\mathbf{1}_A], \ A \in \mathcal{F}(T). \tag{3.10}$$

**3.3 Remark:** We shall denote by $Z_0^*(\cdot), W_0^*(\cdot)$ and $\mathbf{P_0^*}$ the processes and probability measure, respectively, corresponding to the process $\theta_0^*(\cdot)$ of (3.6), via the equations (3.4) (with $Z_0^*(0) = 1$), (3.9) and (3.10). With this notation, (3.3) becomes $dP(t) = P(t)\sigma(t)dW_0^*(t)$, $P(0) = p$.

**3.4 Definition:** Let $\mathcal{D}_\infty$ be the class of positive martingales $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, for which the random variable

$$\text{(3.11)} \qquad \frac{Z_0(T)}{Z_0^*(T)}, \qquad \text{and thus also} \quad \frac{Z_1(T)}{Z_0^*(T)P(T)}\ ,$$

is essentially bounded.

**3.5 Definition:** We shall say that a given trading strategy $(L, M)$ is *admissible* for $(x, y)$, and write $(L, M) \in \mathcal{A}(x, y)$, if

$$\text{(3.12)} \qquad \frac{X(\cdot) + R(\cdot)Y(\cdot)}{B(\cdot)} \quad \text{is a } \mathbf{P_0} - \text{supermartingale}, \ \ \forall\ (Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty.$$

Consider, for example, a trading strategy $(L, M)$ that satisfies the no-bankruptcy conditions

$$X(t) + (1 + \lambda)Y(t) \geq 0 \ \ \text{and} \ \ X(t) + (1 - \mu)Y(t) \geq 0, \ \ \forall\ 0 \leq t \leq T.$$

Then $X(\cdot) + R(\cdot)Y(\cdot) \geq 0$ for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ (recall (3.2), and note Remark 3.6 below); this means that the $\mathbf{P_0}$−local martingale of (3.8) is nonnegative, hence a $\mathbf{P_0}$−supermartingale. But the second and the third terms

$$\int_0^{\cdot} \frac{1 + \lambda - R(s)}{B(s)} dL(s), \quad \int_0^{\cdot} \frac{R(s) - (1 - \mu)}{B(s)} dM(s)$$

in (3.8) are increasing processes, thus the first term $\frac{X(\cdot)+R(\cdot)Y(\cdot)}{B(\cdot)}$ is also a $\mathbf{P_0}$−supermartingale, for every pair $(Z_0(\cdot), Z_1(\cdot))$ in $\mathcal{D}$. The condition (3.12) is actually weaker, in that it requires this property only for pairs in $\mathcal{D}_\infty$. This provides a motivation for Definition 3.4, namely, *to allow for as wide a class of trading strategies as possible, and still exclude* *arbitrage opportunities.* This is usually done by imposing a lower bound on the wealth process; however, that excludes simple strategies of the form "trade only once, by buying a fixed number of shares of the stock at a specified time $t$", which may require (unbounded) borrowing. We shall have occasion, to use such strategies in the sequel; see, for example, (4.20).

**3.6 Remark:** Here is a trivial (but useful) observation: if $x+(1-\mu)y \geq a+(1-\mu)b$ and $x+(1+\lambda)y \geq a+(1+\lambda)b$, then $x+ry \geq a+rb$, $\forall\ 1-\mu \leq r \leq 1+\lambda$.

## 4. HEDGING PRICE.

Suppose that we are given an initial holding $y \in R$ in the stock, and want to hedge a given contingent claim $(C_0, C_1)$ with strategies which are admissible (in the sense of Definitions 2.1, 3.4). What is the smallest amount of holdings in the bank

$$h(C_0, C_1; y) := \inf\{x \in R/\ \exists (L, M) \in \mathcal{A}(x, y) \text{ and } \ (L, M) \text{ hedges } (C_0, C_1)\} \tag{4.1}$$

that allows to do this? We call $h(C_0, C_1; y)$ the *hedging price* of the contingent claim $(C_0, C_1)$ for initial holding $y$ in the stock, and with the convention that $h(C_0, C_1; y) = \infty$ if the set in (4.1) is empty.

Suppose this is not the case, and let $x \in R$ belong to the set of (4.1); then for any $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$ we have from (3.12), the Definition 2.1 of hedging, and Remark 3.6:

$$\begin{aligned} x + \frac{y}{p} E Z_1(T) = x + \frac{y}{p} z &\geq E_0 \left[ \frac{X(T) + R(T)Y(T)}{B(T)} \right] \\ &\geq E_0 \left[ \frac{C_0 + R(T)C_1}{B(T)} \right] = E \left[ \frac{Z_0(T)}{B(T)} (C_0 + R(T)C_1) \right], \end{aligned}$$

so that $x \geq E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{p} Z_1(T)\right]$. Therefore

$$h(C_0, C_1; y) \geq \sup_{\mathcal{D}_\infty} E \left[ \frac{Z_0(T)}{B(T)} (C_0 + R(T)C_1) - \frac{y}{p} Z_1(T) \right], \tag{4.2}$$

and this inequality is clearly also valid if $h(C_0, C_1; y) = \infty$.

**4.1 Lemma:** *If the contingent claim* $(C_0, C_1)$ *is bounded from below, in the sense*

$$(4.3) \qquad C_0 + (1+\lambda)C_1 \geq -K \ \textit{and} \ C_0 + (1-\mu)C_1 \geq -K, \ \textit{for some} \ 0 \leq K < \infty$$

*then*

$$(4.4) \quad \sup_{\mathcal{D}_\infty} E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{p}Z_1(T)\right] = \sup_{\mathcal{D}} E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{p}Z_1(T)\right].$$

**Proof:** Start with arbitrary $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ and define the sequence of stopping times $\{\tau_n\} \uparrow T$ by

$$\tau_n := \inf\{t \in [0,T] \ / \ \frac{Z_0(t)}{Z_0^*(t)} \geq n\} \wedge T, \ n \in N.$$

Consider also, for $i = 0, 1$ and in the notation of (3.6): $\theta_i^{(n)}(t) := \begin{Bmatrix} \theta_i(t), & 0 \leq t < \tau_n \\ \theta_i^*(t), & \tau_n \leq t \leq T \end{Bmatrix}$

and

$$Z_i^{(n)}(t) = z_i \exp\{\int_0^t \theta_i^{(n)}(s)dW(s) - \frac{1}{2}\int_0^t (\theta_i^{(n)}(s))^2 ds\}$$

with $z_0 = 1, \ z_1 = Z_1(0) = EZ_1(T)$. Then, for every $n \in N$, both $Z_0^{(n)}(\cdot)$ and $Z_1^{(n)}(\cdot)$ are positive martingales, $R^{(n)}(\cdot) = \frac{Z_1^{(n)}(\cdot)}{Z_0^{(n)}(\cdot)P(\cdot)} = R(\cdot \wedge \tau_n)$ takes values in $[1-\mu, 1+\lambda]$ (by (3.2) and Remark 3.1), and $Z_0^{(n)}(\cdot)/Z_0^*(\cdot)$ is bounded by $n$ (in fact, constant on $[\tau_n, T]$). Therefore, $(Z_0^{(n)}(\cdot), Z_1^{(n)}(\cdot)) \in \mathcal{D}_\infty$. Now let $\kappa$ denote an upper bound on $K/B(T)$, and observe, from Remark 3.6, (4.3) and Fatou's lemma:

$$\begin{aligned}
&E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{p}Z_1(T)\right] + \frac{y}{p}Z_1(0) + \kappa \\
&= E\left[Z_0(T)\left\{\frac{C_0 + R(T)C_1}{B(T)} + \kappa\right\}\right] \\
(4.5) \qquad &= E\left[\lim_n Z_0^{(n)}(T)\left\{\frac{C_0 + R^{(n)}(T)C_1}{B(T)} + \kappa\right\}\right] \\
&\leq \liminf_n E\left[Z_0^{(n)}(T)\left\{\frac{C_0 + R^{(n)}(T)C_1}{B(T)} + \kappa\right\}\right] \\
&= \liminf_n E\left[\frac{Z_0^{(n)}(T)}{B(T)}(C_0 + R^{(n)}(T)C_1) - \frac{y}{p}Z_1^{(n)}(T)\right] + \frac{y}{p}Z_1(0) + \kappa.
\end{aligned}$$

This shows that the left-hand-side dominates the right-hand-side in (4.4); the reverse inequality is obvious. □

**Remark:** Formally taking $y = 0$ in (4.5), we deduce

$$E_0 \left( \frac{C_0 + R(T)C_1}{B(T)} \right) \leq \liminf_{n\to\infty} E_0^{(n)} \left( \frac{C_0 + R^{(n)}(T)C_1}{B(T)} \right), \tag{4.6}$$

where $E_0, E_0^{(n)}$ denote expectations with respect to the probability measures $\mathbf{P_0}$ of (3.10) and $\mathbf{P_0^{(n)}}(\cdot) = E[Z_0^{(n)}(T)\mathbf{1}.]$, respectively.

Here is the main result of this section.

**4.2 Theorem:** *Under the conditions (4.3) and*

$$E_0^*(C_0^2 + C_1^2) < \infty \;, \tag{4.7}$$

*we have*

$$h(C_0, C_1; y) = \sup_{\mathcal{D}} E\left[ \frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{p} Z_1(T) \right]. \tag{4.8}$$

In (4.7), $E_0^*$ denotes expectation with respect to the probability measure $\mathbf{P_0^*}$ of Remark 3.2. The conditions (4.3), (4.7) are both easily verified for a European call or put (as in (2.7) or (2.8), respectively). In fact, using the first part of the argument in Appendix A, one can show that if a pair of admissible terminal holdings $(X(T), Y(T))$ hedges, in the sense of (2.5) and (2.6), a pair $(\tilde{C}_0, \tilde{C}_1)$ satisfying (4.7) (for example, $(\tilde{C}_0, \tilde{C}_1) \equiv (0, 0)$), then necessarily the pair $(X(T), Y(T))$ also satisfies (4.7) – and so does any other pair of random variables $(C_0, C_1)$ which are bounded from below and are hedged by $(X(T), Y(T))$. In other words, any strategy which satisfies the "no-bankruptcy" condition of hedging $(0, 0)$, necessarily results in a square-integrable final wealth. In this sense, the condition (4.7) is consistent with the standard "no-bankruptcy" condition, hence not very restrictive.

It would be of significant interest to be able to describe the least expensive hedging strategy associated with a general hedgeable contingent claim; this would probably require a purely *probabilistic approach*, using dynamic programming and control-theoretic ideas coupled with martingale methods, in the spirit of our earlier work Cvitanić & Karatzas (1993). Such a proof we have not been able to obtain. Our *functional-analytic proof*, which takes up the remainder of this section and was inspired by similar arguments in Kusuoka (1995), does not provide the construction of such a strategy.

**Proof:** In view of Lemma 4.1 and the inequality (4.2), it suffices to show

$$h(C_0, C_1; y) \le \sup_{\mathcal{D}} E\left[Z_0(T)\frac{C_0}{B(T)} + Z_1(T)\left(\frac{C_1}{S(T)} - \frac{y}{p}\right)\right] =: R. \tag{4.9}$$

And in order to alleviate somewhat the (already rather heavy) notation, we shall take $p = 1$, $r(\cdot) \equiv 0$, thus $B(\cdot) \equiv 1$, for the remainder of the section and in Appendix A; the reader will verify easily that this entails no loss of generality.

We start by taking an arbitrary $b < h(C_0, C_1; y)$ and considering the sets

(4.10)

$$A_0 := \{(U, V) \in (\mathbf{L}_2^*)^2 : \exists (L, M) \in \mathcal{A}(0,0) \text{ that hedges } (U, V) \text{ starting with } x = 0, y = 0\}$$

$$A_1 := \{(C_0 - b, C_1 - yS(T))\}, \tag{4.11}$$

where $\mathbf{L}_2^* = \mathbf{L}_2(\Omega, \mathcal{F}(T), \mathbf{P}_\mathbf{0}^*)$. It is not hard to prove (see below) that

$$A_0 \textit{ is a convex cone, and contains the origin } (0,0), \textit{ in } (\mathbf{L}_2^*)^2, \tag{4.12}$$

$$A_0 \cap A_1 = \emptyset. \tag{4.13}$$

It is, however, considerably harder to establish that

$$A_0 \textit{ is closed in } (\mathbf{L}_2^*)^2. \tag{4.14}$$

(This proof is carried out in Appendix A.) From (4.12)-(4.14) and the Hahn-Banach theorem there exists a pair of random variables $(\rho_0^*, \rho_1^*) \in (\mathbf{L}_2^*)^2$, not equal to $(0,0)$, such that

$$E_0^*[\rho_0^* V_0 + \rho_1^* V_1] = E[\rho_0 V_0 + \rho_1 V_1] \le 0, \quad \forall\ (V_0, V_1) \in A_0 \tag{4.15}$$

$$(4.16) \qquad E_0^*[\rho_0^*(C_0-b)+\rho_1^*(C_1-yS(T))] = E[\rho_0(C_0-b)+\rho_1(C_1-yS(T))] \geq 0,$$

where $\rho_i := \rho_i^* Z_0^*(T),\ i=0,1$. It is also not hard to check (see below) that

$$(4.17) \qquad (1-\mu)E[\rho_0|\mathcal{F}(t)] \leq \frac{E[\rho_1 S(T)|\mathcal{F}(t)]}{S(t)} \leq (1+\lambda)E[\rho_0|\mathcal{F}(t)],\ \forall\ 0\leq t\leq T$$

$$(4.18) \qquad \rho_1 \geq 0,\ \ \rho_0 \geq 0 \text{ and } E\rho_0 > 0,\ \ E(\rho_1 S(T)) > 0.$$

In view of (4.18), we may take $E\rho_0 = 1$, and then (4.16) gives

$$(4.19) \qquad b \leq E[\rho_0 C_0 + \rho_1(C_1 - yS(T))].$$

Consider now *arbitrary* $0<\varepsilon<1,\ (Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, and define

$$\tilde{Z}_0(t) := \varepsilon Z_0(t) + (1-\varepsilon)E[\rho_0|\mathcal{F}(t)],\ \tilde{Z}_1(t) := \varepsilon Z_1(t) + (1-\varepsilon)E[\rho_1 S(T)|\mathcal{F}(t)],\ 0\leq t\leq T.$$

Clearly these are positive martingales, and $\tilde{Z}_0(0)=1$; on the other hand, multiplying in (4.17) by $1-\varepsilon$, and in $(1-\mu)Z_0(t) \leq Z_1(t)/S(t) \leq (1+\lambda)Z_0(t),\ 0\leq t\leq T$ (just (3.2) with $r(\cdot)\equiv 0$) by $\varepsilon$, and adding up, we obtain $(\tilde{Z}_0(\cdot), \tilde{Z}_1(\cdot)) \in \mathcal{D}$. Thus, in the notation of (4.9),

$$\begin{aligned}
R &\geq E\left[\tilde{Z}_0(T)C_0 + \tilde{Z}_1(T)\left(\frac{C_1}{S(T)} - y\right)\right] \\
&= (1-\varepsilon)E[\rho_0 C_0 + \rho_1(C_1 - yS(T))] + \varepsilon E\left[Z_0(T)C_0 + Z_1(T)\left(\frac{C_1}{S(T)} - y\right)\right] \\
&\geq b(1-\varepsilon) + \varepsilon E\left[Z_0(T)C_0 + Z_1(T)\left(\frac{C_1}{S(T)} - y\right)\right]
\end{aligned}$$

from (4.19); letting $\varepsilon \downarrow 0$ and then $b \uparrow h(C_0, C_1; y)$, we obtain (4.9), as required to complete the proof of Theorem 4.2.

**Proof of (4.12):** Suppose that $(U_i, V_i) \in (\mathbf{L}_2^*)^2$ are hedged by $(L_i, M_i) \in \mathcal{A}(0,0)$, respectively, for $i=1,2$; in other words, if we let $(X_i, Y_i)$ be the corresponding holdings as in (2.3), (2.4) (with $x=y=0,\ r(\cdot)\equiv 0$), we have:

$$X_i(T) + (1-\mu)Y_i(T) \geq U_i + (1-\mu)V_i,$$

$$X_i(T) + (1+\lambda)Y_i(T) \geq U_i + (1+\lambda)V_i,$$

and $X_i(\cdot) + R(\cdot)Y_i(\cdot)$ is a $\mathbf{P_0}-$supermartingale, $\forall\ (Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$ for $i = 1, 2$. Now, with $\zeta \geq 0, \eta \geq 0$ and $(U, V) = (\zeta U_1 + \eta U_2, \zeta V_1 + \eta V_2) \in (\mathbf{L}_2^*)^2$, it is straightforward to see (using the linearity of the equations (2.3) and (2.4)) that $(U, V)$ is hedged by $(L, M) = (\zeta L_1 + \eta L_2, \zeta M_1 + \eta M_2) \in \mathcal{A}(0, 0)$. If we take $0 < \eta < 1,\ \zeta = 1 - \eta$ we verify the convexity of $A_0$; if we take $\eta > 0, \zeta = 0$ we verify that $A_0$ is a cone; and we can hedge $(0, 0) \in (\mathbf{L}_2^*)^2$ simply by $L \equiv M \equiv 0$.

**Proof of (4.13):** Suppose that $A_0 \cap A_1$ is not empty, i.e., that there exists $(L, M) \in \mathcal{A}(0, 0)$ such that, with $X(\cdot) = X^{0,L,M}(\cdot)$ and $Y(\cdot) = Y^{0,L,M}(\cdot)$, the process $X(\cdot) + R(\cdot)Y(\cdot)$ is a $\mathbf{P_0}-$supermartingale for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$, and we have:

$$X(T) + (1-\mu)Y(T) \geq (C_0 - b) + (1-\mu)(C_1 - yS(T)),$$

$$X(T) + (1+\lambda)Y(T) \geq (C_0 - b) + (1+\lambda)(C_1 - yS(T)).$$

But then, with

$$\tilde{X}(\cdot) := X^{b,L,M}(\cdot) = b + X(\cdot),\ \tilde{Y}(\cdot) := Y^{y,L,M}(\cdot) = Y(\cdot) + yS(\cdot)$$

we have, from above, that $\tilde{X}(\cdot) + R(\cdot)\tilde{Y}(\cdot) = X(\cdot) + R(\cdot)Y(\cdot) + b + yZ_1(\cdot)/Z_0(\cdot)$ is a $\mathbf{P_0}-$supermartingale for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$, and that

$$\tilde{X}(T) + (1-\mu)\tilde{Y}(T) \geq C_0 + (1-\mu)C_1,$$

$$\tilde{X}(T) + (1+\lambda)\tilde{Y}(T) \geq C_0 + (1+\lambda)C_1.$$

In other words, $(L, M)$ belongs to $\mathcal{A}(b, y)$ and hedges $(C_0, C_1)$ starting with $(b, y)$ – a contradiction to the definition (4.1), and to the fact $h(C_0, C_1; y) > b$.

**Proof of (4.17), (4.18):** Fix $t \in [0, T)$ and let $\xi$ be an arbitrary bounded, nonnegative, $\mathcal{F}(t)-$measurable random variable. Consider the strategy of starting with $(x, y) = (0, 0)$

and buying $\xi$ shares of stock at time $s = t$, otherwise doing nothing ("buy-and-hold strategy"); more explicitly, $M^\xi(\cdot) \equiv 0$, $L^\xi(s) = \xi S(t) \mathbf{1}_{(t,T]}(s)$ and thus

(4.20)

$$X^\xi(s) := X^{0,L^\xi,M^\xi}(\cdot) = -\xi(1+\lambda)S(t)\mathbf{1}_{(t,T]}(s), \quad Y^\xi(s) := Y^{0,L^\xi,M^\xi}(s) = \xi S(s)\mathbf{1}_{(t,T]}(s),$$

for $0 \le s \le T$. Consequently, $Z_0(s)[X^\xi(s)+R(s)Y^\xi(s)] = \xi[Z_1(s)-(1+\lambda)S(t)Z_0(s)]\mathbf{1}_{(t,T]}(s)$ is a $\mathbf{P}-$supermartingale for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, since, for instance with $t < s \le T$:

$$\begin{aligned}
E[Z_0(s)(X^\xi(s) + R(s)Y^\xi(s))|\mathcal{F}(t)] &= \xi\left(E[Z_1(s)|\mathcal{F}(t)] - (1+\lambda)S(t)E[Z_0(s)|\mathcal{F}(t)]\right) \\
= \xi[Z_1(t) - (1+\lambda)S(t)Z_0(t)] &= \xi S(t)Z_0(t)[R(t) - (1+\lambda)] \le 0 = Z_0(t)[X^\xi(t) + R(t)Y^\xi(t)].
\end{aligned}$$

Therefore, $(L^\xi, M^\xi) \in \mathcal{A}(0,0)$, thus $(X^\xi(T), Y^\xi(T))$ belongs to the set $A_0$ of (4.10), and, from (4.15):

$$\begin{aligned}
0 \ge E[\rho_0 X^\xi(T) + \rho_1 Y^\xi(T)] &= E[\xi(\rho_1 S(T) - (1+\lambda)\rho_0 S(t))] \\
&= E\big[\xi\big(E[\rho_1 S(T)|\mathcal{F}(t)] - (1+\lambda)S(t)E[\rho_0|\mathcal{F}(t)]\big)\big].
\end{aligned}$$

From the arbitrariness of $\xi \ge 0$, we deduce the inequality of the right-hand side in (4.17), and a dual argument gives the inequality of the left-hand side, for given $t \in [0,T)$. Now all three processes in (4.17) have continuous paths (recall that martingales of the Brownian filtration are representable as stochastic integrals, and thus have almost all paths continuous); consequently, (4.17) is valid for *all* $t \in [0,T]$.

Next, we notice that (4.17) with $t = T$ implies $(1-\mu)\rho_0 \le \rho_1 \le (1+\lambda)\rho_0$, so that $\rho_0$, hence also $\rho_1$, is nonnegative. Similarly, (4.17) with $t = 0$ implies $(1-\mu)E\rho_0 \le E[\rho_1 S(T)] \le (1+\lambda)E\rho_0$, and therefore, since $(\rho_0, \rho_1)$ is not equal to $(0,0)$, $E\rho_0 > 0$, hence also $E[\rho_1 S(T)] > 0$. This proves (4.18). □

**4.3 Example:** Consider the *European call option* of (2.7), whereby one has to deliver a share of the stock if the price $S(T)$ at time $t = T$ exceeds $q$, and one can still cover the remaining position in the bank by the amount $q > 0$ of the exercise price. From (4.8) with

$y = 0$, we have

$$h(C_0, C_1) \equiv h(C_0, C_1; 0) = \sup_{\mathcal{D}} E\left[Z_1(T)\mathbf{1}_{\{S(T)>q\}} - q\frac{Z_0(T)}{B(T)}\mathbf{1}_{\{S(T)>q\}}\right], \tag{4.21}$$

and therefore, $h(C_0, C_1) \leq \sup_{\mathcal{D}} EZ_1(T) = \sup_{\mathcal{D}} Z_1(0) \leq (1+\lambda)p$. The number $p(1+\lambda)$ corresponds to the cost of the "buy-and-hold strategy", of acquiring one share of the stock at $t = 0$ (at a price $p(1+\lambda)$, due to the transaction cost), and holding on to it until $t = T$. Davis & Clark (1993) conjectured that this hedging strategy is actually the cheapest:

$$h(C_0, C_1) = (1+\lambda)p. \tag{4.22}$$

The conjecture (4.22) was proved by Soner, Shreve & Cvitanić (1995), as well as by Levental & Skorohod (1995). It is an open question *to derive (4.22) directly from the representation (4.21)*; in other words, to find a sequence $\{(Z_0^{(n)}(\cdot), Z_1^{(n)}(\cdot))\}_{n \in N}$ with

$$\mathbf{P_0}^{(n)}[S(T) > q] \to 0, \quad E[Z_1^{(n)}(T)\mathbf{1}_{\{S(T)>q\}}] \to 1, \quad Z_1^{(n)}(0) \to 1+\lambda,$$

as $n \uparrow \infty$. We have not yet been able to accomplish this.

## 5. UTILITY FUNCTIONS.

In the next section we shall use the basic result, Theorem 4.2, to discuss some expected-utility-maximization problems in the context of the model of section 2. For this, we shall need the concept of utility function.

A function $U : (0, \infty) \to R$ will be called *utility function* if it is strictly increasing, strictly concave, contnuously differentiable, and satisfies

$$U'(0+) := \lim_{x \downarrow 0} U'(x) = \infty, \quad U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{5.1}$$

We shall understand $U(x) = -\infty$ for $x < 0$.

The continuous, strictly decreasing function $U'(\cdot)$ has an inverse $I(\cdot)$ with these same properties, which maps $(0,\infty)$ onto itself, and satisfies $I(0+)=\infty,\ I(\infty)=0$. We shall also find useful the convex dual

$$\tilde{U}(\zeta) := \max_{x>0}[U(x) - x\zeta] = U(I(\zeta)) - \zeta I(\zeta), \quad 0<\zeta<\infty \tag{5.2}$$

of $U(\cdot)$, which satisfies

$$\tilde{U}'(\zeta) = -I(\zeta), \quad 0<\zeta<\infty \tag{5.3}$$

**Remark:** For some purposes, we shall need to impose the extra condition

$$xU'(x) \le a + (1-b)U(x), \ \forall\ 0<x<\infty \tag{5.4}$$

(for suitable $a \ge 0, 0 < b \le 1$) on our utility functions. This condition is clearly satisfied by $U(x) = \log x$ and by $U(x) = \frac{1}{\delta}x^{\delta}$, for $0<\delta<1$; it is also satisfied if $U(\infty)=\infty$ and $U(\cdot)$ is bounded from below (cf. Cuoco (1994)).

## 6. MAXIMIZING EXPECTED UTILITY FROM TERMINAL WEALTH.

Consider now a small investor, who can make decisions in the context of the market model of (2.1), (2.2) as described in section 2, and who derives utility $U(X(T+))$ from his *terminal wealth*

$$X(T+) := X(T) + f(Y(T)), \ \text{where} \ \ f(u) := \left\{ \begin{array}{l} (1+\lambda)u\ ;\ u \le 0 \\ (1-\mu)u\ ;\ u>0 \end{array} \right\}. \tag{6.1}$$

In other words, this agent liquidates at the end of the day his position in the stock, incurs the appropriate transaction cost, and collects all the money in the bank-account. For a given initial holding $y \ge 0$ in the stock, his optimization problem is to find an admissible pair $(\hat{L}, \hat{M}) \in \mathcal{A}^+(x,y)$ that *maximizes expected utility from terminal wealth*, i.e., attains the supremum

$$V(x;y) := \sup_{(L,M)\in\mathcal{A}^+(x,y)} EU\big(X^{x,L,M}(T) + f(Y^{y,L,M}(T))\big), \ 0<x<\infty, \tag{6.2}$$

where $\mathcal{A}^+(x,y)$ is the class of processes $(L,M) \in \mathcal{A}(x,y)$ for which $X^{x,L,M}(T)+f(Y^{y,L,M}(T)) \geq 0$. We show in Appendix B that the supremum of (6.2) is attained, i.e., that there exists an optimal pair $(\hat{L},\hat{M})$ for this problem, and that $V(x,y) < \infty$. Our purpose in this section is to describe the nature of this optimal pair, by using results of section 4 in the context of the *dual problem*

$$\tilde{V}(\zeta;y) := \inf_{(Z_0,Z_1)\in\mathcal{D}} E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right],\ 0<\zeta<\infty, \tag{6.3}$$

under the following assumption.

**6.1 Assumption:** There exists a pair $(\hat{Z}_0(\cdot),\hat{Z}_1(\cdot)) \in \mathcal{D}$, that attains the infimum in (6.3), and does so for all $0<\zeta<\infty$. Moreover, for all $0<\zeta<\infty$, we have

$$\tilde{V}(\zeta;y)<\infty \quad \text{and} \quad E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\frac{\zeta Z_0^*(T)}{B(T)}\right)\right] < \infty.$$

**6.2 Remark:** The assumption that the infimum of (6.3) is attained is a big one; we have not yet been able to obtain a general existence result to this effect, only very simple examples that can be solved explicitly (cf. Examples 6.5-6.7). The assumption that the minimization in (6.3) can be carried out for all $0<\zeta<\infty$ simultaneously, is made only for simplicity; it can be dispensed with using methods analogous to those in Cvitanić & Karatzas (1992). Note, however, that this latter assumption is satisfied if $y=0$ and either $U(x)=\log x$ or $U(x)=\frac{1}{\delta}x^\delta$ for $0<\delta<1$. It should also be mentioned that the optimal pair $(\hat{Z}_0(\cdot),\hat{Z}_1(\cdot))$ of Assumption 6.1 need not be unique (we thank the anonimous referee for pointing this out); thus, in the remainder of this section, $(\hat{Z}_0(\cdot),\hat{Z}_1(\cdot))$ will denote *any* pair that attains the infimum in (6.3), as in Assumption 6.1

For any such pair, we have then the following property, proved at the end of this section.

**6.3 Lemma:** *Under the Assumption 6.1 and the condition (5.4), we have*

$$E\left[\frac{Z_0(T)}{B(T)} I\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) - \frac{y}{p}Z_1(T)\right] \leq E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) - \frac{y}{p}\hat{Z}_1(T)\right] < \infty,\ \forall 0<\zeta<\infty \tag{6.4}$$

*for every* $(Z_0(\cdot), Z_1(\cdot))$ *in* $\mathcal{D}$.

Now, because the function $\zeta \mapsto E\left[\frac{\hat{Z}_0(T)}{B(T)} I(\zeta \frac{\hat{Z}_0(T)}{B(T)})\right] : (0,\infty) \to (0,\infty)$ is continuous and strictly decreasing, there exists a unique $\hat{\zeta} = \hat{\zeta}(x; y, U) \in (0,\infty)$ that satisfies

$$E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right] = x + \frac{y}{p} E\hat{Z}_1(T). \tag{6.5}$$

And with

$$\hat{C}_0 := I\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right), \quad \hat{C}_1 := 0, \tag{6.6}$$

it follows from (6.4) that

(6.7)
$$\begin{aligned}\sup_{(Z_0,Z_1)\in\mathcal{D}} E\left[Z_0(T)\frac{\hat{C}_0}{B(T)} + Z_1(T)\left(\frac{\hat{C}_1}{S(T)} - \frac{y}{p}\right)\right] &= E\left[\hat{Z}_0(T)\frac{\hat{C}_0}{B(T)} + \hat{Z}_1(T)\left(\frac{\hat{C}_1}{S(T)} - \frac{y}{p}\right)\right]\\ &= x.\end{aligned}$$

Consequently, if in addition we have $\hat{C}_0 \in \mathbf{L}_2^*$, then Theorem 4.2 gives $h(\hat{C}_0, \hat{C}_1; y) = x$. Now it can be shown, by an argument analogous to that in the Appendix A (see also the appendix in Soner, Shreve & Cvitanić (1995)), that *the infimum in (4.1) is actually attained*; in other words, there exists a pair $(\hat{L}, \hat{M}) \in \mathcal{A}(x,y)$ such that, with $\hat{X}(\cdot) \equiv X^{x,\hat{L},\hat{M}}(\cdot)$, $\hat{Y}(\cdot) \equiv Y^{y,\hat{L},\hat{M}}(\cdot)$, we have

$$\hat{X}(T) + (1-\mu)\hat{Y}(T) \geq \hat{C}_0, \quad \hat{X}(T) + (1+\lambda)\hat{Y}(T) \geq \hat{C}_0. \tag{6.8}$$

**6.4 Theorem:** *Under Assumption 6.1, and the conditions (5.4),*

$$E_0^*[\hat{C}_0^2] = E_0^*\left[I^2(\hat{\zeta}\hat{Z}_0(T)/B(T))\right] < \infty, \tag{6.9}$$

*the above pair* $(\hat{L}, \hat{M}) \in \mathcal{A}(x,y)$ *is optimal for the problem of (6.2), and satisfies*

$$\hat{X}(T+) := \hat{X}(T) + f(\hat{Y}(T)) = I(\hat{\zeta}\hat{Z}_0(T)/B(T)) = \hat{C}_0 \tag{6.10}$$

$$\hat{L}(\cdot) \textit{ is flat off the set } \{0 \leq t \leq T / \hat{R}(t) = 1+\lambda\} \tag{6.11}$$

(6.12) $$\hat{M}(\cdot) \text{ is flat off the set } \{0 \le t \le T / \hat{R}(t) = 1-\mu\}$$

(6.13) $$\frac{\hat{X}(t) + \hat{R}(t)\hat{Y}(t)}{B(t)} = \hat{E}_0 \left[ \frac{I(\hat{\zeta}\hat{Z}_0(T)/B(T))}{B(T)} \middle| \mathcal{F}(t) \right], \ 0 \le t \le T,$$

*where* $\hat{R}(\cdot) := \frac{\hat{Z}_1(\cdot)}{\hat{Z}_0(\cdot)P(\cdot)}$. *Furthermore, we have* $\tilde{V}(\hat{\zeta}; y) = V(x; y) - x\hat{\zeta} < \infty$.

**Proof:** As we just argued, (6.9) and Theorem 4.2 imply the existence of a pair $(\hat{L}, \hat{M}) \in \mathcal{A}(x, y)$, so that (6.8) is satisfied; and from (6.8), we know that both

(6.14) $$\hat{X}(T) + \hat{R}(T)\hat{Y}(T) \ge \hat{C}_0, \quad \hat{X}(T) + f(\hat{Y}(T)) \ge \hat{C}_0$$

hold. On the other hand, (3.12) implies that the process

(6.15) $$\frac{\hat{X}(\cdot) + \hat{R}(\cdot)\hat{Y}(\cdot)}{B(\cdot)} \text{ is a } \hat{\mathbf{P}}_{\mathbf{0}} - \text{supermartingale.}$$

Therefore, from (6.5), (6.14) and (6.15) we have

(6.16) $$\begin{aligned} x + \frac{y}{p} E\hat{Z}_1(T) &= E\left[ \frac{\hat{Z}_0(T)}{B(T)} I\left( \hat{\zeta} \frac{\hat{Z}_0(T)}{B(T)} \right) \right] = \hat{E}_0 \left( \frac{\hat{C}_0}{B(T)} \right) \\ &\le \hat{E}_0 \left( \frac{\hat{X}(T) + \hat{R}(T)\hat{Y}(T)}{B(T)} \right) \le x + \frac{y}{p} E\hat{Z}_1(T), \end{aligned}$$

whence

(6.17) $$\hat{X}(T) + \hat{R}(T)\hat{Y}(T) = \hat{C}_0.$$

But now from (6.8), (6.14) we deduce $\hat{R}(T) = 1 - \mu$ on $\{\hat{Y}(T) > 0\}$, and $\hat{R}(T) = 1 + \lambda$ on $\{\hat{Y}(T) < 0\}$; thus

$$\begin{aligned} \hat{C}_0 &= \hat{X}(T) + \hat{R}(T)\hat{Y}(T) \\ &= \hat{X}(T) + \hat{Y}(T)[(1+\lambda)\mathbf{1}_{\{\hat{Y}(T) \le 0\}} + (1-\mu)\mathbf{1}_{\{\hat{Y}(T) > 0\}}] = \hat{X}(T) + f(\hat{Y}(T)), \end{aligned}$$

and (6.10) follows.

It develops from (6.15), (6.16) that the process $\frac{\hat{X}(\cdot)+\hat{R}(\cdot)\hat{Y}(\cdot)}{B(\cdot)}$ is a $\hat{\mathbf{P}}_{\mathbf{0}}-$supermartingale with constant expectation, thus a $\hat{\mathbf{P}}_{\mathbf{0}}-$martingale; from this and (6.17), we obtain (6.13), as well as the fact that this process is nonnegative, hence that the $\hat{\mathbf{P}}_{\mathbf{0}}-$local martingale

$$(6.18)\quad \frac{\hat{X}(t)+\hat{R}(t)\hat{Y}(t)}{B(t)} + \int_0^t \frac{1+\lambda-\hat{R}(s)}{B(s)}d\hat{L}(s) + \int_0^t \frac{\hat{R}(s)-(1-\mu)}{B(s)}d\hat{M}(s),\ 0 \le t \le T$$

is also nonnegative. Consequently, the process of (6.18) is a $\hat{\mathbf{P}}_{\mathbf{0}}-$supermartingale, with $\hat{\mathbf{P}}_{\mathbf{0}}-$expectation at most $x + \frac{y}{p}E\hat{Z}_1(T)$ at $t = T$; but this is equal to the $\hat{\mathbf{P}}_{\mathbf{0}}-$expectation of $\frac{\hat{X}(T)+\hat{R}(T)\hat{Y}(T)}{B(T)}$ by (6.16), whence the nonnegative terms

$$\int_0^T \frac{1+\lambda-\hat{R}(s)}{B(s)}d\hat{L}(s)\ ,\quad \int_0^T \frac{\hat{R}(s)-(1-\mu)}{B(s)}d\hat{M}(s)$$

must have $\hat{\mathbf{P}}_{\mathbf{0}}-$expectation equal to zero. The claims (6.11), (6.12) follow.

Now for the optimality of the pair $(\hat{L},\hat{M})$: we have from (6.10), (5.2) and (6.5)

$$(6.19)\quad \begin{aligned} EU(\hat{X}(T)+f(\hat{Y}(T)))) &= EU(\hat{C}_0) = EU\left(I\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right) \\ &= E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right) + \hat{\zeta}E\left[\frac{\hat{Z}_0(T)}{B(T)}I\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right] \\ &= E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right) + \hat{\zeta}x + \hat{\zeta}\frac{y}{p}E\hat{Z}_1(T) = \tilde{V}(\hat{\zeta};y) + x\hat{\zeta}. \end{aligned}$$

Consider also the holdings processes $X(\cdot) \equiv X^{x,L,M}(\cdot)$, $Y(\cdot) \equiv Y^{y,L,M}(\cdot)$ corresponding to an *arbitrary* strategy $(L, M) \in \mathcal{A}(x, y)$. Again from (5.2), we have

$$U(X(T)+(1-\mu)Y(T)) \le \tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right) + \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}[X(T)+(1-\mu)Y(T)]$$

$$U(X(T)+(1+\lambda)Y(T)) \le \tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right) + \hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}[X(T)+(1+\lambda)Y(T)]$$

and thus, in conjunction with Remark 3.6, (4.6) and (3.12),

(6.20)
$$\begin{aligned}
EU(X(T)+f(Y(T))) &\le E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}\hat{E}_0\left(\frac{X(T)+\hat{R}(T)Y(T)}{B(T)}\right)\\
&\le E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}\liminf_{n\to\infty}\hat{E}_0^{(n)}\left(\frac{X(T)+\hat{R}^{(n)}(T)Y(T)}{B(T)}\right)\\
&\le E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}(x+\frac{y}{p}E\hat{Z}_1(T))\\
&=\tilde{V}(\hat{\zeta};y)+x\hat{\zeta}.
\end{aligned}$$

The optimality of $(\hat{L},\hat{M})\in\mathcal{A}(x,y)$ for the problem of (6.2), as well as the equality $V(x;y)=\tilde{V}(\hat{\zeta};y)+x\hat{\zeta}$, follow now directly from (6.19) and (6.20). □

Notice that, if $r(\cdot)$ is deterministic, then Jensen's inequality gives

$$\begin{aligned}
E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right)+\frac{y}{p}\zeta Z_1(T)\right] &\ge \tilde{U}\left(\frac{\zeta}{B(T)}EZ_0(T)\right)+\frac{y}{p}\zeta Z_1(0)\\
&\ge \tilde{U}\left(\frac{\zeta}{B(T)}\right)+y\zeta(1-\mu),\quad \forall\ (Z_0(\cdot),Z_1(\cdot))\in\mathcal{D}.
\end{aligned}\tag{6.21}$$

We shall use this observation to find examples, in which the optimal strategy $(\hat{L},\hat{M})$ of Theorem 6.4 trades either not at all, or only at time $t=0$.

**6.5 Example:** $r(\cdot)$ *deterministic,* $y=0$. In this case we see from (6.21) that

$$\tilde{V}(\zeta;0)=\inf_{(Z_0,Z_1)\in\mathcal{D}}E\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right)\ge\tilde{U}(\zeta/B(T)),$$

and the infimum is achieved by taking $\hat{Z}_0(\cdot)\equiv 1$, i.e., by any pair $(1,\hat{Z}_1(\cdot))\in\mathcal{D}$ that satisfies $1-\mu\le\hat{R}(\cdot)=\hat{Z}_1(\cdot)/P(\cdot)\le 1+\lambda$, if such exists. In particular, one can take $\hat{Z}_1(0)=(1+\lambda)p$ and $\hat{\theta}_1(\cdot)\equiv\sigma(\cdot)$, in which case $(1,\hat{Z}_1(\cdot))\in\mathcal{D}$ if and only if

$$0\le\int_0^t(b(s)-r(s))ds\le\log\frac{1+\lambda}{1-\mu}\ ,\forall\ 0\le t\le T.\tag{6.22}$$

Furthermore, from (6.10) and (6.5),(6.6) we have

$$\hat{X}(T)+f(\hat{Y}(T))=I(\hat{\zeta}/B(T))=\hat{C}_0=xB(T).$$

All the conditions (6.4), (6.9) and the Assumption 6.1 are satisfied rather trivially; and the *no-trading-strategy* $\hat{L} \equiv 0$, $\hat{M} \equiv 0$ is optimal, from Theorem 6.4 (and gives $\hat{X}(T) = xB(T)$, $\hat{Y}(T) = 0$). The condition (6.22) is satisfied, for instance, if

$$r(\cdot) \leq b(\cdot) \leq r(\cdot) + \rho\ , \ \text{ for some } 0 \leq \rho \leq \frac{1}{T} \log \frac{1+\lambda}{1-\mu}. \tag{6.23}$$

If $b(\cdot) = r(\cdot)$ the result is not surprising – even without transaction costs, it is then optimal not to trade. However, for $b(\cdot) > r(\cdot)$ the optimal portfolio always invests a positive amount in the stock, if there are no transaction costs; the same is true even in the presence of transaction costs, if one is maximizing expected discounted utility from consumption over an infinite time-horizon, and if the market coefficients are constant – see Shreve & Soner (1994), Theorem 11.6.

The situation here, on the finite time-horizon $[0, T]$, is quite different: if the excess rate of return $b(\cdot) - r(\cdot)$ is positive but small relative to the transaction costs, and/or if the time-horizon is small, in the sense of (6.23), then it is optimal not to trade.

**Remark:** In the infinite time-horizon case with constant market coefficients, as in Shreve & Soner (1994), the ratio $\hat{X}/\hat{Y}$ of optimal holdings is a reflected diffusion process in a fixed interval; more precisely, one trades only when this ratio hits the endpoints of the interval, and in such a way as to keep the ratio inside the interval. In our case, under the assumptions of Example 6.5, and with $U(x) = \log x$, one obtains from (6.13) that

$$(\hat{X}(t) + \hat{R}(t)\hat{Y}(t))/B(t) = (\hat{\zeta}\hat{Z}_0(t))^{-1}, \quad 0 \leq t \leq T.$$

Comparing the stochastic integral representation of $(\hat{\zeta}\hat{Z}_0(\cdot))^{-1}$ with the equation (3.8), one obtains

$$\frac{\hat{R}(t)\hat{Y}(t)}{B(t)}(\hat{\theta}_1(t) - \hat{\theta}_0(t)) = -\frac{\hat{\theta}_0(t)}{\hat{\zeta}\hat{Z}_0(t)}, \quad 0 \leq t \leq T.$$

The last two equations imply

$$\hat{X}(t)/\hat{Y}(t) = -\hat{R}(t)\left(\frac{\hat{\theta}_1(t)}{\hat{\theta}_0(t)}\right), \quad 0 \leq t \leq T,$$

provided $\hat{Y}(t)\hat{\theta}_0(t) \neq 0$, $\forall\ t \in [0,T]$. While $\hat{R}(\cdot)$ is a reflected process in a fixed interval, it is not clear what happens to the second factor, either for fixed $T$ or as $T \to \infty$.

**6.6 Example:** *$b(\cdot) \equiv r(\cdot)$ deterministic, $y > 0$.* In this case we can take $\hat{Z}_0(\cdot) \equiv 1$ and $\hat{Z}_1(t) = p(1-\mu)\exp\{\int_0^t \sigma(s)dW(s) - \frac{1}{2}\int_0^t \sigma^2(s)ds\}$ (i.e., $\hat{\theta}_1(\cdot) \equiv \sigma(\cdot)$) to deduce from (6.21) that

$$\begin{aligned}\tilde{V}(\zeta;y) &= \inf_{(Z_0,Z_1)\in\mathcal{D}} E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right] \\ &= E\left[\tilde{U}\left(\zeta\frac{\hat{Z}_0(T)}{B(T)}\right) + \frac{y}{p}\zeta \hat{Z}_1(T)\right] = \tilde{U}(\zeta/B(T)) + y\zeta(1-\mu).\end{aligned}$$

It is easily checked that $(\hat{Z}_0(\cdot), \hat{Z}_1(\cdot)) \in \mathcal{D}_\infty$; that Assumption 6.1 and (6.4), (6.9) hold; and that

$$\hat{X}(T) + f(\hat{Y}(T)) = \hat{C}_0 = I(\hat{\zeta}/B(T)) = B(T)(x + y(1-\mu)).$$

In fact, the strategy

$$\hat{L}(\cdot) \equiv 0, \quad \hat{M}(\cdot) = y\mathbf{1}_{(0,T]}(\cdot),$$

which liquidates immediately (i.e., at $t = 0$) the position in the stock, leads precisely to $\hat{X}(t) = [x + y(1-\mu)\mathbf{1}_{(0,T]}(t)]B(t)$, $\hat{Y}(t) = y\mathbf{1}_{\{0\}}(t)$, $0 \le t \le T$, and is optimal in $\mathcal{A}(x,y)$ for the problem of (6.2).

**6.7 Example:** $b(\cdot) \equiv r(\cdot)$, $U(x) = \log x$. In the case of a logarithmic utility function $U(x) = \log x$, the dual problem of (6.3) becomes

$$\tilde{V}(\zeta;y) = -(1+\log\zeta) + E\int_0^T r(t)dt + \inf_{(Z_0,Z_1)\in\mathcal{D}} E\left[\int_0^T \theta_0^2(t)dt + \frac{y}{p}\zeta Z_1(T)\right].$$

We have been able to solve this last minimization problem only in the case $b(\cdot) \equiv r(\cdot)$, when we can take $\hat{\theta}_0(\cdot) \equiv 0$, $\hat{\theta}_1(\cdot) \equiv \sigma(\cdot)$ and thus $\hat{Z}_0(\cdot) \equiv 1$, $\hat{Z}_1(\cdot) = p(1-\mu)\exp\{\int_0^\cdot \sigma(s)dW(s) - \frac{1}{2}\int_0^\cdot \sigma^2(s)ds\}$ as well as $\hat{L}(\cdot) \equiv 0$, $\hat{M}(\cdot) \equiv y\mathbf{1}_{(0,T]}(\cdot)$ ("sell stock holdings, if any, at time $t = 0$; otherwise do nothing"). Compared to the similar results of Examples 6.5, 6.6, the advantage here is that $b(\cdot) \equiv r(\cdot)$ is allowed to be random.

**Proof of Lemma 6.3:** For simplicity of notation, we shall take again $p = 1$ and prove, for any given $\zeta \in (0, \infty)$ and $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$:

$$E\left[\frac{Z_0(T)}{B(T)} I\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) - y Z_1(T)\right] \le E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) - y \hat{Z}_1(T)\right] < \infty. \tag{6.24}$$

From condition (5.4) we have $\eta I(\eta) \le a + (1-b) U(I(\eta))$, $0 < \eta < \infty$, and by subtracting $(1-b)\eta I(\eta)$ from both sides:

$$b \eta I(\eta) \le a + (1-b) \tilde{U}(\eta).$$

It follows that

$$\begin{aligned} b\zeta E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) - y \hat{Z}_1(T)\right] &\le a + (1-b) E\tilde{U}\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) - by\zeta \hat{Z}_1(0) \\ &\le a + (1-b)\left[\tilde{V}(\zeta; y) - \zeta y(1-\mu)\right] - b\zeta y(1-\mu) < \infty, \end{aligned}$$

which proves the second inequality in (6.24).

To prove the first inequality, we use a *perturbation argument*: for fixed but arbitrary $0 < \varepsilon < 1$ and $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, let

$$\tilde{Z}_i^{(\varepsilon)}(\cdot) := (1-\varepsilon)\hat{Z}_i(\cdot) + \varepsilon Z_i(\cdot), \; i = 0, 1$$

and note that $(\tilde{Z}_0^{(\varepsilon)}(\cdot), \tilde{Z}_1^{(\varepsilon)}(\cdot)) \in \mathcal{D}$. Because the pair $(\hat{Z}_0(\cdot), \hat{Z}_1(\cdot)) \in \mathcal{D}$ attains the infimum in (6.3), we have $E(G^{(\varepsilon)}) \le 0$, where

$$\begin{aligned} G^{(\varepsilon)} &:= \frac{1}{\varepsilon}\left[\tilde{U}\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) - \tilde{U}\left(\zeta \frac{\tilde{Z}_0^{(\varepsilon)}(T)}{B(T)}\right)\right] + \frac{\zeta y}{\varepsilon}(\hat{Z}_1(T) - \tilde{Z}_1^{(\varepsilon)}(T)) \\ &= \frac{\zeta}{B(T)} I\left(\frac{\zeta F_\varepsilon}{B(T)}\right)(Z_0(T) - \hat{Z}_0(T)) + \zeta y(\hat{Z}_1(T) - Z_1(T)) \\ &\ge \frac{\zeta}{B(T)} I\left(\frac{\zeta Z_0(T)}{B(T)}\right)(Z_0(T) - \hat{Z}_0(T)) + \zeta y(\hat{Z}_1(T) - Z_1(T)) \\ &\ge -\frac{\zeta}{B(T)} \hat{Z}_0(T) I\left(\frac{\zeta Z_0(T)}{B(T)}\right) + \zeta y(\hat{Z}_1(T) - Z_1(T)), \end{aligned} \tag{6.25}$$

where $F_\varepsilon$ is a random variable with values between $\hat{Z}_0(T)$ and $\tilde{Z}_0^{(\varepsilon)}(T)$; in particular, $\lim_{\varepsilon \downarrow 0} F_\varepsilon = \hat{Z}_0(T)$.

Suppose first that $Z_0(\cdot)/Z_0^*(\cdot) \geq K$ for some constant $0 < K < \infty$. Then, by Assumption 6.1,

$$E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\frac{\zeta Z_0(T)}{B(T)}\right)\right] \leq E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\frac{\zeta K Z_0^*(T)}{B(T)}\right)\right] < \infty, \tag{6.26}$$

so that the last random variable in (6.25) is integrable. Then from Fatou's lemma, we have

$$\begin{aligned}
&E\left[\frac{\zeta}{B(T)} I\left(\frac{\zeta \hat{Z}_0(T)}{B(T)}\right)(Z_0(T) - \hat{Z}_0(T)) + \zeta y(\hat{Z}_1(T) - Z_1(T))\right] \\
&= E\left[\lim_{\varepsilon \downarrow 0} \frac{\zeta}{B(T)} I\left(\frac{\zeta F_\varepsilon}{B(T)}\right)(Z_0(T) - \hat{Z}_0(T)) + \zeta y(\hat{Z}_1(T) - Z_1(T))\right] \\
&\leq \liminf_{\varepsilon \downarrow 0} E(G^{(\varepsilon)}) \leq 0 \; ;
\end{aligned} \tag{6.27}$$

the inequality (6.24) follows.

Now for an arbitrary $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, define $\tau_n := \inf\{t \in [0,T]/Z_0(t)/Z_0^*(t) \leq \frac{1}{n}\} \wedge T$. Proceed as in the proof of Lemma 4.1, to obtain a sequence $\{(Z_0^{(n)}(\cdot), Z_1^{(n)}(\cdot))\} \in \mathcal{D}$ such that $Z_0^{(n)}(\cdot)/Z_0^*(\cdot) \geq 1/n$. Therefore, the first inequality in (6.24) is valid for $(Z_0^{(n)}(T), Z_1^{(n)}(T))$, $\forall\, n \in N$, and we can let $n \to \infty$ and use Fatou's lemma to obtain the result for $(Z_0(T), Z_1(T))$. $\square$

## 7. PRICING CONTINGENT CLAIMS.

We indicate here a possible way of pricing contingent claims in a market with transaction costs. The minimal hedging price is only an upper bound for the price of a claim, and is typically too high (see Example 4.3). Given a utility function $U(\cdot)$ and initial wealth $x$, Davis (1994) defines the *fair price* of a claim in a market with frictions to be the price which makes the agent's utility neutral with respect to a small (infinitesimal) diversion of funds into the claim (if such exists). See Davis (1994) and Karatzas & Kou (1994) for the precise mathematical formulations. Assuming $y = 0$, it can be argued, using the methods of those papers, that the fair price $V(0)$ of a claim $C = (C_0, C_1)$ in our setting should be the expected value of the discounted claim evaluated under the optimal shadow state-price densities of the dual problem, i.e., by

$$V(0) = E\left[\hat{Z}_0(T)\frac{C_0}{B(T)} + \hat{Z}_1(T)\frac{C_1}{S(T)}\right],$$

provided that the dual optimization problem of (6.3) has a *unique* solution $(\hat{Z}_0(\cdot), \hat{Z}_1(\cdot))$. Notice that this price does not depend on the initial holdings $x$ in stock, if the solution to the dual problem does not depend on $\zeta$, as in Assumption 6.1. However, $V(0)$ *does* depend in general on the return rate of the stock $b(\cdot)$. For example, if $b(\cdot) \equiv r(\cdot)$ and deterministic, and the claim is a European call, then it follows from Example 6.6 that the fair price is the Black-Scholes price, independently of the utility function $U(\cdot)$ that is being considered. If $b(\cdot) - r(\cdot)$ is nonnegative and small, the price would be close to the Black-Scholes price. Constantinides (1993) and Constantinides & Zariphopoulou (1995) show that the price of a European call, under not too large transaction costs, is always close to the Black-Scholes price. However, they use a different definition of (bounds for) the fair price, based on maximizing utility from consumption plus utility from terminal wealth, adjusted for the value of the claim at time $t = T$.

## A. APPENDIX.

The purpose of this section is to prove the closedness property (4.14), for the set $A_0$ of (4.10), using a method similar to the one in the appendix of Shreve, Soner & Cvitanić (1995). To this effect, let us consider a sequence $\{(U_n, V_n)\}_{n\in N} \subset A_0$ converging in $(\mathbf{L}_2^*)^2$ to some $(U, V) \in (\mathbf{L}_2^*)^2$, i.e.,

$$E_0^*[(U_n - U)^2 + (V_n - V)^2] \longrightarrow 0, \text{ as } n \to \infty, \tag{A.1}$$

and observe that, as a result, the expected values $E_0^*(U_n^2), E_0^*(V_n^2)$ are bounded uniformly in $n$. Let also $\{(L_n, M_n)\}_{n\in N} \subset \mathcal{A}(0,0)$ denote the corresponding hedging strategies, so that with $X_n(\cdot), Y_n(\cdot)$ defined by

$$\begin{aligned} X_n(t) &= -(1+\lambda)L_n(t) + (1-\mu)M_n(t) \ \ , \ 0 \le t \le T \\ Y_n(t) &= L_n(t) - M_n(t) + \int_0^t \sigma(u)Y_n(u)dW_0^*(u), \ 0 \le t \le T \end{aligned} \tag{A.2}$$

we have the following hedging and admissibility properties, for every $n \in N$:

$$\left\{ \begin{array}{l} X_n(T) + (1-\mu)Y_n(T) \ge U_n + (1-\mu)V_n \\ X_n(T) + (1+\lambda)Y_n(T) \ge U_n + (1+\lambda)V_n \end{array} \right\}, \tag{A.3}$$

$$Q_n(\cdot) := X_n(\cdot) + R(\cdot)Y_n(\cdot) \ \text{ is a } \ \mathbf{P_0} - \text{supermartingale}, \ \forall \ (Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty. \tag{A.4}$$

The question then is, whether we can find $(L, M) \in \mathcal{A}(0,0)$, so that $(L_n, M_n) \to (L, M)$ and $(X_n, Y_n) \to (X, Y) \equiv (X^{0,L,M}, Y^{0,L,M})$, in a suitable sense, as $n \to \infty$, and still have the analogues of (A.3) and (A.4):

$$\left\{ \begin{array}{l} X(T) + (1-\mu)Y(T) \ge U + (1-\mu)V \\ X(T) + (1+\lambda)Y(T) \ge U + (1+\lambda)V \end{array} \right\}, \tag{A.5}$$

$$Q(\cdot) := X(\cdot) + R(\cdot)Y(\cdot) \ \text{ is a } \ \mathbf{P_0} - \text{supermartingale}, \ \forall \ (Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty. \tag{A.6}$$

Let us start by introducing some notation:

$$
(A.7)\qquad
\begin{aligned}
I_n(t) &:= \int_0^t \sigma(u)Y_n(u)dW_0^*(u)\\
S_n^\lambda(t) &:= E_0^*[U_n + (1+\lambda)V_n|\mathcal{F}(t)]\\
S_n^\mu(t) &:= E_0^*[U_n + (1-\mu)V_n|\mathcal{F}(t)],
\end{aligned}
$$

for $0 \le t \le T$, and by noticing the inequalities (proved below):

$$
(A.8)\qquad S_n^\lambda(t) \le X_n(t) + (1+\lambda)Y_n(t),\quad S_n^\mu(t) \le X_n(t) + (1-\mu)Y_n(t),\ 0 \le t \le T
$$

$$
(A.9)\ (\lambda+\mu)L_n(t) \le (1-\mu)I_n(t) - S_n^\mu(t),\ (\lambda+\mu)M_n(t) \le (1+\lambda)I_n(t) - S_n^\lambda(t),\ 0 \le t \le T
$$

$$
(A.10)\qquad |Y_n(t)| \le \frac{1+\lambda}{\lambda+\mu}|I_n(t)| + \frac{|S_n^\mu(t)| + |S_n^\lambda(t)|}{\lambda+\mu}\ ,\ 0 \le t \le T
$$

$$
(A.11)\qquad \sup_{0\le t\le T, n\in N} E_0^*\left[(S_n^\lambda(t))^2 + (S_n^\mu(t))^2 + I_n^2(t)\right] =: C < \infty.
$$

From (A.8)-(A.11) it develops that the sequences $\{L_n(\cdot)\}_{n\in N}, \{M_n(\cdot)\}_{n\in N}, \{X_n(\cdot)\}_{n\in N}$ and $\{Y_n(\cdot)\}_{n\in N}$ are *bounded* in $\mathcal{H}$, the Hilbert space of progressively measurable, real-valued processes $\xi(t),\ 0 \le t \le T$ with $E_0^* \int_0^T \xi^2(t)dt < \infty$ and $<\eta, \xi> = E_0^* \int_0^T \eta(t)\xi(t)dt$. Thus, there exist processes $L(\cdot), M(\cdot)$ and $Y(\cdot)$ in $\mathcal{H}$, such that

$$
(A.12)\qquad L_n(\cdot) \to L(\cdot),\ M_n(\cdot) \to M(\cdot),\ Y_n(\cdot) \to Y(\cdot) \text{ weakly in } \mathcal{H},\ \text{as } n \to \infty
$$

(possibly only along a subsequence, which is then relabelled). We can define then

$$
(A.13)\quad X(\cdot) := (1-\mu)M(\cdot) - (1+\lambda)L(\cdot),\ \text{and notice that } X_n(\cdot) \to X(\cdot) \text{ weakly in } \mathcal{H}.
$$

From Lemmata 4.5-4.7 in Karatzas & Shreve (1984), we can assume that $L(\cdot), M(\cdot)$ have increasing, left-continuous paths, and that

$$
(A.14)\qquad \left\{ \begin{array}{c} L_n(t) \to L(t),\ M_n(t) \to M(t) \text{ as } n \to \infty \\ \text{weakly in } \mathbf{L}^1(\Omega, \mathcal{F}(T), \mathbf{P}_0^*),\ \text{for a.e. } t \in [0,T]. \end{array} \right\}
$$

It follows from (A.14) that, for every $h \in (0,T)$, we have

$$\lim_{n\to\infty} E_0^*(L_n(T)\mathbf{1}_A) \geq \frac{1}{h}\lim_{n\to\infty}\int_{T-h}^{T} E_0^*(L_n(s)\mathbf{1}_A)ds = \frac{1}{h}\int_{T-h}^{T} E_0^*(L(s)\mathbf{1}_A)ds \geq E_0^*(L(T-h)\mathbf{1}_A),$$

and letting $h \downarrow 0$:

$$(A.15) \qquad \lim_{n\to\infty} E_0^*(L_n(T)\mathbf{1}_A) \geq E_0^*(L(T)\mathbf{1}_A), \quad \forall\ A \in \mathcal{F}(T).$$

Similarly,

$$(A.16) \qquad \lim_{n\to\infty} E_0^*(M_n(T)\mathbf{1}_A) \geq E_0^*(M(T)\mathbf{1}_A), \quad \forall\ A \in \mathcal{F}(T).$$

Recall now the definition (A.7) of $I_n(\cdot)$ and define the process

$$(A.17) \qquad I(t) := \int_0^t \sigma(s)Y(s)dW_0^*(s),\ \ 0 \leq t \leq T.$$

We can show (see below) that

$$(A.18) \qquad I_n(t) \to I(t) \text{ as } n \to \infty \text{ weakly in } \mathbf{L}_2^*,\ \forall\ t \in [0,T].$$

It develops then, by taking weak limits in (A.2), that the processes $X(\cdot), Y(\cdot)$ of (A.13), (A.12) satisfy

$$\begin{aligned} X(t) &= -(1+\lambda)L(t) + (1-\mu)M(t) \quad , \ 0 \leq t \leq T \\ Y(t) &= L(t) - M(t) + \int_0^t \sigma(u)Y(u)dW_0^*(u),\ \ 0 \leq t \leq T. \end{aligned}$$

(This is verified first for fixed $t \in [0,T]$, and then for all $0 \leq t \leq T$ simultaneously by the left-continuity of the processes involved.) In other words,

$$X(\cdot) \equiv X^{0,L,M}(\cdot) \ \text{ and } \ Y(\cdot) \equiv Y^{0,L,M}(\cdot).$$

To finish the argument it remains to verify the properties (A.5) and (A.6) of hedging and admissibility, respectively.

**PROOF OF (A.5): Hedging.** In view of (A.2), we may write (A.3) as

$$(1-\mu)I_n(T) - (\lambda+\mu)L_n(T) \geq U_n + (1-\mu)V_n$$

$$(1+\lambda)I_n(T) - (\lambda+\mu)M_n(T) \geq U_n + (1+\lambda)V_n;$$

We want to deduce from this (A.5), or equivalently

$$(A.19) \qquad (1-\mu)I(T) - (\lambda+\mu)L(T) \geq U + (1-\mu)V$$

$$(A.20) \qquad (1+\lambda)I(T) - (\lambda+\mu)M(T) \geq U + (1+\lambda)V.$$

Recall from (A.16), (A.18), (A.1) that

$$\begin{aligned}(\lambda+\mu)E_0^*[M(T)\mathbf{1}_A] &\leq (\lambda+\mu)\lim_{n\to\infty} E_0^*[M_n(T)\mathbf{1}_A] \\ &\leq \lim_{n\to\infty} E_0^*\big[\{(1+\lambda)(I_n(T)-V_n)-U_n\}\mathbf{1}_A\big] \\ &= E_0^*\big[\{(1+\lambda)(I(T)-V)-U\}\mathbf{1}_A\big]\ , \quad \forall\ A \in \mathcal{F}(T)\end{aligned}$$

and (A.20) follows; a similar argument gives (A.19).

**PROOF OF (A.6): Admissibility.** Fix an arbitrary $(Z_0(\cdot), Z_1(\cdot))$ in $\mathcal{D}_\infty$; from (A.3) and Remark 3.6 we have $Q_n(T) := X_n(T) + R(T)Y_n(T) \geq U_n + R(T)V_n$, and (A.4) gives then

$$(A.21) \qquad Q_n(t) \geq E_0[Q_n(T)|\mathcal{F}(t)] \geq E_0[U_n + R(T)V_n|\mathcal{F}(t)] \geq -\xi_n(t),\ 0 \leq t \leq T.$$

Here

$$(A.22)\ \ \xi_n(t) := E_0[|U_n| + (1+\lambda)|V_n|\ |\mathcal{F}(t)],\ \xi(t) := E_0[|U| + (1+\lambda)|V|\ |\mathcal{F}(t)],\ 0 \leq t \leq T$$

are $\mathbf{P_0}-$martingales, with continuous paths and

$$(A.23) \qquad \sup_{0\leq t\leq T, n\in N} E_0[\xi_n^2(t) + \xi^2(t)] < \infty,\ E_0[\max_{0\leq t\leq T}|\xi_n(t) - \xi(t)|]^2 \underset{n\to\infty}{\longrightarrow} 0.$$

Indeed,

$$E_0\xi_n^2(t) \leq E_0\xi_n^2(T) \leq 2KE_0^*[U_n^2 + (1+\lambda)^2V_n^2] \leq KC_\lambda < \infty,$$

where $C_\lambda$ is a constant depending only on $\lambda$ and $\sup_n E_0^*[U_n^2 + V_n^2]$, and $K$ is an upper bound on $Z_0(T)/Z_0^*(T)$. This proves the first claim in (A.23); similar arguments, together with Doob's inequality and (A.1), yield the second claim.

Consider now a set $D \subset [0,T]$ with $meas([0,T]\backslash D) = 0$, for which we have

$$(A.24) \quad \left\{ \begin{array}{c} X_n(t) \to X(t), \; Y_n(t) \to Y(t) \text{ and thus } Q_n(t) \to Q(t) = X(t) + R(t)Y(t), \; \forall t \in D, \\ \text{weakly in } \mathbf{L}^1(\Omega, \mathcal{F}(T), \mathbf{P_0^*}) \text{ and thus also in } \mathbf{L}^1(\Omega, \mathcal{F}(T), \mathbf{P_0}). \end{array} \right\}$$

by virtue of (A.14) and (A.18). Clearly, from (A.21), (A.23) and (A.24),

$$(A.25) \qquad Q(\cdot) + \xi(\cdot) \geq 0, \text{ on } D.$$

The supermartingale property (A.4) written as

$$E_0[\eta Q_n(s)] \geq E_0[\eta Q_n(t)], \; \forall 0 \leq s < t \leq T, \; n \in N$$

for every bounded, $\mathcal{F}(s)-$measurable random variable $\eta \geq 0$, leads via (A.24) to $E_0[\eta Q(s)] \geq E_0[\eta Q(t)]$, or equivalently

$$(A.26) \qquad Q(s) \geq E_0[Q(t)|\mathcal{F}(s)], \; \forall s < t \text{ in } D.$$

On the other hand, for $t \in D$ and *arbitrary* $s \in [0,t)$, let $\{s_k\}$ be a sequence in $D$ that increases (strictly) to $s$, write (A.26) as $Q(s_k) \geq E_0[Q(t)|\mathcal{F}(s_k)]$, $\forall k \in N$, and let $k \to \infty$. The left-continuity of $Q(\cdot)$, along with Lévy's martingale convergence theorem (e.g. Chung (1974)) and the continuity of $\mathbf{F}$ (e.g. Karatzas & Shreve (1991), §2.8), yield

$$(A.27) \qquad Q(s) \geq E_0[Q(t)|\mathcal{F}(s-)] = E_0[Q(t)|\mathcal{F}(s)], \; \forall t \in D, s \in [0,t).$$

Now consider *arbitrary* $0 \leq s < t \leq T$, as well as a sequence $\{t_m\}$ in $D$ with $s < t_m$ and $t_m$ (strictly) increasing to $t$. We have

$$Q(s) + \xi(s) \geq E_0[Q(t_m) + \xi(t_m)|\mathcal{F}(s)], \; \forall \; m \in N$$

from (A.27) and the martingale property of $\xi(\cdot)$; recall (A.25) and let $m \to \infty$ to conclude, from Fatou's lemma, the continuity of $\xi(\cdot)$ and the left-continuity of $Q(\cdot)$, that

$$E_0[Q(t)|\mathcal{F}(s)] + \xi(s) = E_0[Q(t) + \xi(t)|\mathcal{F}(s)] = E_0[\lim_m (Q(t_m) + \xi(t_m))|\mathcal{F}(s)]$$
$$\leq \liminf_m E_0[Q(t_m) + \xi(t_m)|\mathcal{F}(s)] \leq Q(s) + \xi(s), \ \forall \ 0 \leq s < t \leq T,$$

which establishes (A.6). The proof of the closedness property (4.14) is now complete. □

**Proof of (A.8):** From (A.7), (A.3), (A.4) and Remark 3.1 with $z = 1 + \lambda$, we have $R^*(\cdot) \equiv 1+\lambda$, $S_n^\lambda(t) \leq E_0^*[X_n(T) + (1+\lambda)Y_n(T)|\mathcal{F}(t)] = E_0^*[X_n(T) + R^*(T)Y_n(T)|\mathcal{F}(t)] \leq X_n(t) + R^*(t)Y_n(t) = X_n(t) + (1+\lambda)Y_n(t)$, first for fixed $t$ and then, by continuity of $S_n^\lambda(\cdot)$ and left-continuity of $X_n(\cdot), Y_n(\cdot)$, for all $t \in [0,T]$ simultaneously. Similarly for $S_n^\mu(\cdot)$ (but with $z = 1 - \mu$ in Remark 3.1).

**Proof of (A.9), (A.10):** From (A.2) and (A.8), we obtain (A.9), as well as

$$Y_n(t) \leq I_n(t) + L_n(t) \leq \frac{1+\lambda}{\lambda+\mu} I_n(t) - \frac{S_n^\mu(t)}{\lambda+\mu}$$

$$Y_n(t) \geq I_n(t) - M_n(t) \geq -\frac{1-\mu}{\lambda+\mu} I_n(t) + \frac{S_n^\lambda(t)}{\lambda+\mu} \ ,$$

which lead to (A.10).

**Proof of (A.11):** For $S_n^\lambda(\cdot), S_n^\mu(\cdot)$ we have in fact the stonger property

$$(A.28) \qquad \sup_{\tau \in \mathcal{S}, n \in N} E_0^*[(S_n^\lambda(\tau)^2 + (S_n^\mu(\tau))^2] \leq 2 \sup_{n \in N} E_0^*[U_n^2 + (1+\lambda)V_n^2] \leq K < \infty$$

from Jensen's inequality and (A.7), (A.1), where $\mathcal{S}$ is the class of stopping times of $\mathbf{F}$ with values in $[0,T]$. For fixed $n \in N$, $k \in N$, $t \in [0,T]$, define

$$\tau_n^{(k)} := \inf\{s \in [0,t]/|Y_n(s)| \geq k\} \wedge t; \ Y_n^{(k)}(s) := Y_n(s) \text{ if } s \leq \tau_n^{(k)}, \ Y_n^{(k)}(s) = 0 \text{ if } s > \tau_n^{(k)}$$

$$I_n^{(k)}(s) := I_n(s \wedge \tau_n^{(k)}) = \int_0^s \sigma(u) Y_n^{(k)}(u) dW_0^*(u), \ 0 \leq s \leq t.$$

Now from the boundedness of $\sigma(\cdot)$, (A.10) and (A.28) we obtain

$$E_0^*(I_n^{(k)}(s))^2 = \int_0^s E_0^*(\sigma(u)Y_n^{(k)}(u))^2 du \leq A + B \int_0^s E_0^*(I_n^{(k)}(u))^2 du, \quad 0 \leq s \leq t,$$

and from Gronwall's inequality $E_0^*(I_n^{(k)}(s))^2 \leq C$ where $A, B, C$ are positive real constants that do not depend on $(k, n, s, t)$. For every $n \in N$, we have $\tau_n^{(k)} \uparrow t$ and $I_n^{(k)}(t) = I_n(\tau_n^{(k)}) \to I_n(t)$ a.s. as $k \to \infty$; thus, from Fatou's lemma, $E_0^*(I_n^2(t)) \leq C$.

**Proof of (A.18):** Consider an arbitrary $\xi \in \mathbf{L}_2^*$ and assume, without loss of generality, that $E_0^*\xi = 0$. Then $\xi = \int_0^T \eta(s) dW_0^*(s)$ for some (unique) $\eta(\cdot) \in \mathcal{H}$, and $E_0^*(\xi|\mathcal{F}(t)) = \int_0^T \eta_t(s) dW_0^*(s)$ where $\eta_t(s) := \eta(s)\mathbf{1}_{[0,t]}(s)$, for any given $t \in [0, T]$. Thus

$$\begin{aligned} E_0^*(\xi I_n(t)) &= E_0^*[E_0^*(\xi|\mathcal{F}(t)) I_n(t)] \\ &= E_0^* \int_0^T \sigma(s) Y_n(s) \eta_t(s) ds \underset{n\to\infty}{\longrightarrow} E_0^* \int_0^T \sigma(s) Y(s) \eta_t(s) ds = \ldots = E_0^*(\xi I(t)) \end{aligned}$$

from (A.12) and the boundedness of $\sigma(\cdot)$, establishing (A.18).

## B. APPENDIX.

We establish in this section the *existence of an optimal pair* $(\hat{L}, \hat{M}) \in \mathcal{A}^+(x,y)$ *for the expected utility maximization problem of (6.2):*

$$V(x;y) = EU(X^{x,\hat{L},\hat{M}}(T) + f(Y^{y,\hat{L}.\hat{M}}(T))) < \infty. \tag{B.1}$$

For simplicity of notation, we shall take again $p = 1$ and $r(\cdot) \equiv 0$. The key idea is to consider the set

$$A_{x,y} := \{H \in \mathbf{L}_2^* \ / \ \exists (L,M) \in \mathcal{A}^+(x,y) \text{ that hedges } (H,0)\} \tag{B.2}$$

of "terminal bank account holdings hedgeable by admissible strategies", and to show that

$$A_{x,y} \text{ is a convex, closed and bounded subset of } \mathbf{L}_2^*. \tag{B.3}$$

These properties can be established by using the methodology of Appendix A, almost line-by-line (with very few, and obvious, changes), so we leave the details to the care of the diligent reader. Let us denote

$$J(H) := -EU(H), \ H \in \mathbf{L}_2^*; \tag{B.4}$$

we will show that the value function of (6.2) can be re-written as

$$-V(x;y) = \inf_{H \in A_{x,y}} J(H), \tag{B.5}$$

and that the infimum in (B.5) is attained (and thus is the supremum in (6.2)).

It is clear that the functional

$$J : \mathbf{L}_2^* \longrightarrow R \cup \{+\infty\} \tag{B.6}$$

defined by (B.4) is *convex*; let us verify that it is also *proper* (as indicated in (B.6)), i.e., that

$$EU(H) < \infty, \ \forall \ H \in \mathbf{L}_2^*. \tag{B.7}$$

Indeed, the function $U(\cdot)$ is sublinear: $U(x) \le a + bx$, $\forall\ 0 \le x < \infty$ for some $a > 0, b > 0$. Thus $EU(H) \le \text{const.}(1 + E|H|) < \infty$, since $E_0^*(H^2) < \infty$, because

$$
\begin{aligned}
E|H| &= E_0^* \left[ |H| \exp\left\{ -\int_0^T \theta_0^*(s)dW(s) + \frac{1}{2}\int_0^T |\theta_0^*(s)|^2 ds \right\} \right] \\
&= E_0^* \left[ |H| \exp\left\{ -\int_0^T \theta_0^*(s)dW_0^*(s) - \frac{1}{2}\int_0^T |\theta_0^*(s)|^2 ds \right\} \right] \\
&\le \left( E_0^*(H^2) E_0^* \left[ \exp\left\{ -\int_0^T 2\theta_0^*(s)dW_0^*(s) - \frac{1}{2}\int_0^T |2\theta_0^*(s)|^2 ds \right\} \exp\left\{ \int_0^T |\theta_0^*(s)|^2 ds \right\} \right] \right)^{\frac{1}{2}} \\
&\le \left( E_0^*(H^2) \right)^{\frac{1}{2}} e^{\frac{1}{2}k^2 T} < \infty,
\end{aligned}
$$

where $k$ is an upper bound on $|\theta_0^*(\cdot)|$ of (3.6), Remark 3.1.

Finally, the functional $J$ of (B.4), (B.6) is *lower-semicontinuous* in the topology of $\mathbf{L}_2^*$; indeed, if $\{H_n\}_{n \in N}$ converges to $H$ in the topology of $\mathbf{L}_2^*$, we have

$$
\lim_{n\to\infty} E|H_n - H| = 0. \tag{B.8}
$$

Thus, from Fatou's lemma, $E[a + bH - U(H)] \le \liminf_{n\to\infty} E[a + bH_n - U(H_n)]$, and we obtain the lower-semicontinuity property $J(H) \le \liminf_{n\to\infty} J(H_n)$ in conjunction with (B.8), (B.4).

To recapitulate: in (B.5), we are minimizing the convex, proper, lower-semicontinuous functional $J$, over the closed, convex and bounded subset $A_{x,y}$ of $\mathbf{L}_2^*$. From a basic result of convex analysis (e.g. Ekeland & Temam (1976), p. 35) the functional $J$ attains its infimum over $A_{x,y}$, at a point $\hat{H}$ of $A_{x,y}$. Now $(\hat{H}, 0)$ is hedged by some pair $(\hat{L}, \hat{M}) \in \mathcal{A}^+(x, y)$ (recall (B.2)) with corresponding terminal holdings $(\hat{X}(T), \hat{Y}(T))$, which implies

$$
G := \hat{X}(T) + f(\hat{Y}(T)) = \hat{H}. \tag{B.9}
$$

Indeed, if this were not the case, we would have $G \ge \hat{H}$ a.s., and $G > \hat{H}$ with positive probability, because of the hedging property of Definition 2.1; moreover, since $(\hat{L}, \hat{M})$ obviously hedges $(G, 0)$, this would contradict the optimality of $\hat{H}$ and the strict monotonicity of $U(\cdot)$, provided that $G \in \mathbf{L}_2^*$ – but this follows from the remarks preceding the proof of Theorem 4.2, since $G \geq 0$ and $(\hat{X}(T), \hat{Y}(T))$ hedges $(G, 0)$.

It is now clear from (B.9) and the optimality of $\hat{H}$ that $(\hat{L}, \hat{M})$ is optimal for the problem of (6.2), and that (B.5) holds; in particular, $V(x, y) = -EU(\hat{H}) < \infty$.

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