THEORY OF PORTFOLIO OPTIMIZATION IN MARKETS WITH FRICTIONS *

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September 24, 1999

Abstract

This is a survey paper on portfolio optimization problems in continuous time market models. The tools of convex duality and martingales are used to solve these problems in the complete market case, as well as the markets which are incomplete, due to portfolio constraints or other market frictions such as different interest rates for borrowing and lending, or presence of transaction costs. Also presented is the problem of finding the minimal cost of superreplicating a given claim in such markets.

*Research supported in part by the National Science Foundation Grant DMS-97-32810.

1 Introduction

The main topic of this survey is the problem of utility maximization from terminal wealth for a single agent in various financial markets. Namely, given agent's utility function $U(\cdot)$ and initial capital $x > 0$, he is trying to maximize expected utility $E[U(X^{x,\pi}(T))]$ from his "terminal wealth", over all "admissible" portfolio strategies $\pi(\cdot)$. The same mathematical techniques that we employ here can be used to get similar results for maximizing expected utility from consumption; we refer the interested reader to the rich literature on that problem, some of which is cited below.

The seminal papers on these problems in the continuous-time complete market model are Merton (1969, 1971). Using Ito calculus and stochastic control/partial differential equations approach, Merton finds a solution to the problem in the Markovian model driven by Brownian motion process, for logarithmic and power utility functions. A comprehensive survey of his work is Merton (1990). For non-Markovian models one cannot deal with the problem using partial differential equations. Instead, a martingale approach using convex duality has been developed, with remarkable success in solving portfolio optimization problems in diverse frameworks. The approach is particularly well suited for incomplete markets (in which not all contingent claims can be perfectly replicated). It consists of solving an appropriate dual problem over a set of "state-price densities" corresponding to "shadow markets" associated with the incompleteness of the original market. Given the optimal solution $\hat{Z}$ to the dual problem, it is usually possible to show that the optimal terminal wealth for the primal problem is represented as the inverse of "marginal utility" (the derivative of the utility function) evaluated at $\hat{Z}$. Early work in this spirit includes Foldes (1978a,b) and Bismut (1975), based on his stochastic duality theory Bismut (1973). The first paper using (implicitly) the technique in its modern form, in the complete market, is Pliska (1986), followed by Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989, 1991). The explicit use of the duality method, and in incomplete and/or constrained market models, was applied by Xu (1990), He and Pearson (1991), Xu and Shreve (1992), Karatzas, Lehoczky, Shreve and Xu (1991), Cvitanić and Karatzas (1992, 1993), El Karoui and Quenez (1995), Jouini and Kallal (1995a), Karatzas and Kou (1996), Broadie, Cvitanić and Soner (1998). An excellent exposition of these methods can be found in Karatzas and Shreve (1998), and that of discrete-time models in Pliska (1997); see also Korn (1997). A definite treatment in a very general semimartingale framework is provided in Kramkov and Schachermayer (1998).

A similar approach works in models in which the drift of the wealth process of the agent is concave in his portfolio strategy $\pi(\cdot)$. This includes models with different borrowing and lending rates as well as some "large investor" models. Analytical approach is used in Fleming and Zariphopulou (1991), Bergman (1995), while the tools of duality are essential in El Karoui, Peng and Quenez (1997), Cvitanić (1997), Cuoco and Cvitanić (1998).

Portfolio optimization problems under transaction costs, usually on infinite horizon $T = \infty$, have been studied mostly in Markovian models, using PDE/variational inequalities methods. The literature includes Magill and Constantinides (1976), Constantinides (1979), Taksar, Klass and Assaf (1988), Davis & Norman (1990), Zariphopoulou (1992), Shreve and Soner (1994), Morton and Pliska (1995). We follow the martingale/duality approach of Cvitanić and Karatzas (1996) and Cvitanić and Wang (1999), on the finite horizon $T < \infty$. While this method is powerful enough to guarantee existence and a characterization of the optimal solution, algorithms for actually finding the optimal strategy are still lacking.

In order to apply the martingale approach to portfolio optimization, we first have to resolve the problem of (super)replication of contingent claims in a given market. After presenting the continuous-time complete market model and recalling the classical Black-Scholes-Merton pricing in Sections 2 and 3, we find the minimal cost of superreplicating a given claim B under convex constraints on the proportions of wealth the agent invests in stocks, in Sections 4 and 5 (for much more general results of this kind see Föllmer and Kramkov 1997). In the complete market this cost of superreplication of B is equal to the Black-Scholes price of B, namely equal to the expected value of B (discounted), under a change of probability measure that makes the discounted prices of stocks martingales.

In the case of a constrained market, in which the agent's hedging portfolio has to take values in a given closed convex set K, it is shown that the minimal cost of superreplication is now a supremum of Black-Scholes prices, taken over a family of auxiliary markets, parametrized by processes $\nu(\cdot)$, taking values in the domain of the support function of the set $-K$. These markets are chosen so that the wealth process becomes a supermartingale, under the appropriate change of measure. In the constant market parameters framework, the minimal cost for superreplicating B under constraints can be calculated as the Black-Scholes (unconstrained) price of an appropriately modified contingent claim $\hat{B} \geq B$, and the hedging portfolio for $\hat{B}$ automatically satisfies the constraints.

In Section 6 we show how the same methodology can be used to get analogous results in a market in which the drift of the wealth process is a concave function of the portfolio process.

Section 7 introduces the concept of utility functions, and Section 8 proves existence of an optimal constrained portfolio strategy for maximizing expected utility from terminal wealth. This is done indirectly, by first solving a dual problem, which is, loosely speaking, a problem to find an optimal change of probability measure associated to the constrained market. The optimal portfolio policy is the one that replicates the inverse of marginal utility, evaluated at the Radon-Nikodym derivative corresponding to the optimal change of measure in the dual problem. Explicit solutions are provided in Section 9, for the case of logarithmic and power utilities. Next, in Section 10 we argue that it makes sense to price contingent claims in the constrained market by calculating the Black-Scholes price in the unconstrained auxiliary market that corresponds to the optimal dual change of measure. Although in general this price depends on the utility of the agent and his initial capital, in many cases it does not. In particular, if the contraints are given by a cone, and the market parameters are constant, the optimal dual process is independent of utility and initial capital. This approach to pricing in incomplete markets was suggested in Davis (1997) and further developed in Karatzas and Kou (1996).

In Sections 11-14 we study the superreplication and utility maximization problems in the presence of proportional transaction costs. Similarly as in the case of constraints, we identify the family of (pairs of) changes of probability measure, under which the "wealth process" is a supermartingale, and the supremum over which gives the minimal superreplication cost of a claim in this market. Representations of this type were obtained in various models in Jouini and Kallal (1995b), Kusuoka (1995), Kabanov (1999). (It is known that in standard diffusion models this cost is simply the cost of the least expensive static (buy-and-hold) strategy which superreplicates the claim. For the case of the European call it is then equal to the price of one share of the underlying, the result which was conjectured by Davis and Clarke (1994) and proved by Soner, Shreve and Cvitanić (1995). The same result was shown to hold for more general models and claims in Leventhal and Skorohod (1997) and Cvitanić, Pham and Touzi (1998).) Next, we consider the utility maximization problem under transaction costs, and its dual. The nature of the optimal terminal wealth in the primal problem is shown to be the same as in the case of constraints - it is equal to the inverse of the marginal utility evaluated at the optimal dual solution. This result is used to get sufficient conditions for the optimal policy to be the one of no trade at all - this is the case if the return rate of the stock is not very different from the interest rate of the bank account and the transaction costs are large relative to the time horizon.

The important topic which is not considered here is approximate hedging and pricing under transaction costs. Articles dealing with this problem in continuous-time include Leland (1985) Avellaneda and Parás (1993), Davis, Panas and Zariphopoulou (1993), Davis and Panas (1994), Davis and Zariphopoulou (1995), Barles and Soner (1998), Constantinides and Zariphopoulou (1999). Other related works on the the subject of transaction costs, that the reader may find useful to consult are: Bensaid, Lesne, Pagès & Scheinkman (1992), Boyle & Vorst (1992), Edirisinghe, Naik & Uppal (1993), Flesaker & Hughston (1994), Gilster & Lee (1984), Grannan and Swindle (1996), Hodges & Neuberger (1989), Hoggard, Whalley & Wilmott (1993), Merton (1989), Morton & Pliska (1993).

2 The Complete Market Model

We introduce here the standard, Ito processes model for a financial market $\mathcal{M}$. It consists of one *bank account* and d *stocks*. Price processes $S_0(\cdot)$ and $S_1(\cdot),\ldots,S_d(\cdot)$ of these instruments are modeled by the equations

$$\begin{aligned} dS_0(t) &= S_0(t)r(t)dt\ , \quad S_0(0)=1 \\ dS_i(t) &= S_i(t)\left[b_i(t)dt+\sum_{j=1}^{d}\sigma_{ij}(t)dW^j(t)\right]\ , \quad S_i(0)=s_i>0\ , \end{aligned} \tag{2.1}$$

for $i=1,\ldots,d$, on some given time horizon $[0,T]$, $0<T<\infty$. Here $W(\cdot)=(W^1(\cdot),\ldots,W^d(\cdot))'$ is a standard $d-$dimensional Brownian motion on a complete probability space $(\Omega,\mathcal{F},P)$, endowed with a filtration $\mathbf{F}=\{\mathcal{F}_t\}_{0\le t\le T}$, the P-augmentation of $\mathcal{F}^W(t):=\sigma(W(s);\ 0\le s\le t)\ ,\ \ 0\le t\le T$, the filtration generated by the Brownian motion $W(\cdot)$. The *coefficients* $r(\cdot)$ (interest rate), $b(\cdot)=(b_1(\cdot),\ldots,b_d(\cdot))'$ (vector of stock return rates) and

$\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\le i,j\le d}$ (matrix of stock-volatilities) of the model $\mathcal{M}$, are all assumed to be progressively measurable with respect to $\mathbf{F}$. Furthermore, the matrix $\sigma(\cdot)$ is assumed to be invertible, and all processes $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$, $\sigma^{-1}(\cdot)$ are assumed to be bounded, uniformly in $(t,\omega) \in [0,T]\times\Omega$.

The "risk premium" process

$$\theta_0(t) := \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}] \ , \quad 0 \le t \le T \tag{2.2}$$

where $\mathbf{1} = (1,\ldots,1)' \in \mathbb{R}^d$, is then bounded and $\mathbf{F}-$progressively measurable. Therefore, the process

$$Z_0(t) := \exp\left[-\int_0^t \theta_0'(s)dW_0(s) - \frac{1}{2}\int_0^t ||\theta_0(s)||^2 ds\right], \ 0 \le t \le T \tag{2.3}$$

is a $P-$martingale, and

$$P_0(\Lambda) := E[Z_0(T)1_\Lambda], \quad \Lambda \in \mathcal{F}_T \tag{2.4}$$

is a probability measure equivalent to P on $\mathcal{F}_T$. Under this *risk-neutral equivalent martingale measure* P_0, the discounted stock prices $\frac{S_1(\cdot)}{S_0(\cdot)}, \ldots, \frac{S_d(\cdot)}{S_0(\cdot)}$ become martingales, and the process

$$W_0(t) \ := \ W(t) + \int_0^t \theta_0(s)ds \ , \quad 0 \le t \le T \ , \tag{2.5}$$

becomes Brownian motion, by the Girsanov theorem.

We also introduce the discount process

$$\gamma_0(t) := e^{-\int_0^t r(u)du}, \quad 0 \le t \le T. \tag{2.6}$$

and "state price density" process

$$H_0(t) := \gamma_0(t)Z_0(t), \quad 0 \le t \le T. \tag{2.7}$$

Consider now a financial agent whose actions cannot affect market prices, and who can decide, at any time $t \in [0,T]$, what proportion $\pi_i(t)$ of his (nonnegative) wealth $X(t)$ to invest in the i^{th} stock $(1 \le i \le d)$. Of course these decisions can only be based on the current information $\mathcal{F}_t$, without anticipation of the future. With $\pi(t) = (\pi_1(t), \ldots, \pi_d(t))'$ chosen, the amount $X(t)[1 - \sum_{i=1}^d \pi_i(t)]$ is invested in the bank. Thus, in light of the dynamics (2.1), the wealth process $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$ satisfies the linear stochastic differential equation

$$\begin{aligned} dX(t) &= -dc(t) + \left[X(t)(1 - \sum_{i=1}^d \pi_i(t))\right] r(t)dt + \sum_{i=1}^d \pi_i(t)X(t)\left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW^j(t)\right] \\ &= -dc(t) + r(t)X(t)dt + \pi'(t)\sigma(t)X(t)dW_0(t) \ ; \quad X(0) = x \ . \end{aligned}$$

where the real number $x > 0$ represents initial capital and $c(\cdot) \geq 0$ denotes the agent's *cumulative consumption process.*

We formalize the above discussion as follows.

Definition 2.1 (i) A *portfolio process* $\pi : [0,T] \times \Omega \to \mathbb{R}^d$ is $\mathbf{F}-$progressively measurable and satisfies $\int_0^T ||X(t)\pi(t)||^2 dt < \infty$, almost surely (here, X is the corresponding wealth process defined below). A *consumption process* $c(\cdot)$ is a nonnegative, nondecreasing, progressively measurable process with RCLL paths, with $c(0) = 0$ and $c(T) < \infty$.

(ii) For a given portfolio and consumption processes $\pi(\cdot)$, $c(\cdot)$, the process $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$ defined by (2.9) below, is called the *wealth process* corresponding to strategy (π, c) and initial capital x.

(iii) A portfolio-consumption process pair $(\pi(\cdot), c(\cdot))$ is called *admissible* for the initial capital x, and we write $(\pi, c) \in \mathcal{A}_0(x)$, if

$$X^{x,\pi,c}(t) \geq 0, \qquad 0 \leq t \leq T \tag{2.8}$$

holds almost surely.

For the discounted version of process $X(\cdot)$, we get the equation

$$d(\gamma_0(t)X(t)) \;\; = \;\; -\gamma_0(t)dc(t) + \pi'(t)\sigma(t)\gamma_0(t)X(t)dW_0(t). \tag{2.9}$$

It follows that $\gamma_0(\cdot)X(\cdot)$ is a nonnegative local P_0-supermartingale, hence also a P_0-supermartingale, by Fatou's lemma. Therefore, if τ_0 is defined to be the first time it hits zero, we have $X(t) = 0$ for $t \geq \tau_0$, so that the portfolio values $\pi(t)$ are irrelevant after that happens. Accordingly, we can and do set $\pi(t) \equiv 0$ for $t \geq \tau_0$. The supermartingale property implies

$$E^0[\gamma_0(T)X^{x,\pi,c}(T)] \leq x, \quad \forall\ \pi\ \in\ \mathcal{A}_0(x)\ . \tag{2.10}$$

Here, E^0 denotes the expectation operator under the measure P_0.

We say that a strategy $(\pi(\cdot), c(\cdot))$ results in *arbitrage* if with the initial investment $x = 0$ we have $X^{0,\pi,c}(T) \geq 0$ almost surely, but $X^{0,\pi,c}(T) > 0$ with positive probability. Notice that inequality (2.10) implies that an admissible strategy $(\pi(\cdot), c(\cdot)) \in \mathcal{A}_0(0)$ cannot result in arbitrage.

3 Pricing in the complete market

Let us suppose now that the agent promises to pay a random amount $B(\omega) \geq 0$ at time $t = T$ and that he wants to invest x dollars in the market in such a way that his profit "hedges away" all the risk, namely that $X^{x,\pi,c}(T) \geq B$, almost surely. What is the smallest value of $x > 0$ for which such "hedging" is possible? This smallest value will then be the "price" of the *contingent claim* B at time $t = 0$.

We say that B is a contingent claim if it is a nonnegative, $\mathcal{F}_T$-measurable random variable such that $0 < E^0[\gamma_0(T)B] < \infty$. The *super-replication price* of this contingent claim is defined by

$$h(0) := \inf\{x > 0;\ \exists(\pi, c) \in \mathcal{A}_0(x)\ \ s.t.\ \ X^{x,\pi,c}(T) \geq B\ \ a.s.\}. \tag{3.1}$$

The following classical result identifies $h(0)$ as the expectation, under the risk-neutral probability measure, of the claim's discounted value; see Harrison & Kreps (1979), Harrison & Pliska (1981, 83).

Proposition 3.1 *The infimum in* (3.1) *is attained, and we have*

$$h(0) = E^0[\gamma_0(T)B]\ . \tag{3.2}$$

Furthermore, there exists a portfolio $\pi_B(\cdot)$ *such that* $X_B(\cdot) \equiv X^{h(0),\pi_B,o}(\cdot)$ *is given by*

$$X_B(t) = \frac{1}{\gamma_0(t)} E^0[\gamma_0(T)B|\mathcal{F}_t] \ , \quad 0 \le t \le T \ . \tag{3.3}$$

Proof: Suppose $X^{x,\pi,c}(T) \ge B$ holds a.s. for some $x \in (0,\infty)$ and a suitable $(\pi, c) \in \mathcal{A}_0(x)$. Then from (2.10) we have $x \ge z := E^0[\gamma_0(T)B]$ and thus $h(0) \ge z$.

On the other hand, from the martingale representation theorem, the process

$$X_B(t) := \frac{1}{\gamma_0(t)} E^0[\gamma_0(T)B|\mathcal{F}_t] \ , \quad 0 \le t \le T$$

can be represented as

$$X_B(t) = \frac{1}{\gamma_0(t)}[z + \int_0^t \psi'(s) dW_0(s)]$$

for a suitable $\{\mathcal{F}_t\}$-progressively measurable process $\psi(\cdot)$ with values in $\mathbb{R}^d$ and $\int_0^T ||\psi(t)||^2 dt < \infty$, a.s. Then $\pi_B(t) := \frac{1}{\gamma_0(t)X_B(t)}(\sigma'(t))^{-1}\psi(t)$ is a well defined portfolio process, and we have $X_B(\cdot) \equiv X^{z,\pi_B,0}(\cdot)$, by comparison with (2.9). Therefore, $z \ge h(0)$. $\diamond$

Notice that

$$X_B^{h(0),\pi_B,0}(T) = B \ ,$$

almost surely. We express this by saying that contingent claim B is *attainable*, with initial capital $h(0)$ and portfolio π_B. In this complete market model, we call $h(0)$ the *Black-Scholes price* of B and $\pi_B(\cdot)$ the Black-Scholes hedging portfolio.

Example 3.1 *Constant* $r(\cdot) \equiv r > 0, \sigma(\cdot) \equiv \sigma$ *nonsingular.* In this case, the solution $S(t) = (S_1(t), \ldots, S_d(t))'$ is given by $S_i(t) = f_i(t-s, S(s), \sigma(W_0(t) - W_0(s)))$, $0 \le s \le t$ where $f : [0,\infty) \times \mathbb{R}_+^d \times \mathbb{R}^d \to \mathbb{R}_+^d$ is the function defined by

$$f_i(t, s, y; r) := s_i \exp[(r - \frac{1}{2}a_{ii})t + y_i] \ , \quad i = 1, \ldots, d$$

where $a = \sigma\sigma'$.

Consider now a contingent claim of the type $B = \varphi(S(T))$, where $\varphi : \mathbb{R}_+^d \to [0, \infty)$ is a given continuous function, that satisfies polynomial growth conditions in both $||s||$ and $1/||s||$. Then the value process of this claim is given by

$$\begin{aligned} X_B(t) &= e^{-r(T-t)} E^0[\varphi(S(T))|\mathcal{F}_t] \\ &= e^{-r(T-t)} \int_{\mathbb{R}^d} \varphi(f(T-t, S(t), \sigma z)) \frac{1}{(2\pi(T-t))^{d/2}} \exp\{-\frac{\|z\|^2}{2(T-t)}\} dz \\ &= V(T-t, S(t)), \end{aligned}$$

where

$$V(t,p) := \begin{Bmatrix} e^{-rt} \int_{\mathcal{R}^d} \varphi(h(t,s,\sigma z; r)) \frac{e^{-||z||^2/2t}}{(2\pi t)^{d/2}} dz & ;\ t > 0,\ s \in \mathbb{R}_+^d \\ \varphi(s) & ;\ t = 0,\ s \in \mathbb{R}_+^d \end{Bmatrix}.$$

In particular, the price $h(0)$ of the claim B is given, in terms of the function V, by

$$h(0) = X_B(0) = V(T, S(0)) \ .$$

Moreover, function V is the unique solution to the Cauchy problem (by Feynman-Kac theorem)

$$\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij} x_i x_j \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^d r(x_i \frac{\partial V}{\partial x_i} - V) = \frac{\partial V}{\partial t},$$

with the initial condition $V(0, x) = \varphi(x)$. Applying Ito's rule, we obtain

$$dV(T-t, S(t)) = rV(T-t, S(t)) + \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} S_i(t) \frac{\partial S}{\partial x_i}(T-t, S_i(t)) dW_0^{(j)}(t).$$

Comparing this with (2.9), we get that the hedging portfolio is given by

$$\pi_i(t) V(T-t, S(t)) = S_i(t) \frac{\partial V}{\partial x_i}(T-t, S(t)),\ i = 1, \ldots, d.$$

It should be noted that none of the above depends on vector $b(\cdot)$ of return rates.

If, for example, we have $d = 1$ and in the case $\varphi(s) = (s-k)^+$ of a *European call option*, with $\sigma = \sigma_{11} > 0$, exercise price $k > 0, N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du$ and $d_{\pm}(t,s) := \frac{1}{\sigma\sqrt{t}}\Big[\log(\frac{s}{k}) + (r \pm \frac{\sigma^2}{2})t\Big]$, we have the famous *Black & Scholes (1973) formula*

$$V(t,s) = \left\{ \begin{array}{cl} sN(d_+(t,s)) - ke^{-rt}N(d_-(t,s)) & ;\ t>0,\ s \in (0,\infty) \\ (s-k)^+ & ;\ t=0,\ s \in (0,\infty) \end{array} \right\}.$$

4 Portfolio Constraints

We fix throughout a nonempty, closed, convex set K in $\mathbb{R}^d$, and denote by

$$\delta(x) := \sup_{\pi \in K} \{-\pi' x\} \tag{4.1}$$

the support function of the set $-K$. This is a closed, positively homogeneous, proper convex function on $\mathbb{R}^d$ (Rockafellar (1970), p.114). It is finite on its *effective domain*

$$\tilde{K} := \{x \in \mathbb{R}^d \ / \ \delta(x) < \infty\} \tag{4.2}$$

which is a convex cone (called the "barrier cone" of $-K$). For the rest of the paper we assume the following mild conditions.

Assumption 4.1 The closed convex set $K \subset \mathbb{R}^d$ contains the origin; in other words, the agent is allowed not to invest in stocks at all. In particular, $\delta(\cdot) \geq 0$ on $\tilde{K}$. Moreover, the set K is such that $\delta(\cdot)$ is continuous on the barrier cone $\tilde{K}$ of (4.2).

The role of the closed, convex set K that we just introduced, is to model reasonable constraints on portfolio choice. One may, for instance, consider the following examples.

(i) *Unconstrained case:* $K = \mathbb{R}^d$. Then $\tilde{K} = \{0\}$, and $\delta \equiv 0$ on $\tilde{K}$.

(ii) *Prohibition of short-selling:* $K = [0,\infty)^d$. Then $\tilde{K} = K$, and $\delta \equiv 0$ on $\tilde{K}$.

(iii) *Incomplete Market:* $K = \{\pi \in \mathbb{R}^d; \pi_i = 0,\ \forall\ i = m+1, \ldots, d\}$ for some fixed $m \in \{1, \ldots, d-1\}$. Then $\tilde{K} = \{x \in \mathbb{R}^d;\ x_i = 0,\ \forall\ i = 1, \ldots, m\}$ and $\delta \equiv 0$ on $\tilde{K}$.

(iv) *K is a closed, convex cone* in $\mathbb{R}^d$. Then $\tilde{K} = \{x \in \mathbb{R}^d;\ \pi' x \geq 0,\ \forall\ \pi \in K\}$ is the polar cone of $-K$, and $\delta \equiv 0$ on $\tilde{K}$. This case obviously generalizes (i) - (iii).

(v) *Prohibition of borrowing:* $K = \{\pi \in \mathbb{R}^d; \sum_{i=1}^d \pi_i \leq 1\}$. Then $\tilde{K} = \{x \in \mathbb{R}^d;\ x_1 = \ldots = x_d \leq 0\}$, and $\delta(x) = -x_1$ on $\tilde{K}$.

(vi) *Rectangular constraints:* $K = \times_{i=1}^d I_i$, $I_i = [\alpha_i, \beta_i]$ for some fixed numbers $-\infty \leq \alpha_i \leq 0 \leq \beta_i \leq \infty$, with the understanding that the interval I_i is open to the right (left) if $b_i = \infty$ (respectively, if $\alpha_i = -\infty$). Then $\delta(x) = \sum_{i=1}^d (\beta_i x_i^- - \alpha_i x_i^+)$ and $\tilde{K} = \mathbb{R}^d$ if all the $\alpha_i^{\prime} s$, $\beta_i^{\prime} s$ are real. In general, $\tilde{K} = \{x \in \mathbb{R}^d; x_i \geq 0,\ \forall\ i \in \mathcal{S}_+$ and $x_j \leq 0,\ \forall\ j \in \mathcal{S}_-\}$ where $\mathcal{S}_+ := \{i = 1, \ldots, d\ /\ \beta_i = \infty\}$, $\mathcal{S}_- := \{i = 1, \ldots, d\ /\ \alpha_i = -\infty\}$. $\diamond$

We consider now only portfolios that take values in the given, convex, closed set $K \subset \mathbb{R}^d$, i.e., we replace the set of admissible policies $\mathcal{A}_0(x)$ with

$$\mathcal{A}'(x) := \{(\pi, c) \in \mathcal{A}_0(x);\ \pi(t, \omega) \in K \quad \text{for} \quad \ell \times \mathbf{P} - a.e.\ (t, \omega)\} .$$

Here, ℓ stands for Lebesgue measure on $[0, T]$.

Denote by $\mathcal{D}$ the set of all *bounded* progressively measurable process $\nu(\cdot)$ taking values in $\tilde{K}$ a.e. on $\Omega \times [0, T]$. In analogy with (2.2)-(2.5), introduce

$$\theta_\nu(t) := \sigma^{-1}(t)[\nu(t) + b(t) - r(t)\mathbf{1}] , \quad 0 \leq t \leq T, \tag{4.3}$$

$$Z_\nu(t) := \exp\left[-\int_0^t \theta'_\nu(s) dW(s) - \frac{1}{2}\int_0^t ||\theta_\nu(s)||^2 ds\right],\ 0 \leq t \leq T, \tag{4.4}$$

$$P_\nu(\Lambda) := E[Z_\nu(T) 1_\Lambda], \quad \Lambda \in \mathcal{F}_T \tag{4.5}$$

$$W_\nu(t) \ := \ W(t) + \int_0^t \theta_\nu(s) ds \ , \quad 0 \leq t \leq T, \tag{4.6}$$

a $P^\nu-$Brownian motion. Also denote

$$\gamma_\nu(t) := e^{-\int_0^t [r(u) + \delta(\nu(u))] du} \tag{4.7}$$

and

$$H_\nu(t) := \gamma_\nu(t) Z_\nu(t). \tag{4.8}$$

Proposition 4.1 *The (nonnegative) process*

$$M_\nu(t) := H_\nu(t)X(t) + \int_0^t H_\nu(s)\left[X(s)(\delta(\nu_s) + \nu'(s)\pi(s))ds + dc(s)\right]$$

is a $P-$ supermartingale for every $\nu \in \mathcal{D}$ and $(\pi, c) \in \mathcal{A}'(x)$. In particular,

$$\sup_{\nu \in \mathcal{D}} E\left[H_\nu(T)X(T) + \int_0^T H_\nu(s)X(s)\{\delta(\nu_s) + \pi'(s)\nu(s)\}ds\right] \leq x \ . \tag{4.9}$$

Proof: Ito's rule implies

$$M_\nu(t) = x + \int_0^t H_\nu(s)X(s)\left[\pi'(s)\sigma(s) - \theta'_\nu(s)\right] dW(s).$$

In particular, the process on the right-hand side is a nonnegative local martingale, hence a supermartingale. $\diamond$

In general, there are several interpretations for the processes $\nu \in \mathcal{D}$: they are stochastic "Lagrange multipliers" associated with the portfolio constraints; in economics jargon, they correspond to the shadow prices relevant to the incompleteness of the market introduced by constraints. The number $h_\nu(0) := E^\nu[\gamma_\nu(T)B] = E[H_\nu(T)B]$ is the unconstrained hedging price for B in an *auxiliary market* $\mathcal{M}_\nu$; this market consists of a bank account with interest rate $r^{(\nu)}(t) := r(t) + \delta(\nu(t))$ and d stocks, with the same volatility matrix $\{\sigma_{ij}(t)\}_{1 \leq i,j \leq d}$ as before and return rates $b_i^{(\nu)}(t) := b_i(t) + \nu_i(t) + \delta(\nu(t))$, $1 \leq i \leq d$, for any given $\nu \in \mathcal{D}$. We shall show that the price for superreplicating B with a constrained portfolio in the market $\mathcal{M}$, is given by the supremum of the unconstrained hedging prices $h_\nu(0)$ in these auxiliary markets $\mathcal{M}_\nu$, $\nu \in \mathcal{D}$.

5 Superreplication under portfolio constraints

Consider the minimal cost of superreplication of the claim B in the market with constraints:

$$(6.1) \quad h(0) := \left\{ \begin{array}{c} \inf\{x > 0; \exists(\pi, c) \in \mathcal{A}'(x),\ s.t.\ X^{x,\pi,c}(T) \geq B\ a.s.\} \\ \infty \quad \text{, if the above set is empty} \end{array} \right\}.$$

Let us denote by $\mathcal{S}$ the set of all $\{\mathcal{F}_t\}$-stopping times τ with values in $[0, T]$, and by $\mathcal{S}_{\rho,\sigma}$ the subset of $\mathcal{S}$ consisting of stopping times τ s.t. $\rho \leq \tau \leq \sigma$, for any two $\rho \in \mathcal{S}, \sigma \in \mathcal{S}$ such that $\rho \leq \sigma$, a.s. For every $\tau \in \mathcal{S}$ consider also the $\mathcal{F}_\tau$-measurable random variable

$$V(\tau) := ess \sup_{\nu \in \mathcal{D}} E^\nu[B\gamma_0(T) \exp\{-\int_\tau^T \delta(\nu(s))ds\}|\mathcal{F}_\tau]. \tag{5.1}$$

We will show that $h(0) = V(0)$. We first need

Proposition 5.1 *If $V(0) = \sup_{\nu \in \mathcal{D}} E^\nu[\gamma_\nu(T)B] < \infty$, then the family of random variables $\{V(\tau)\}_{\tau \in \mathcal{S}}$ satisfies the equation of Dynamic Programming*

$$V(\tau) = ess \sup_{\nu \in \mathcal{D}_{\tau,\theta}} E^\nu[V(\theta) \exp\{-\int_\tau^\theta \delta(\nu(u))du\}|\mathcal{F}_\tau] \ ; \quad \forall \ \ \theta \in \mathcal{S}_{\tau,T} \ , \tag{5.2}$$

where $\mathcal{D}_{\tau,\theta}$ is the restriction of $\mathcal{D}$ to the stochastic interval $[\![\tau, \theta]\!]$.

Proposition 5.2 *The process $V = \{V(t), \mathcal{F}_t; 0 \leq t \leq T\}$ can be considered in its RCLL modification and, for every $\nu \in \mathcal{D}$,*

$$\left\{ \begin{array}{c} Q_\nu(t) := V(t)e^{-\int_0^t \delta(\nu(u))du}, \mathcal{F}_t;\ 0 \leq t \leq T \\ \\ \textit{is a } \mathbf{P}^\nu\textit{-supermartingale with RCLL paths} \end{array} \right\}. \tag{5.3}$$

Furthermore, V is the smallest adapted, RCLL process that satisfies (5.3) *as well as*

$$V(T) = B\gamma_0(T), \qquad a.s. \tag{5.4}$$

Proof of Proposition 5.1: Let us start by observing that, for any $\theta \in \mathcal{S}$, the random variable

$$J_\nu(\theta) \ \ := \ \ E^\nu[V(T)e^{-\int_\theta^T \delta(\nu(s))ds}|\mathcal{F}_\theta]$$

$$
\begin{aligned}
&= \frac{E[Z_\nu(\theta)Z_\nu(\theta,T)V(T)e^{-\int_\theta^T \delta(\nu(s))ds}|\mathcal{F}_\theta]}{E[Z_\nu(\theta)Z_\nu(\theta,T)|\mathcal{F}_\theta]} \\
&= E[Z_\nu(\theta,T)V(T)e^{-\int_\theta^T \delta(\nu(s))ds}|\mathcal{F}_\theta]
\end{aligned}
$$

depends only on the restriction of ν to $[\![\theta,T]\!]$ (we have used the notation $Z_\nu(\theta,T) = \frac{Z_\nu(T)}{Z_\nu(\theta)}$). It is also easy to check that the family of random variables $\{J_\nu(\theta)\}_{\nu\in\mathcal{D}}$ is directed upwards; indeed, for any $\mu \in \mathcal{D}, \nu \in \mathcal{D}$ and with $A = \{(t,\omega);\ J_\mu(t,\omega) \geq J_\nu(t,\omega)\}$ the process $\lambda := \mu\mathbf{1}_A + \nu\mathbf{1}_{A^c}$ belongs to $\mathcal{D}$ and we have a.s. $J_\lambda(\theta) = \max\{J_\mu(\theta), J_\nu(\theta)\}$; then from Neveu (1975), p.121, there exists a sequence $\{\nu_k\}_{k\in\mathbf{N}} \subseteq \mathcal{D}$ such that $\{J_{\nu_k}(\theta)\}_{k\in\mathbf{N}}$ is increasing and

$$
(i) \qquad V(\theta) = \lim_{k\to\infty} \uparrow J_{\nu_k}(\theta), \quad a.s.
$$

Returning to the proof itself, let us observe that

$$
\begin{aligned}
V(\tau) &= esssup_{\nu\in\mathcal{D}_{\tau,T}} E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds}E^\nu\{V(T)e^{-\int_\theta^T \delta(\nu(s))ds}|\mathcal{F}_\theta\}|\mathcal{F}_\tau] \\
&\leq esssup_{\nu\in\mathcal{D}_{\tau,T}} E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds}V(\theta)|\mathcal{F}_\tau], \quad a.s.
\end{aligned}
$$

To establish the opposite inequality, it certainly suffices to pick $\mu \in \mathcal{D}$ and show that

$$
(ii) \qquad V(\tau) \geq E^\mu[V(\theta)e^{-\int_\tau^\theta \delta(\mu(s))ds}|\mathcal{F}_\tau]
$$

holds almost surely.

Let us denote by $M_{\tau,\theta}$ the class of processes $\nu \in \mathcal{D}$ which agree with μ on $[\![\tau,\theta]\!]$. We have

$$
\begin{aligned}
V(\tau) &\geq \operatorname*{ess\ sup}_{\nu\in M_{\tau,\theta}} E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds-\int_\theta^T \delta(\nu(s))ds}V(T)|\mathcal{F}_\tau] \\
&= \operatorname*{ess\ sup}_{\nu\in M_{\tau,\theta}} E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds}E^\nu\{e^{-\int_\theta^T \delta(\nu(s))ds}V(T)|\mathcal{F}_\theta\}|\mathcal{F}_\tau].
\end{aligned}
$$

Thus, for every $\nu \in M_{\tau,\theta}$, we have

$$
V(\tau) \;\; \geq \;\; E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds}J_\nu(\theta)|\mathcal{F}_\tau]
$$

$$
\begin{aligned}
&= \frac{E[Z_\nu(\tau)Z_\nu(\tau,\theta)E\{Z_\nu(\theta,T)|\mathcal{F}_\theta\}e^{-\int_\tau^\theta \delta(\nu(s))ds}J_\nu(\theta)|\mathcal{F}_\tau]}{E[Z_\nu(\tau)Z_\nu(\tau,\theta)E\{Z_\nu(\theta,T)|\mathcal{F}_\theta\}|\mathcal{F}_\tau]} \\
&= E[Z_\nu(\tau,\theta)e^{-\int_\tau^\theta \delta(\nu(s))ds}J_\nu(\theta)|\mathcal{F}_\tau] \\
&= E[Z_\mu(\tau,\theta)e^{-\int_\tau^\theta \delta(\mu(s))ds}J_\nu(\theta)|\mathcal{F}_\tau] \\
&= \ldots = E^\mu[e^{-\int_\tau^\theta \delta(\mu(s))ds}J_\nu(\theta)|\mathcal{F}_\tau].
\end{aligned}
$$

Now clearly we may take $\{\nu_k\}_{k\epsilon\mathbf{N}} \subseteq M_{\tau,\theta}$ in (i), as $J_\nu(\theta)$ depends only on the restriction of ν on $[\![\theta, T]\!]$; and from the above,

$$
\begin{aligned}
V(\tau) &\geq \lim_{k\to\infty} \uparrow E^\mu[e^{-\int_\tau^\theta \delta(\mu(s))ds}J_{\nu_k}(\theta)|\mathcal{F}_\tau] \\
&= E^\mu[e^{-\int_\tau^\theta \delta(\mu(s))ds}\lim_{k\to\infty} \uparrow J_{\nu_k}(\theta)|\mathcal{F}_\tau] \\
&= E^\mu[e^{-\int_\tau^\theta \delta(\mu(s))ds}V(\theta)|\mathcal{F}_\tau], \ a.s.
\end{aligned}
$$

by Monotone Convergence. $\diamond$

It is an immediate consequence of this proposition that

$$
(iii) \qquad V(\tau)e^{-\int_0^\tau \delta(\nu(u))du} \geq E^\nu[V(\theta)e^{-\int_0^\theta \delta(\nu(u))du}|\mathcal{F}_\tau], \quad a.s.
$$

holds for any given $\tau \in \mathcal{S}, \theta \in \mathcal{S}_{\tau,T}$ and $\nu \in \mathcal{D}$.

Proof of Proposition 5.2: Let us consider the positive, adapted process $\{V(t,\omega), \mathcal{F}_t;\ t \in [0,T] \cap \mathcal{Q}\}$ for $\omega \in \Omega$. From (iii), the process

$$
\{V(t,\omega)e^{-\int_0^t \delta(\nu(s,\omega))ds},\ \mathcal{F}_t;\ t \in [0,T] \cap \mathcal{Q}\} \text{ for } \quad \omega \in \Omega
$$

is a $\mathbf{P}^\nu$- supermartingale on $[0,T] \cap \mathcal{Q}$, where $\mathcal{Q}$ is the set of rational numbers, and thus has a.s. finite limits from the right and from the left (recall Proposition 1.3.14 in Karatzas & Shreve (1991), as well as the right-continuity of the filtration $\{\mathcal{F}_t\}$). Therefore,

$$
V(t+,\omega) := \left\{ \begin{array}{lcc} \lim_{\substack{s\downarrow t \\ s\in\mathcal{Q}}} V(s,\omega) & ; & 0 \leq t < T \\ V(T,\omega) & ; & t = T \end{array} \right\}
$$

$$
V(t-,\omega) := \left\{ \begin{array}{lcc} \lim_{\substack{s\uparrow t \\ s\in\mathcal{Q}}} V(s,\omega) & ; & 0 < t \leq T \\ V(0) & ; & t = 0 \end{array} \right\}
$$

are well-defined and finite for every $\omega \in \Omega^*, P(\Omega^*) = 1$, and the resulting processes are adapted. Furthermore (loc.cit.), $\{V(t+)e^{-\int_0^t \delta(\nu(s))ds},\ \mathcal{F}_t;\ 0 \le t \le T\}$ is a RCLL, $\mathbf{P}^\nu$-supermartingale, for all $\nu \in \mathcal{D}$; in particular,

$$V(t+) \ge E^\nu[V(T)e^{-\int_t^T \delta(\nu(s))ds}|\mathcal{F}_t],\ a.s.$$

holds for every $\nu \in \mathcal{D}$, whence $V(t+) \ge V(t)$ a.s. On the other hand, from Fatou's lemma we have for any $\nu \in \mathcal{D}$:

$$\begin{aligned} V(t+) &= E^\nu[\lim_{n\to\infty} V(t+\frac{1}{n})\ e^{-\int_t^{t+1/n} \delta(\nu(u))du}|\mathcal{F}_t] \\ &\le \lim_{n\to\infty} E^\nu[V(t+\frac{1}{n})\ e^{-\int_t^{t+1/n} \delta(\nu(u))du}|\mathcal{F}_t] \le V(t),\ a.s. \end{aligned}$$

and thus $\{V(t+),\ \mathcal{F}_t;\ 0 \le t \le T\}, \{V(t),\ \mathcal{F}_t;\ 0 \le t \le T\}$ are modifications of one another.

The remaining claims are immediate. $\diamond$

Theorem 5.1 *For an arbitrary contingent claim B, we have $h(0) = V(0)$. Furthermore, if $V(0) < \infty$, there exists a pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}'(V(0))$ such that $X^{V(0),\hat{\pi},\hat{c}}(T) = B,\ a.s.$*

Proof: Proposition 4.1 implies $x \ge E^\nu[\gamma_\nu(T)B]$ for every $\nu \in \mathcal{D}$, hence $h(0) \ge V(0)$.

We now show the more difficult part: $h(0) \le V(0)$. Clearly, we may assume $V(0) < \infty$. From (5.3), the martingale representation theorem and the Doob-Meyer decomposition, we have for every $\nu \in \mathcal{D}$:

$$Q_\nu(t) = V(0) + \int_0^t \psi_\nu'(s)dW_\nu(s) - A_\nu(t), \quad 0 \le t \le T\ , \tag{5.5}$$

where $\psi_\nu(\cdot)$ is an $\mathbb{R}^d$-valued, $\{\mathcal{F}_t\}$-progressively measurable and a.s. square-integrable process and $A_\nu(\cdot)$ is adapted with increasing, RCLL paths and $A_\nu(0) = 0, EA_\nu(T) < \infty$ a.s. The idea then is to consider the positive, adapted, RCLL process

$$\hat{X}(t) := \frac{V(t)}{\gamma_0(t)} = \frac{Q_\nu(t)}{\gamma_\nu(t)}, \quad 0 \le t \le T \quad (\forall\ \nu \in \mathcal{D}) \tag{5.6}$$

with $\hat{X}(0) = V(0), \hat{X}(T) = B$ a.s., and to find a pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}'(V(0))$ such that $\hat{X}(\cdot) = X^{V(0),\hat{\pi},\hat{c}}(\cdot)$. This will prove that $h(0) \leq V(0)$.

In order to do this, let us observe that for any $\mu \in \mathcal{D}, \nu \in \mathcal{D}$ we have from (5.3)

$$Q_\mu(t) = Q_\nu(t) \exp\left[\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}ds\right] ,$$

and from (5.5):

$$\begin{aligned} dQ_\mu(t) &= \exp[\textstyle\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}ds] \cdot [Q_\nu(t)\{\delta(\nu(t)) - \delta(\mu(t))\}dt \\ &\qquad + \psi_\nu'(t)dW_\nu(t) - dA_\nu(t)] \qquad (5.7) \\ &= \exp[\textstyle\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}ds] \cdot [\hat{X}(t)\gamma_\nu(t)\{\delta(\nu(t)) - \delta(\mu(t))\}dt \\ &\qquad - dA_\nu(t) + \psi_\nu'(t)\sigma^{-1}(t)(\nu(t) - \mu(t))dt + \psi_\nu'(t)dW_\mu(t)] . \end{aligned}$$

Comparing this decomposition with

$$dQ_\mu(t) = \psi_\mu'(t)dW_\mu(t) - dA_\mu(t) , \qquad (5.8)$$

we conclude that

$$\psi_\nu'(t)\ e^{\int_0^t \delta(\nu(s))ds} = \psi_\mu'(t)\ e^{\int_0^t \delta(\mu(s))ds}$$

and hence that this expression is independent of $\nu \in \mathcal{D}$:

$$\psi_\nu'(t)\ e^{\int_0^t \delta(\nu(s))ds} = \hat{X}(t)\gamma_0(t)\hat{\pi}'(t)\sigma(t); \quad \forall\ 0 \leq t \leq T,\ \nu \in \mathcal{D} \qquad (5.9)$$

for some adapted, $\mathbb{R}^d$-valued, a.s. square integrable process $\hat{\pi}$ (we do not know yet that $\hat{\pi}$ takes values in K). If $X(t) = 0$, then $X(s) = 0$ for all $s \geq t$, and we can set, for example, $\pi(s) = 0,\ s \geq t$ (in fact, one can show that $\int_0^T \mathbf{1}_{\{\hat{X}(t)=0\}} \|\psi_\nu(t)\|^2 dt = 0$, a.s; see Karatzas and Kou (1996)).

Similarly, we conclude from (5.7), (5.9) and (5.8):

$$e^{\int_0^t \delta(\nu(s))ds} dA_\nu(t) - \gamma_0(t)\hat{X}(t)[\delta(\nu(t)) + \hat{\pi}'(t)\nu(t)]dt$$

$$= e^{\int_0^t \delta(\mu(s))ds} dA_\mu(t) - \gamma_0(t)\hat{X}(t)[\delta(\mu(t)) + \hat{\pi}'(t)\mu(t)]dt$$

and hence this expression is also independent of $\nu \in \mathcal{D}$:

$$\hat{c}(t) := \int_0^t \gamma_\nu^{-1}(s) dA_\nu(s) - \int_0^t \hat{X}(s)[\delta(\nu(s)) + \nu'(s)\hat{\pi}(s)]ds \ , \tag{5.10}$$

for every $0 \le t \le T, \nu \in \mathcal{D}$. Setting $\nu \equiv 0$, we obtain $\hat{c}(t) = \int_0^t \gamma_0^{-1}(s)dA_0(s), 0 \le t \le T$ and hence

$$\left\{ \begin{array}{c} \hat{c}(\cdot) \text{ is an increasing, adapted, RCLL process} \\ \text{with } \hat{c}(0) = 0 \quad \text{and} \quad \hat{c}(T) < \infty, a.s. \end{array} \right\}. \tag{5.11}$$

Next, we claim that

$$\delta(\nu) + \nu'\hat{\pi}(t,\omega) \ge 0, \quad \ell \otimes \mathbf{P} - a.e. \tag{5.12}$$

holds for every $\nu \in \tilde{K}$. Then Theorem 13.1 of Rockafellar (1970) (together with continuity of $\delta(\cdot)$ and closedness of K) leads to the fact that

$$\hat{\pi}(t,\omega) \in K \quad \text{holds} \quad \ell \otimes \mathbf{P} - a.e. \ \text{ on } \ [0,T] \times \Omega.$$

In order to verify (5.12), notice that from (5.10) we obtain

$$\int_0^t \gamma_\nu^{-1}(s) A_\nu(s) ds \ = \hat{c}(t) + \textstyle\int_0^t \hat{X}(s)\{\delta(\nu_s) + \nu_s'\hat{\pi}_s\}ds; \ 0 \le t \le T, \ \nu \in \mathcal{D}.$$

Fix $\nu \in \tilde{K}$ and define the set $F_\nu := \{(t,\omega) \in [0,T] \times \Omega; \ \delta(\nu)) + \nu'\hat{\pi}(t,\omega) < 0\}$. Let $\mu(t) := [\nu 1_{F_\nu^c} + n\nu 1_{F_\nu}], \ n \in \mathbf{N}$; then $\mu \in \mathcal{D}$, and assuming that (5.12) does not hold, we get for n large enough

$$\begin{aligned} E[\int_0^T \gamma_\mu^{-1}(s) A_\mu(s) ds] \ &= E\left[\hat{c}(T) + \textstyle\int_0^T \hat{X}(t) 1_{F_\nu^c}\{\delta(\nu) + \nu'\hat{\pi}(t)\}dt\right] \\ &\quad + nE\left[\textstyle\int_0^T \hat{X}(t) 1_{F_\nu}\{\delta(\nu) + \nu'\hat{\pi}(t)\}dt\right] < 0 \ , \end{aligned}$$

a contradiction.

Now we can put together (5.5)-(5.10) to deduce

$$\begin{aligned} d(\gamma_\nu(t)\hat{X}(t)) &= dQ_\nu(t) = \psi_\nu'(t)dW_\nu(t) - dA_\nu(t) \\ &= \gamma_\nu(t)[-d\hat{c}(t) - \hat{X}(t)\{\delta(\nu(t)) + \nu'(t)\hat{\pi}(t)\}dt \\ &\qquad + \hat{X}(t)\hat{\pi}'(t)\sigma(t)dW_\nu(t)] \ , \end{aligned} \tag{5.13}$$

for any given $\nu \in \mathcal{D}$. As a consequence, the process

$$\begin{aligned}\hat{M}_\nu(t) &:= \gamma_\nu(t)\hat{X}(t) + \int_0^t \gamma_\nu(s)d\hat{c}(s) + \int_0^t \gamma_\nu(s)\hat{X}(s)[\delta(\nu(s)) + \nu'(s)\hat{\pi}(s)]ds \qquad (5.14)\\ &= V(0) + \int_0^t \gamma_\nu(s)\hat{X}(s)\hat{\pi}'(s)\sigma(s)dW_\nu(s)\ , \quad 0 \le t \le T\end{aligned}$$

is a nonnegative, $\mathbf{P}^\nu$-local martingale, hence supermartingale. In particular, for $\nu \equiv 0$, (5.13) gives:

$$\begin{aligned}d(\gamma_0(t)\hat{X}(t)) &= -\gamma_0(t)d\hat{c}(t) + \gamma_0(t)\hat{X}(t)\hat{\pi}'(t)\sigma(t)dW_0(t),\\ \hat{X}(0) &= V(0)\ , \quad \hat{X}(T) = B\ ,\end{aligned}$$

which is equation (2.9) for the process $\hat{X}(\cdot)$ of (5.6). This shows $\hat{X}(\cdot) \equiv X^{V(0),\hat{\pi},\hat{c}}(\cdot)$, and hence $h(0) \le V(0) < \infty$. $\diamond$

Definition 5.1 We say that claim B is $K-$*hedgeable* if its minimal cost of superreplication is finite, $V(0) < \infty$; we say it is $K-$*attainable* if there exists a portfolio process π with values in K such that $(\pi, 0) \in \mathcal{A}'(V(0))$ and $X^{V(0),\pi,0}(T) = B$, a.s.

Theorem 5.2 *For a given K-hedgeable contingent claim B, and any given $\lambda \in \mathcal{D}$, the conditions*

$$\{Q_\lambda(t) = V(t)e^{-\int_0^t \delta(\lambda(u))du}, \mathcal{F}_t;\ 0 \le t \le T\} \quad \textit{is a } \mathbf{P}^\lambda\textit{-martingale} \qquad (5.15)$$

$$\lambda \ \ \textit{achieves the supremum in} \ \ V(0) = \sup_{\nu \epsilon \mathcal{D}} E^\nu[B\gamma_\nu(T)] \qquad (5.16)$$

$$\left\{\begin{array}{c} B \textit{ is } K\textit{-attainable (by a portfolio } \pi\textit{), and the} \\ \textit{corresponding } \gamma_\lambda(\cdot)X^{V(0),\pi,0}(\cdot) \textit{ is a } \mathbf{P}^\lambda\textit{-martingale}\end{array}\right\} \qquad (5.17)$$

are equivalent, and imply

$$\hat{c}(t,\omega) = 0,\ \ \delta(\lambda(t,\omega)) + \lambda'(t,\omega)\hat{\pi}(t,\omega) = 0; \quad \ell \otimes P - a.e. \qquad (5.18)$$

for the pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}'(V(0))$ of Theorem 5.1.

Proof: The $\mathbf{P}^\lambda$-supermartingale $Q_\lambda(\cdot)$ is a $\mathbf{P}^\lambda$-martingale, if and only if $Q_\lambda(0) = E^\lambda Q_\lambda(T) \Leftrightarrow V(0) = E^\lambda[B\gamma_\lambda(T)] \Leftrightarrow$ (5.16).

On the other hand, (5.15) implies $A_\lambda(\cdot) \equiv 0$, and so from (5.10): $\hat{c}(t) = -\int_0^t \hat{X}(s)[\delta(\lambda(s)) + \lambda'(s)\hat{\pi}(s)]ds$. Now (5.18) follows from the increase of $\hat{c}(\cdot)$ and the nonnegativity of $\delta(\lambda) + \lambda'\hat{\pi}$, since $\hat{\pi}$ takes values in K.

From (5.16) (and its consequences (5.15), (5.18)), the process $\hat{X}(\cdot)$ of (5.6) and (5.13) coincides with $X^{V(0),\hat{\pi},0}(\cdot)$, and we have: $\hat{X}(T) = B$ almost surely, $\gamma_\lambda(\cdot)\hat{X}(\cdot)$ is a $\mathbf{P}^\lambda$-martingale; thus (5.17) is satisfied with $\pi \equiv \hat{\pi}$. On the other hand, suppose that (5.17) holds; then $V(0) = E^\lambda[B\gamma_\lambda(T)]$, so (5.16) holds.

Theorem 5.3 *Let B be a K-hedgeable contingent claim. Suppose that, for any $\nu \in \mathcal{D}$ with $\delta(\nu) + \nu'\hat{\pi} \equiv 0$,*

$$Q_\nu(\cdot) \text{ in (5.3) is of class } DL[0,T], \quad \text{under } \mathbf{P}^\nu. \tag{5.19}$$

Then, for any given $\lambda \in \mathcal{D}$, the conditions (5.15), (5.16), (5.18) *are equivalent, and imply*

$$\left\{ \begin{array}{c} B \text{ is } K\text{-attainable (by a portfolio } \pi\text{), and the} \\ \text{corresponding } \gamma_0(\cdot)X^{V(0),\pi,0}(\cdot) \text{ is a } \mathbf{P}^0\text{-martingale} \end{array} \right\}. \tag{5.20}$$

Proof: We have already shown the implications (5.15) $\Leftrightarrow$ (5.16) $\Rightarrow$ (5.18). To prove that these three conditions are actually *equivalent* under (5.19), suppose that (5.18) holds; then from (5.10): $A_\lambda(\cdot) \equiv 0$, whence the $\mathbf{P}^\lambda$-local martingale $Q_\lambda(\cdot)$ is actually a $\mathbf{P}^\lambda$-martingale (from (5.5) and the assumption (5.19)); thus (5.15) is satisfied.

Clearly then, if (5.15), (5.16), (5.18) are satisfied for some $\lambda \in \mathcal{D}$, they are satisfied for $\lambda \equiv 0$ as well; and from Theorem (5.2), we know then that (5.20) (i.e., (5.17) with $\lambda \equiv 0$) holds.

Remark 5.1 (i) Loosely speaking, Theorems 5.2, 5.3 say that *the supremum in* (5.16) *is attained if and only if it is attained by $\lambda \equiv 0$, if and only if the Black-Scholes (unconstrained) portfolio happens to satisfy constraints.*

(ii) It can be shown that the conditions $V(0) < \infty$ and (5.19) are satisfied (the latter, in fact, for every $\nu \in \mathcal{D}$) in the case of the simple European call option $B = (S_1(T) - k)^+$, provided

$$\text{the function} \quad x \mapsto \delta(x) + x_1 \quad \text{is bounded from below on } \tilde{K}. \tag{5.21}$$

The same is true for any contingent claim B that satisfies $B \leq \alpha S_1(T)$ a.s., for some $\alpha \in (0, \infty)$. Note that the condition (5.21) is indeed satisfied, if the convex set K contains both the origin and the point $(1, 0, \ldots, 0)$ (and thus also the line-segment adjoining these points); for then $x_1 + \delta(x) \geq x_1 + \sup_{0 \leq \alpha \leq 1}(-\alpha x_1) = x_1^+ \geq 0,\ \forall x \in \tilde{K}$.

We would like now to have a method for calculating the price $h(0)$. In order to do that, we assume constant market coefficients r, b, σ and consider only the claims of the form $B = b(S(T))$, for a given, lower-semicontinuous function b. Similarly as in the no-constraints case, the minimal hedging process will be given as $X(t) = V(t, S(t))$, for some function $V(t,s)$, depending on the constraints. Introduce also, for a given process $\nu(\cdot)$ in $\mathbb{R}^d$, the auxiliary, shadow economy vector of stock prices $S^\nu(\cdot)$ by

$$dS_i^\nu(t) = S_i^\nu(t)\left[rdt + \sum_{j=1}^d \sigma_{ij} dW_\nu^{(j)}(t)\right]$$

and notice that its distribution under measure P_ν is the same as the one of $S(\cdot)$ under P_0. From Theorem 5.1 we know that

$$V(t,s) = \sup_{\nu \in \mathcal{D}} E^\nu \left[b(S(T)) e^{-\int_t^T (r + \delta(\nu(s)))ds} \,\middle|\, S(t) = s \right]. \tag{5.22}$$

We will show that this complex looking stochastic control problem has a simple solution. First, we modify the value of the claim by considering the following function:

$$\hat{b}(s) = \sup_{\nu \in \tilde{K}} b(se^{-\nu}) e^{-\delta(\nu)}.$$

Here, $se^{-\nu} = (s_1 e^{-\nu_1}, \ldots, s_d e^{-\nu_d})'$, and we use the same notation for the componentwise product of two vectors throughout.

Theorem 5.4 *The minimal K-hedging price function $V(t,s)$ of the claim $b(S(T))$ is the Black-Scholes cost function for replicating $\hat{b}(S(T))$. In particular, under technical assumptions, it is the solution to the PDE*

$$V_t + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}s_is_jV_{s_is_j} + r\left(\sum_{i=1}^{d} s_iV_{s_i} - V\right) = 0, \tag{5.23}$$

with the terminal condition

$$V(T,s) = \hat{b}(s), \quad s \in \mathbb{R}_+^d, \tag{5.24}$$

and the corresponding hedging strategy π satisfies the constraints. Under technical assumptions, it is given by

$$\pi_i(t) = s_i(t)V_{s_i}(t,s(t))/V(t,s(t)), \quad i = 1,\ldots,d. \tag{5.25}$$

Proof: (a) We first show that hedging $b(S(T))$ under constraints is no more expensive than hedging $\hat{b}(S(T))$ without constraints. Let $\nu \in \mathcal{D}$ and observe that, from the properties of the support function and the cone property of $\tilde{K}$,

$$(i)\ \ \hat{\hat{b}} = \hat{b}$$

$$(ii) \int_t^T \delta(\nu_s)ds \geq \delta(\int_t^T \nu_s ds),$$

$$(iii) \int_t^T \nu_s ds \ \text{ is an element of } \ \tilde{K},$$

where $\int_t^T \nu(s)ds := (\int_t^T \nu_1(s)ds, ..., \int_t^T \nu_d(s)ds)'$. Moreover, we have

$$(iv)\ S_i^\nu(t) = S_i(t)e^{\int_0^t \nu_i(s)ds},$$

because the processes on the left-hand side and the right-hand side satisfy the same linear SDE. Then, for every $\nu \in \mathcal{D}$ we have

$$\begin{aligned} E^\nu[\hat{b}(S(T))e^{-\int_0^T(r+\delta(\nu(s)))ds}] &\leq E^\nu[\hat{b}(S^\nu(T)e^{-\int_0^T \nu(s)ds})e^{-\delta(\int_0^T \nu(s)ds)}e^{-rT}] \\ &\leq E^\nu[\sup_{\nu\in\tilde{K}} \hat{b}(S^\nu(T)e^{-\nu})e^{-\delta(\nu)}e^{-rT}] \qquad (5.26) \\ &= E^\nu[\hat{\hat{b}}(S^\nu(T))e^{-rT}] = E^0[\hat{b}(S(T))e^{-rT}]. \end{aligned}$$

Similarly for conditional expectations of (5.22), hence $V(t,s)$ is no larger than the Black-Scholes price process of the claim $\hat{b}(S(T))$.

(b) To conclude we have to show that to superreplicate $b(S(T))$ we have to hedge at least $\hat{b}(S(T))$. It is sufficient to prove that the left limit of $V(t,s)$ at $t=T$ is larger than $\hat{b}(s)$. For this, let $\{\nu^k\}$ be the maximizing sequence in the cone $\tilde{K}$ attaining $\hat{b}(s)$, i.e., such that $b(se^{-\nu^k})e^{-\delta(\nu^k)}$ converges to $\hat{b}(s)$ as k goes to infinity. Then, using (for fixed $t<T$) constant deterministic controls $\nu_k(t)=\nu^k/(T-t)$ in (5.22), we get

$$V(t,s) \geq E^0[b(S(T)e^{-\nu^k})e^{-\delta(\nu^k)}e^{-r(T-t)} \mid S(t)=s],$$

hence

$$\varliminf_{t\to T} V(t,s) \geq b(se^{-\nu^k})e^{-\delta(\nu^k)}$$

and letting k to infinity, we finish the proof. Here is a sketch of a PDE proof for part (a) in the proof above: Let V be the solution to (5.23), (5.24). For a given $\nu\in\tilde{K}$, consider the function $W_\nu=(sV_s)'\nu+\delta(\nu)V$, where V_s is the vector of partial derivatives of V with respect to s_i, $i=1,\ldots,d$. By Theorem 13.1 in Rockafellar (1970), to prove that portfolio π of (5.25) takes values in K, it is sufficient to prove that W_ν is non-negative, for all $\nu\in\tilde{K}$. It is not difficult to see (assuming enough smoothness) that W_ν solves PDE (5.23), too. Moreover, it is also possible to check that $W_\nu(s,T)\geq 0$. So, by the maximum principle, $W_\nu\geq 0$ everywhere.

Example 5.2 We restrict ourselves to the case of only one stock, $d=1$, and the constraints of the type

$$K=[-l,u], \tag{5.27}$$

with $0\leq l,u\leq+\infty$, with the understanding that the interval K is open to the right (left) if $u=+\infty$ (respectively, if $l=+\infty$). It is straightforward to see that

$$\delta(\nu)=l\nu^+ + u\nu^-,$$

and $\tilde{K} = \mathbb{R}$ if both l and u are finite. In general,

$$\tilde{K} = \{x \in \mathbb{R} \ : \ x \geq 0 \text{ if } u = +\infty, \ x \leq 0 \text{ if } l = +\infty\}.$$

For the European call $b(s) = (s-k)^+$, one easily gets that $\hat{b}(s) \equiv \infty$, if $u < 1$, $\hat{b}(s) = s$ if $u = 1$ (no-borrowing) and $\hat{b}(s) = b(s)$ if $u = \infty$ (short-selling constraints don't matter for the call option). For $1 < u < \infty$ we have (by ordinary calculus)

$$\hat{b}(s) = \begin{cases} s-k & ; \ s \geq \frac{ku}{u-1} \\ \frac{k}{u-1}\left(\frac{(u-1)s}{ku}\right)^u & ; \ s < \frac{ku}{u-1} \end{cases}.$$

For the European put $b(s) = (k-s)^+$, one gets $\hat{b} = b$ if $l = \infty$ (borrowing constraints don't matter), $\hat{b} \equiv k$ if $l = 0$ (no short-selling), and otherwise

$$\hat{b}(s) = \begin{cases} k-s & ; \ s \leq \frac{kl}{l+1} \\ \frac{k}{l+1}\left(\frac{ku}{(l+1)s}\right)^l & ; \ s > \frac{kl}{l+1} \end{cases}.$$

Numerical results on hedging these (and other) options under the above constraints can be found in Broadie, Cvitanić and Soner (1998).

6 The Case of Concave Drift

In this section we consider the case of an agent whose drift is a concave function of his trading strategy. The most prominent example is the case in which the borrowing rate R is larger than the lending rate r. Moreover, it also includes examples of a "large investor" who can influence the drift of the asset prices by trading in the market (see Cuoco and Cvitanić 1998).

We assume that the wealth process $X(t)$ satisfies the stochastic differential equation

$$dX(t) = X(t)g(t,\pi_t)dt + X(t)\pi'(t)\sigma(t)dW(t) - dc(t)\ , \quad X(0) = x > 0, \ (6.1)$$

where function $g(t,\cdot)$ is concave for all $t \in [0,T]$, and uniformly (with respect to t) Lipschitz:

$$|g(t,x) - g(t,y)| \le k\|x-y\|, \quad \forall\, t \in [0,T];\ x,y \in \mathbb{R}^d,$$

for some $0 < k < \infty$. Moreover, we assume $g(\cdot,0) \equiv 0$.

In analogy with the case of constraints we define the convex conjugate function $\tilde{g}$ of g by

$$\tilde{g}(t,\nu) := \sup_{\pi \in \mathbb{R}^d} \{g(t,\pi) + \pi'\nu\}, \tag{6.2}$$

on its *effective domain* $\mathcal{D}_t := \{\nu : \tilde{g}(\nu,t) < \infty\}$. Introduce also the class $\mathcal{D}$ of processes $\nu(t)$ taking values in $\mathcal{D}_t$, for all t. It is clear that under above assumptions $\mathcal{D}$ is not empty. We also assume, for simplicity, that function $\tilde{g}(t,\cdot)$ is bounded on its effective domain, uniformly in t.

For a given $\{\mathcal{F}_t\}$–progressively measurable process $\nu(\cdot)$ with values in $\mathbb{R}^d$ we introduce

$$\gamma_\nu(t,u) := \exp\{-\int_t^u \tilde{g}(s,\nu_s)ds\}, \quad \gamma_\nu(t) := \gamma_\nu(0,t),$$

$$dZ_\nu(t) := -\sigma^{-1}(t)\nu(t)Z_\nu(t)dW(t),\ Z_\nu(0) = 1, \quad H_\nu(t) := Z_\nu(t)\gamma_\nu(t)\ . \tag{6.3}$$

For every $\nu \in \mathcal{D}$ we have (by Ito's rule)

$$\begin{aligned} H_\nu(t)X(t) \quad &+ \quad \int_0^t H_\nu(s)\left[X(s)(\tilde{g}(s,\nu_s) - g(s,\pi_s) - \pi'(s)\nu(s))ds + dc(s)\right] \\ &= \quad x + \int_0^t H_\nu(s)X(s)\left[\pi'(s)\sigma(s) + \sigma^{-1}(s)\nu(s)\right] dW(s). \end{aligned} \tag{6.4}$$

In particular, the process on the right-hand side is a nonnegative local martingale, hence a supermartingale. Therefore we get the following *necessary condition for π to be admissible*:

$$\sup_{\nu \in \mathcal{D}} E\left[H_\nu(T)X(T) + \int_0^T H_\nu(s)X(s)\{\tilde{g}(s,\nu_s) - g(s,\pi_s) - \pi'(s)\nu(s)\}ds\right] \le x\ . \tag{6.5}$$

The supermartingale property excludes arbitrage opportunities from this market: if $x = 0$, then necessarily $X(t) = 0$, $\forall\ 0 \le t \le T$, almost surely.

Next, for a given $\nu \in \mathcal{D}$, introduce the process

$$W_\nu(t) := W(t) - \int_0^t \sigma^{-1}(s)\nu(s)ds \ ,$$

as well as the measure

$$\mathbf{P}_\nu(A) := E[Z_\nu(T)1_A] = E^\nu[1_A], \quad A \in \mathcal{F}_T \ .$$

It can be shown under our assumptions that the sets $\mathcal{D}_t$ are uniformly bounded. Therefore, if $\nu \in \mathcal{D}$, then $Z_\nu(\cdot)$ *is a martingale.* Thus, for every $\nu \in \mathcal{D}$, the measure $\mathbf{P}_\nu$ is a probability measure and the process $W_\nu(\cdot)$ is a $\mathbf{P}_\nu-$Brownian motion, by Girsanov theorem.

Given a contingent claim B, consider, for every stopping time τ, the $\mathcal{F}_\tau$-measurable random variable

$$V(\tau) := ess \sup_{\nu \in \mathcal{D}} E^\nu[B\gamma_\nu(\tau, T)|\mathcal{F}_\tau].$$

The proof of the following theorem is similar to the corresponding theorem in the case of constraints.

Theorem 6.1 *For an arbitrary contingent claim B, we have $h(0) = V(0)$. Furthermore, there exists a pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}_0(V(0))$ such that $X^{V(0),\hat{\pi},\hat{c}}(\cdot) = V(\cdot)$.*

The theorem gives the minimal hedging price for a claim B; in fact, it is easy to see (using the same supermartingale argument as before) that the process $V(\cdot)$ is the minimal wealth process that hedges B. There remains the question of whether consumption is necessary. We show that, in fact, $\hat{c}(\cdot) \equiv 0$.

Theorem 6.2 *Every contingent claim B is attainable, namely the process $\hat{c}(\cdot)$ from Theorem 6.1 is a zero-process.*

Proof : Let $\{\nu_n; n \in \mathbb{N}\}$ be a maximizing sequence for achieving $V(0)$, i.e., $\lim_{n\to\infty} E^{\nu_n}[B\gamma_{\nu_n}(T)] = V(0)$. Similarly to (6.5), one can get

$$\sup_{\nu\in\mathcal{D}} E^{\nu}\left[\gamma_\nu(T)V(T) + \int_0^T \gamma_\nu(t)d\hat{c}(t)\right] \leq V(0).$$

Since $V(T) = B$, this implies $\lim_{n\to\infty} E^{\nu_n}\int_0^T \gamma_{\nu_n}(t)d\hat{c}(t) = 0$ and, since the processes $\gamma_{\nu_n}(\cdot)$ are bounded away from zero (uniformly in n), $\lim_{n\to\infty} E[Z_{\nu_n}(T)\hat{c}(T)] = 0$. Using weak compactness arguments as in Cvitanić & Karatzas (1993, Theorem 9.1) we can show that there exists $\nu \in \mathcal{D}$ such that $\lim_{n\to\infty} E[Z_{\nu_n}\hat{c}(T)] = E[Z_\nu(T)\hat{c}(T)] = 0$ (along a subsequence). It follows that $\hat{c}(\cdot) \equiv 0$. $\diamond$

The theorems above also follow from the general theory of Backward Stochastic Differential Equations, as presented in El Karoui, Peng and Quenez (1997).

Example 6.3 *Different borrowing and lending rates.* We have studied so far a model in which one is allowed to borrow money, at an interest rate $R(\cdot)$ equal to the bank rate $r(\cdot)$. In this section we consider the more general case of a financial market $\mathcal{M}^*$ in which $R(\cdot) \geq r(\cdot)$, without constraints on portfolio choice. We assume that the progressively measurable process $R(\cdot)$ is also bounded.

In this market $\mathcal{M}^*$ it is not reasonable to borrow money and to invest money in the bank at the same time. Therefore, we restrict ourselves to policies for which the relative amount borrowed at time t is equal to $\Big(1 - \sum_{i=1}^d \pi_i(t)\Big)^-$. Then, the wealth process $X = X^{x,\pi,c}$ corresponding to initial capital $x > 0$ and portfolio/consumption pair (π, c), satisfies

$$\begin{aligned} dX(t) &= r(t)X(t)dt - dc(t) \\ &\quad + X(t)\left[\pi'(t)\sigma(t)dW_0(t) - (R(t) - r(t))\Big(1 - \sum_{i=1}^d \pi_i(t)\Big)^- dt\right]. \end{aligned}$$

We get $\tilde{g}(\nu(t)) = r(t) - \nu_1(t)$ for $\nu \in \mathcal{D}$, where

$\mathcal{D} := \{\quad \nu;\ \nu$ progressively measurable, $\mathbb{R}^d -$ valued process with

$$r - R \leq \nu_1 = \ldots = \nu_d \leq 0, \quad \ell \otimes \mathbf{P} - a.e.\}$$

We also have

$$\begin{aligned} \tilde{g}(\nu(t)) - g(t,\pi(t)) - \pi'(t)\nu(t) = \quad & [R(t) - r(t) + \nu_1(t)]\Big(1 - \textstyle\sum_{i=1}^d \pi_i(t)\Big)^- \\ & -\nu_1(t)\Big(1 - \textstyle\sum_{i=1}^d \pi_i(t)\Big)^+, \end{aligned}$$

for $0 \leq t \leq T$. It can be shown, in analogy to the case of constraints, that the optimal dual process $\hat{\lambda}(\cdot) \in \mathcal{D}$ can be taken as the one that attains zero in this equation, namely as

$$\hat{\lambda}(t) = \hat{\lambda}_1(t)\mathbf{1}, \quad \hat{\lambda}_1(t) := [r(t) - R(t)]\; 1_{\{\sum_{i=1}^d \hat{\pi}_i(t) > 1\}}. \qquad \diamond$$

Assume now constant coefficients, and observe that the stock price processes vector satisfies the equations

$$\begin{aligned} dS_i(t) &= S_i(t)[b_i(t)dt + \sum_{i=1}^d \sigma_{ij} dW^j(t)] \\ &= S_i(t)[(r - \nu_1(t))dt + \sum_{i=1}^d \sigma_{ij} dW_\nu^j(t)], \quad 1 \leq i \leq d, \end{aligned}$$

for every $\nu \in \mathcal{D}$. Consider now a contingent claim of the form $B = \varphi(S(T))$, for a given continuous function $\varphi : \mathbb{R}_+^d \to [0, \infty)$ that satisfies a polynomial growth condition, as well as the *value function*

$$Q(t,s) := \sup_{\nu \in \mathcal{D}} E^\nu[\varphi(S(T))e^{-\int_t^T (r - \nu_1(s))ds} | S(t) = s]$$

on $[0,T] \times \mathbb{R}_+^d$. Clearly, the processes $\hat{X}, V$ are given as

$$\hat{X}(t) = Q(t, S(t)), \quad V(t) = e^{-rt}\hat{X}(t)\ ; \quad 0 \leq t \leq T,$$

where Q solves the semilinear parabolic partial differential equation of Hamilton-Jacobi-Bellman (HJB) type

$$\frac{\partial Q}{\partial t} + \frac{1}{2}\sum_i \sum_j a_{ij} s_i s_j \frac{\partial^2 Q}{\partial s_i \partial s_j} + \max_{r - R \leq \nu_1 \leq 0} \Big[(r - \nu_1)\{\sum_i s_i \frac{\partial Q}{\partial s_i} - Q\} \Big] = 0,$$

for $0 \leq t < T,\ s \in \mathbb{R}^d_+$,

$$Q(T,s) = \varphi(s);\ s \in \mathbb{R}^d_+$$

(see Ladyženskaja, Solonnikov & Ural'tseva 1968 for the basic theory of such equations, and Fleming & Rishel 1975, Fleming and Soner 1993 for the connections with stochastic control). The maximization in the HJB equation is achieved by $\nu_1^* = (r-R)1_{\{\sum_i s_i \frac{\partial Q}{\partial s_i} \geq Q\}}$; the portfolio $\hat{\pi}(\cdot)$ and the process $\hat{\lambda}_1(\cdot)$ are then given, respectively, by

$$\hat{\pi}_i(t) = \frac{S_i(t) \cdot \frac{\partial}{\partial p_i} Q(t, S(t))}{Q(t, S(t))}, \qquad i = 1, \ldots, d$$

and

$$\hat{\lambda}_1(t) = (r-R)1_{\{\sum_i \hat{\pi}_i(t) \geq 1\}}.$$

The HJB PDE becomes

$$\frac{\partial Q}{\partial t} + \frac{1}{2}\sum_i \sum_j s_i s_j a_{ij} \frac{\partial^2 Q}{\partial s_i \partial s_j} + R\left(\sum_i s_i \frac{\partial Q}{\partial s_i} - Q\right)^+ - r\left(\sum_i s_i \frac{\partial Q}{\partial s_i} - Q\right)^- = 0.$$

Suppose now that the function φ satisfies $\sum_i s_i \frac{\partial \varphi(s)}{\partial s_i} \geq \varphi(s),\ \forall\ s \in \mathbb{R}^d_+$. Then the solution Q also satisfies this inequality:

$$\sum_i s_i \frac{\partial Q(t,s)}{\partial s_i} \geq Q(t,s), \qquad 0 \leq t \leq T$$

for all $s \in \mathbb{R}^d_+$ and is given as the solution to the Black-Scholes equation with r replaced with R

$$\begin{aligned} \frac{\partial Q}{\partial t} + \frac{1}{2}\sum_i \sum_j s_i s_j a_{ij} \frac{\partial^2 Q}{\partial s_i \partial s_j} + R\left(\sum_i s_i \frac{\partial Q}{\partial s_i} - Q\right) = 0\ ; t < T \quad , s > 0 \\ Q(T,s) = \varphi(s) \quad ; s > 0 \end{aligned}$$

In this case the seller's hedging portfolio $\hat{\pi}(\cdot)$ always borrows: $\sum_{i=1}^d \hat{\pi}_i(t) \geq 1,\ 0 \leq t \leq T$, and it was to be expected that all he has to do is use R as the interest rate. Note, however, that this price may be too high for the buyer of the option.

7 Utility functions

A function $U : (0, \infty) \to \mathbb{R}$ will be called a *utility function* if it is strictly increasing, strictly concave, of class C^1, and satisfies

$$U'(0+) := \lim_{x \downarrow 0} U'(x) = \infty \ , \quad U'(\infty) \ := \ \lim_{x \to \infty} U'(x) = 0 \ .$$

We shall denote by I the (continuous, strictly decreasing) inverse of the function U'; this function maps $(0, \infty)$ onto itself, and satisfies $I(0+) = \infty, I(\infty) = 0$. We also introduce the Legendre-Fenchel transform

$$\tilde{U}(y) := \max_{x>0} [U(x) - xy] = U(I(y)) - yI(y), \quad 0 < y < \infty$$

of $-U(-x)$; this function $\tilde{U}$ is strictly decreasing and strictly convex, and satisfies

$$\tilde{U}'(y) = -I(y), \quad 0 < y < \infty,$$

$$U(x) = \min_{y>0} [\tilde{U}(y) + xy] = \tilde{U}(U'(x)) + xU'(x) \ , \quad 0 < x < \infty \ .$$

It is now readily checked that

$$U(I(y)) \geq U(x) + y[I(y) - x]$$

$$\tilde{U}(U'(x)) + x[U'(x) - y] \leq \tilde{U}(y) \ ,$$

are valid for all $x > 0, y > 0$. It is also easy to see that

$$\tilde{U}(\infty) = U(0+), \quad \tilde{U}(0+) = U(\infty)$$

hold; see Karatzas et al. (1991), Lemma 4.2.

For some of the results that follow, we will need to impose the following conditions on our utility functions:

$$c \mapsto cU'(c) \ \text{ is nondecreasing on } \ (0, \infty) \ ; \tag{7.1}$$

for some $\alpha \in (0,1), \gamma \in (1,\infty)$ we have : $\alpha U'(x) \geq U'(\gamma x), \quad \forall\, x \in (0,\infty)$. (7.2)

Condition (7.1) is equivalent to

$$y \mapsto yI(y) \text{ is nonincreasing on } (0,\infty) ,$$

and implies that

$$x \mapsto \tilde{U}(e^x) \text{ is convex on } \mathbb{R} .$$

(If U is of class C^2, then condition (7.1) amounts to the statement that $\frac{-cU''(c)}{U'(c)}$, the so-called "Arrow-Pratt measure of relative risk - aversion", does not exceed 1. For the general treatment under the weakest possible conditions on the utility function see Kramkov and Schachermayer 1998.)

Similarly, condition (7.2) is equivalent to having

$$I(\alpha y) \leq \gamma I(y), \quad \forall\, y \in (0,\infty) \quad \text{for some } \alpha \in (0,1), \quad \gamma > 1 .$$

Iterating this, we obtain the apparently stronger statement

$$\forall\, \alpha \in (0,1),\ \exists\, \gamma \in (1,\infty) \text{ such that } \quad I(\alpha y) \leq \gamma I(y), \quad \forall\, y \in (0,\infty) .$$

8 Portfolio optimization under constraints

In this section we consider the optimization problem of maximizing utility from terminal wealth for an investor subject to the portfolio constraints given by set K , i.e., we want to maximize

$$J(x;\pi) := EU(X^{x,\pi}(T)) ,$$

over the class $\mathcal{A}'_0$ of constrained portfolios π for which $(\pi, 0) \in \mathcal{A}'(x)$ and which satisfy

$$EU^-(X^{x,\pi}(T)) < \infty.$$

The value function of this problem will be denoted by

$$V(x) := \sup_{\pi \in \mathcal{A}'_0(x)} J(x;\pi) , \quad x \in (0,\infty) . \tag{8.1}$$

We assume that $V(x) < \infty, \ \forall\, x \in (0,\infty)$. It is fairly straightforward that the function $V(\cdot)$ is increasing and concave on $(0,\infty)$. and that this assumption is satisfied if the function U is nonnegative and satisfies the growth condition

$$0 \le U(x) \le \kappa(1 + x^{\alpha}) ; \quad \forall\, x \in (0,\infty) \tag{8.2}$$

for some constants $\kappa \in (0,\infty)$ and $\alpha \in (0,1)$ - see Karatzas et al. (1991) for details.

Recall the notation

$$H_\nu(t) = \gamma_\nu(t) Z_\nu(t)$$

of (4.8). We introduce the function

$$\mathcal{X}_\nu(y) := E\Big[H_\nu(T) I(y H_\nu(T))\Big] , \quad 0 < y < \infty,$$

and the class $\mathcal{H}$ of $\tilde{K}$-valued, progressively measurable processes $\nu(\cdot)$ such that $E\int_0^T \|\nu(t)\|^2 dt + E\int_0^T \delta(\nu(t)) dt < \infty$. Consider the subclass $\mathcal{D}'$ of $\mathcal{H}$ given by

$$\mathcal{D}' := \{\nu \in \mathcal{H}; \ \ \mathcal{X}_\nu(y) < \infty, \quad \forall\, y \in (0,\infty)\} .$$

For every $\nu \in \mathcal{D}'$, the function $\mathcal{X}_\nu(\cdot)$ is continuous and strictly decreasing, with $\mathcal{X}_\nu(0+) = \infty$ and $\mathcal{X}_\nu(\infty) = 0$; we denote its inverse by $\mathcal{Y}_\nu(\cdot)$.

Next, we prove a crucial lemma, which provides sufficient conditions for optimality in the problem of (8.1). The duality approach of the lemma and subsequent analysis was implicitly used in Pliska (1986), Karatzas, Lehoczky & Shreve (1987), Cox & Huang (1989) in the case of no constraints, and explicitly in He & Pearson (1991), Karatzas et al. (1991), Xu & Shreve (1992), Cvitanić and Karatzas (1993) for various types of constraints.

Lemma 8.1 *For any given $x > 0$, $y > 0$ and $\pi \in \mathcal{A}'(x)$, we have*

$$EU(X^{x,\pi}(T)) \leq E\tilde{U}(yH_\nu(T)) + yx, \quad \forall\ \nu \in \mathcal{H}. \tag{8.3}$$

In particular, if $\hat{\pi} \in \mathcal{A}'(x)$ is such that equality holds in (8.3), for some $\lambda \in \mathcal{H}$ and $\hat{y} > 0$, then $\hat{\pi}$ is optimal for our (primal) optimization problem, while λ is optimal for the dual problem

$$\tilde{V}(\hat{y}) := \inf_{\nu \in \mathcal{H}} E\tilde{U}(\hat{y}H_\nu(T)) =: \inf_{\nu \in \mathcal{H}} \tilde{J}(\hat{y}; \nu). \tag{8.4}$$

Furthermore, equality holds in (8.3) if

$$X^{x,\pi}(T) = I(yH_\nu(T)) \ a.s., \tag{8.5}$$

$$\delta(\nu_t) = -\nu'(t)\pi(t) \ a.e., \tag{8.6}$$

$$E[H_\nu(T)X^{x,\pi}(T)] = x \tag{8.7}$$

(the latter being equivalent to $\nu \in \mathcal{D}'$ and $y = \mathcal{Y}_\nu(x)$, if (8.5) holds).

Proof: By definitions of $\tilde{U}, \delta$ we get

$$U(X(T)) \leq \tilde{U}(yH_\nu(T)) + yH_\nu(T)X(T) + \int_0^T H_\nu(t)X(t)[\delta(\nu_t) + \nu'(t)\pi(t)]dt.$$

The upper bound of (8.3) follows from Proposition 4.1 (also valid for $\nu(\cdot) \in \mathcal{H}$); condition (8.5) follows from the definition of $\tilde{U}(\cdot)$, conditions (8.6) and (8.7) correspond to $H_\nu(\cdot)X(\cdot)$ being a martingale, not only a supermartingale. $\diamond$

Remark 8.1 Lemma 8.1 suggests the following strategy for solving the optimization problem:

(i) show that the dual problem (8.4) has an optimal solution $\lambda_y \in \mathcal{D}'$ for all $y > 0$;

(ii) using Theorem 5.1, find the minimal hedging price $h_y(0)$ and a corresponding portfolio $\hat{\pi}_y$ for hedging $B_{\lambda_y} := I(yH_{\lambda_y}(T))$;

(iii) prove (8.6) for the pair $(\hat{\pi}_y, \lambda_y)$;

(iv) show that, for every $x > 0$, you can find $\hat{y} = y_x > 0$ such that $x = h_{\hat{y}}(0) = E[H_{\lambda_{\hat{y}}}(T)I(\hat{y}H_{\lambda_{\hat{y}}}(T))]$.

Then (i)-(iv) would imply that $\hat{\pi}_{\hat{y}}$ is the optimal portfolio process for the utility maximization problem of an investor starting with initial capital equal to x.

To verify that step (i) can be accomplished, we impose the following condition:

$$\forall\ y\ \in\ (0,\infty),\ \ \exists\ \nu \in \mathcal{H}\ \text{ such that }\ \tilde{J}(y;\nu) := E\tilde{U}(yH_\nu(T)) < \infty \tag{8.8}$$

We also impose the assumption

$$U(0+) > -\infty\ \ ,\qquad U(\infty) = \infty\ . \tag{8.9}$$

Under the condition (8.2), the requirement (8.8) is satisfied. Indeed, we get

$$0 \le\ \tilde{U}(y) \le \tilde{\kappa}(1 + y^{-\rho})\ ;\quad \forall\ \ y\ \in\ (0,\infty)$$

for some $\tilde{\kappa}\ \in\ (0,\infty)$ and $\rho = \frac{\alpha}{1-\alpha}$.

Even though the log function does not satisfy (8.9), we solve that case directly in examples below.

Theorem 8.1 *Assume that* (7.1), (7.2), (8.8) *and* (8.9) *are satisfied. Then condition (i) of Remark 8.1 is true, i.e. the dual problem admits a solution in the set* $\mathcal{D}'$, *for every* $y > 0$.

The fact that the dual problem admits a solution under the conditions of Theorem 8.1 follows almost immediately (by standard weak compactness arguments) from Proposition 8.1 below. The details, as well as a relatively straightforward proof of Proposition 8.1, can be found in Cvitanić and Karatzas (1992). Denote by $\mathcal{H}'$ the Hilbert space of progressively measurable processes ν with norm $[\![\nu]\!] = E\int_0^T \nu^2(s)ds < \infty$.

Proposition 8.1 *Under the assumptions of Theorem 8.1, the functional* $\tilde{J}(y;\cdot):\mathcal{H}'\to\mathbb{R}\cup\{+\infty\}$ *of* (8.4) *is (i) convex, (ii) coercive:* $\lim_{[\![\nu]\!]\to\infty}\tilde{J}(y;\nu)=\infty$*, and (iii) lower-semicontinuous: for every* $\nu\epsilon\mathcal{H}'$ *and* $\{\nu_n\}_{n\epsilon\mathbf{N}}\subseteq\mathcal{H}'$ *with* $[\![\nu_n-\nu]\!]\to 0$ *as* $n\to\infty$*, we have*

$$\tilde{J}(y;\nu)\le\underline{\lim}_{n\to\infty}\tilde{J}(y;\nu_n)\ .$$

We move now to step (ii) of Remark 8.1. We have the following useful fact:

Lemma 8.2 *For every* $\nu\in\mathcal{H}$*,* $0<y<\infty$*, we have*

$$E[H_\nu(T)B_{\lambda_y}]\le E[H_{\lambda_y}(T)B_{\lambda_y}]\ . \tag{8.10}$$

In fact, (8.10) is equivalent to λ_y being optimal for the dual problem, but we do not need that result here; its proof is quite lengthy and technical (see Cvitanić and Karatzas 1992, Theorem 10.1). We are going to provide a simpler proof for Lemma 8.2, but under the additional assumption that

$$E[H_{\lambda_y}(T)I(yH_\nu(T))]<\infty,\ \forall\nu\in\mathcal{H},y>0. \tag{8.11}$$

Proof of Lemma 8.2: Fix $\varepsilon\in(0,1),\nu\in\mathcal{H}$ and define (supressing dependence on t)

$$G_\varepsilon:=(1-\varepsilon)H_{\lambda_y}+\varepsilon H_\nu,\quad \mu_\varepsilon:=G_\varepsilon^{-1}((1-\varepsilon)H_{\lambda_y}\lambda_y+\varepsilon H_\nu\nu),$$
$$\tilde{\mu}_\varepsilon:=G_\varepsilon^{-1}((1-\varepsilon)H_{\lambda_y}\delta(\lambda_y)+\varepsilon H_\nu\delta(\nu)).$$

Then $\mu_\varepsilon\in\mathcal{H}$, because of the convexity of $\tilde{K}$. Moreover, we have

$$dG_\varepsilon=(\theta+\sigma^{-1}\mu_\varepsilon)G_\varepsilon dW-\tilde{\mu}_\varepsilon G_\varepsilon dt,$$

and convexity of δ implies $\delta(\mu_\varepsilon)\le\tilde{\mu}_\varepsilon$, and therefore, comparing the solutions to the respective (linear) SDE's, we get

$$G_\varepsilon(\cdot)\le H_{\mu_\varepsilon}(\cdot),\ a.s..$$

Since λ_y is optimal and $\tilde{U}$ is decreasing, this implies

$$\varepsilon^{-1}\left(E[\tilde{U}(yH_{\lambda_y}(T)) - \tilde{U}(yG_\varepsilon(T))]\right) \leq 0. \tag{8.12}$$

Next, recall that $I = -\tilde{U}'$ and denote by V_ε the random variable inside the expectation operator in (8.12). Fix $\omega \in \Omega$, and assume, supressing the dependence on ω and T, that $H_\nu \geq H_{\lambda_y}$. Then $\varepsilon^{-1}V_\varepsilon = I(F)y(H_\nu - H_{\lambda_y})$, where $yH_{\lambda_y} \leq F \leq yH_{\lambda_y} + \varepsilon y(H_\nu - H_{\lambda_y})$. Since I is decreasing we get $\varepsilon^{-1}V_\varepsilon \geq yI(yH_\nu)(H_\nu - H_{\lambda_y})$. We get the same result when assuming $H_\nu \leq H_{\lambda_y}$. This and assumption (8.11) imply that we can use Fatou's lemma when taking the limit as $\varepsilon \downarrow 0$ in (8.12), which gives us (8.10). $\diamond$

Now, given $y > 0$ and the optimal λ_y for the dual problem, let π_y be the portfolio of Theorem 5.1 for hedging the claim $B_{\lambda_y} = I(yH_{\lambda_y}(T))$. Lemma 8.2 implies that, in the notation of Section 5,

$$h_y(0) = V_y(0) = E[H_{\lambda_y}(T)I(yH_{\lambda_y}(T))] = \text{ initial capital for portfolio } \pi_y,$$

so (8.7) is satisfied for $x = h_y(0)$. It also implies, by (5.18), that (8.6) holds for the pair (π_y, λ_y). Therefore we have completed both steps (ii) and (iii). Step (iv) is a corollary of the following result.

Proposition 8.2 *Under the assumptions of Theorem 8.1, for any given* $x > 0$*, there exists* $\hat{y} > 0$ *that achieves* $\inf_{y>0}[\tilde{V}(y) + xy]$ *and satisfies*

$$x = \mathcal{X}_{\lambda_{\hat{y}}}(\hat{y})$$

For the (straightforward) proof see Cvitanić and Karatzas (1992, Proposition 12.2). We now put together the results of this section:

Theorem 8.2 *Under the assumptions of Theorem 8.1, for any given* $x > 0$ *there exists an optimal portfolio process* $\hat{\pi}$ *for the utility maximization problem* (8.1)*. Process* $\hat{\pi}$ *is equal to the portfolio of Theorem 5.1 for minimaly hedging the claim* $I(\hat{y}H_{\lambda_{\hat{y}}}(T))$*, where* $\hat{y}$ *is given by Proposition 8.2 and* $\lambda_{\hat{y}}$ *is the optimal process for the dual problem* (8.4)*.*

9 Examples

Example 9.4 *Logarithmic utility.* If $U(x) = \log x$, we have $I(y) = \frac{1}{y}$, $\tilde{U}(y) = -(1 + \log y)$ and

$$\mathcal{X}_\nu(y) = \frac{1}{y}\ , \quad \mathcal{Y}_\nu(x) = \frac{1}{x}$$

and therefore, the optimal terminal wealth is

$$X_\lambda(T) = x\frac{1}{H_\lambda(T)} \tag{9.1}$$

for $\lambda \epsilon \mathcal{H}$ optimal. (In particular $\mathcal{D}' = \mathcal{H}$ in this case). Therefore,

$$E\Big[\tilde{U}(\mathcal{Y}_\lambda(x)H_\nu(T))\Big] = -1 - \log\frac{1}{x} + E\Big(\log\frac{1}{H_\nu(T)}\Big)\ .$$

But

$$E\Big(\log\frac{1}{H_\nu(T)}\Big) = E\int_0^T \Big[r(s) + \delta(\nu(s)) + \frac{1}{2}||\theta(s) + \sigma^{-1}(s)\nu(s)||^2\Big]ds\ ,$$

and thus the dual problem amounts to a point-wise minimization of the convex function $\delta(x) + \frac{1}{2}||\theta(t) + \sigma^{-1}(t)x||^2$ over $x\epsilon\tilde{K}$, for every $t\ \epsilon\ [0,T]$:

$$\lambda(t) = arg\min_{x\epsilon\tilde{K}}\ [\ 2\delta(x) + ||\theta(t) + \sigma^{-1}(t)x||^2\ ]\ .$$

Furthermore, (9.1) gives

$$H_\lambda(t)X_\lambda(t) = x; \quad 0 \le t \le T\ ,$$

and using Ito's rule to get the SDE for $H_\lambda(\cdot)X_\lambda(\cdot)$ we get, by equating the integrand in the stochastic integral term to zero, $\sigma'(t)\hat{\pi}(t) = \theta_\lambda(t)$, $\ell \otimes P$ - a.e.

We conclude that the optimal portfolio is given by

$$\hat{\pi}(t) = (\sigma(t)\sigma'(t))^{-1}[\lambda(t) + b(t) - r(t)\mathbf{1}].$$

Example 9.5 *(Constraints on borrowing)* From the point of view of applications, an interesting example is the one in which the total proportion $\sum_{i=1}^{d} \pi_i(t)$ of wealth invested in stocks is bounded from above by some real constant $a > 0$. For example, if we take $a = 1$, we exclude borrowing; with $a \in (1, 2)$, we allow borrowing up to a fraction $1 - a$ of wealth. If we take $a = 1/2$, we have to invest at least half of the wealth in the bank.

To illustrate what happens in this situation, let again $U(x) = \log x$, and, for the sake of simplicity, $d = 2$, $\sigma =$ unit matrix, and the constraints on the portfolio be given by

$$K = \{x \epsilon \mathbb{R}^2;\ x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 \leq a\}$$

for some $a \epsilon (0, 1]$ (obviously, we also exclude short-selling with this K). We have here $\delta(x) \equiv a \max\{x_1^-, x_2^-\}$, and thus $\tilde{K} = \mathbb{R}^2$. By some elementary calculus and/or by inspection, and omitting the dependence on t, we can see that the optimal dual process λ that minimizes $\frac{1}{2}||\theta_t + \nu_t||^2 + \delta(\nu_t)$, and the optimal portfolio $\pi_t = \theta_t + \lambda_t$, are given respectively by

$$\lambda = -\theta\ ;\ \pi = (0,0)' \ \text{ if } \ \theta_1, \theta_2 \leq 0$$

(do not invest in stocks if the interest rate is larger than the stocks return rates),

$$\begin{aligned}
\lambda &= (0, -\theta_2)';\ \ \pi = (\theta_1, 0)' \ \text{ if } \ \theta_1 \geq 0, \theta_2 \leq 0,\ a \geq \theta_1\ , \\
\lambda &= (a - \theta_1, -\theta_2)';\ \ \pi = (a, 0)' \ \text{ if } \ \theta_1 \geq 0, \theta_2 \leq 0,\ a < \theta_1\ , \\
\lambda &= (-\theta_1, 0)';\ \ \pi = (0, \theta_2)' \ \text{ if } \ \theta_1 \leq 0, \theta_2 \geq 0,\ a \geq \theta_2\ , \\
\lambda &= (-\theta_1, a - \theta_2)';\ \ \pi = (0, a)' \ \text{ if } \ \theta_1 \leq 0, \theta_2 \geq 0,\ a < \theta_2\ ,
\end{aligned}$$

(do not invest in the stock whose rate is less than the interest rate, invest $X \min\{a, \theta_i\}$ in the i-th stock whose rate is larger than the interest rate),

$$\lambda = (0,0)';\ \ \pi = \theta \ \text{ if } \ \theta_1, \theta_2 \geq 0,\ \theta_1 + \theta_2 \leq a$$

(invest $\theta_i X$ in the respective stocks - as in the no constraints case - whenever the optimal portfolio of the no constraints case happens to take values in K),

$$\lambda = (a - \theta_1, -\theta_2)';\ \pi = (a, 0)' \ \text{ if } \ \theta_1, \theta_2 \geq 0,\ a \leq \theta_1 - \theta_2\ ,$$

$$\lambda \quad = (-\theta_1, a-\theta_2)'; \ \pi = (0,a)' \ \text{ if } \ \theta_1, \theta_2 \geq 0, \ a \leq \theta_2 - \theta_1$$

(with both $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 > a$ do not invest in the stock whose rate is smaller, invest aX in the other one if the absolute value of the difference of the stocks rates is larger than a),

$$\lambda_1 = \lambda_2 = \frac{a - \theta_1 - \theta_2}{2} \ ; \ \ \pi_1 = \frac{a + \theta_1 - \theta_2}{2} \ , \ \pi_2 = \frac{a + \theta_2 - \theta_1}{2}$$

if $\theta_1, \theta_2 \geq 0, \theta_1 + \theta_2 > a > |\theta_1 - \theta_2|$ (if none of the previous conditions is satisfied, invest the amount $\frac{a}{2}X$ in the stocks, corrected by the difference of their rates).

Let us consider now the case, where the coefficients $r(\cdot), b(\cdot), \sigma(\cdot)$ of the market model are deterministic functions on $[0,T]$, which we shall take for simplicity to be continuous. Then there is a formal HJB (Hamilton-Jacobi-Bellman) equation associated with the dual optimization problem, namely,

$$Q_t + \inf_{x \epsilon \tilde{K}} [\frac{1}{2} y^2 Q_{yy} ||\theta(t) + \sigma^{-1}(t)x||^2 - yQ_y \delta(x)] - yQ_y r(t) = 0 \ , \qquad (9.2)$$

in $[0,T) \times (0,\infty)$;

$$Q(T,y) = \tilde{U}(y) \ ; \quad y \ \epsilon \ (0,\infty) \ .$$

If there exists a classical solution $Q \ \epsilon \ C^{1,2}([0,T) \times (0,\infty))$ of this equation, that satisfies appropriate growth conditions, then standard verification theorems in stochastic control (e.g. Fleming and Soner 1993) lead to the representation

$$\tilde{V}(y) = Q(0,y), \quad 0 < y < \infty$$

for the dual value function.

Example 9.6 *(Cone constraints)* Suppose that $\delta \equiv 0$ on $\tilde{K}$. Then

$$\lambda(t) = arg \min_{x \epsilon \tilde{K}} ||\theta(t) + \sigma^{-1}(t)x||^2$$

is *deterministic*, the same for all $y \epsilon (0,\infty)$, and the equation (9.2) becomes

$$Q_t + \frac{1}{2}||\theta_\lambda(t)||^2 y^2 Q_{yy} - r(t) y Q_y + \tilde{U}_1(t,y) = 0 \ ; \quad \text{in} \quad [0,T) \times (0,\infty) \ .$$

Example 9.7 *(Power utility)* Consider the case $U(x) = \frac{x^\alpha}{\alpha}$, $x \ \epsilon \ (0,\infty)$ for some $\alpha \ \epsilon \ (0,1)$. Then $\tilde{U}(y) = \frac{1}{\rho} y^{-\rho}$, $0 < y < \infty$ with $\rho := \frac{\alpha}{1-\alpha}$. Again, the process $\lambda(\cdot)$ is *deterministic*, namely

$$\lambda(t) = arg \min_{x \epsilon \tilde{K}} \ [\ ||\theta(t) + \sigma^{-1}(t) x||^2 + 2(1-\alpha)\delta(x) \ ] \ ,$$

and is the same for all $y \epsilon (0,\infty)$. In this case one finds

$$\pi_\lambda(t) = \frac{1}{1-\alpha}(\sigma(t)\sigma'(t))^{-1}[b(t) - r(t)\mathbf{1} + \lambda(t)] \ .$$

Example 9.8 *(Different interest rates for borrowing and lending)* We consider the market with different interest rates for borrowing R, and lending r, $R(\cdot) \geq r(\cdot)$. The methodology of the previous section can still be used in the context of the models introduced in Section 6, of which the different interest rates case is just one example. We are looking for an optimal process $\lambda_y \in \mathcal{H}$ for the corresponding dual problem, in which function $\delta(\cdot)$ is replaced by function $\tilde{g}(\cdot)$ (see Cvitanić 1997 for details), and, for any given $x \in (0,\infty)$, for an optimal portfolio $\hat{\pi}$ for the original primal control problem. In the case of logaritmic utility $U(x) = \log x$, we see that $\lambda(t) = \lambda_1(t)\mathbf{1}$, where

$$\lambda_1(t) = arg \min_{r(t)-R(t) \leq x \leq 0} (-2x + ||\theta(t) + \sigma^{-1}(t)\mathbf{1}x||^2) \ .$$

With $A(t) := tr[(\sigma^{-1}(t))'(\sigma^{-1}(t))]$, $B(t) := \theta'(t)\sigma^{-1}(t)\mathbf{1}$, this minimization is achieved as follows:

$$\lambda_1(t) = \left\{ \begin{array}{ccc} \frac{1-B(t)}{A(t)} & ; \ if & 0 < B(t) - 1 < A(t)(R(t) - r(t)) \\ 0 & ; \ if & B(t) \leq 1 \\ r(t) - R(t) & ; \ if & B(t) - 1 \geq A(t)(R(t) - r(t)) \end{array} \right\} \ .$$

The optimal portfolio is then computed as

$$\hat{\pi}_t = \begin{cases} (\sigma_t\sigma_t')^{-1}[b_t - (r_t + \frac{B_t - 1}{A_t})\mathbf{1}] & ; \quad 0 < B_t - 1 \le A_t(R_t - r_t) \\ (\sigma_t\sigma_t')^{-1}[b_t - r_t\mathbf{1}] & ; \quad B_t \le 1 \\ (\sigma_t\sigma_t')^{-1}[b_t - R_t\mathbf{1}] & ; \quad B_t - 1 \ge A_t(R_t - r_t) \end{cases}$$

In the case $U(x) = \frac{x^\alpha}{\alpha}$, for some $\alpha \in (0,1)$, we get $\lambda(t) = \lambda_1(t)\mathbf{1}$ with

$$\lambda_1(t) = arg \min_{r(t)-R(t)\le x \le 0} \left[-2(1-\alpha)x + ||\theta(t) + \sigma^{-1}(t)\mathbf{1}x||^2\right]$$

$$= \left\{ \begin{array}{lll} \frac{1-\alpha-B(t)}{A(t)} & ; \; if & 0 < B(t) - 1 + \alpha < A(t)(R(t) - r(t)) \\ 0 & ; \; if & B(t) \le 1 - \alpha \\ r(t) - R(t) & ; \; if & B(t) - 1 + \alpha \ge A(t)(R(t) - r(t)). \end{array} \right\}$$

The optimal portfolio is given as

$$\hat{\pi}_t = \begin{cases} \frac{(\sigma_t\sigma_t')^{-1}}{A_t}[b_t - (r_t + \frac{B_t - 1 + \alpha}{A_t})\mathbf{1}] & ; \quad 0 < B_t - 1 + \alpha < A_t(R_t - r_t) \\ \frac{(\sigma_t\sigma_t')^{-1}}{1-\alpha}[b_t - r_t\mathbf{1}] & ; \quad B_t \le 1 - \alpha \\ \frac{(\sigma_t\sigma_t')^{-1}}{1-\alpha}[b_t - R_t\mathbf{1}] & ; \quad B_t - 1 + \alpha \ge A_t(R_t - r_t) \end{cases}$$

10 Utility based pricing

How to choose a price of a contingent claim B in the no-arbitrage pricing interval $[\tilde{h}(0), h(0)]$ in the case of incomplete markets, i.e., when the interval is non-degenerate (consists of more than just the Black-Scholes price)? (Here, $\tilde{h}(0)$ is the maximal price at which the buyer of the option would still be able to hedge away all the risk.) There have been many attempts to provide a satisfactory answer to this question. We describe one suggested by Davis (1997), as presented in Karatzas & Kou (1996), to which we refer for the proofs of the results presented below. The approach is based on the following "zero marginal rate of substitution" principle: Given the agent's utility function U and initial wealth x, the "utility based price" $\hat{p}$ is the one that makes the agent neutral with respect to diversion of a small amount of funds into the contingent claim at time zero, while maximizing the utility from total wealth at the exercise time T. It can be shown that

$$\hat{p} = E[H_{\lambda_x}(T)B], \tag{10.1}$$

where λ_x is the associated optimal dual process. In particular, this price can be calculated in the context of examples of the previous section, and *does not depend on* U and x, in the case of cone constraints ($\delta \equiv 0$) and constant coefficients (Example 9.6). It can also be shown that, in this case, it gives rise to the probability measure P_{λ_x} which minimizes the relative entropy with respect to the original measure P, among all measures P_ν, $\nu \in \mathcal{D}$.

We describe now more precisely what we mean by "utility based price". For a given $-x < \delta < x$ and price p of the claim, we introduce the value function

$$Q(\delta, p, x) := \sup_{\pi \in \mathcal{A}'(x-\delta)} EU(X^{x-\delta}(T) + \frac{\delta}{p}B). \tag{10.2}$$

In other words, the agent acquires δ/p units of the claim B at price p at time zero, and maximizes his/her terminal wealth at time T. Davis (1997) suggests to use price $\hat{p}$ for which

$$\frac{\partial Q}{\partial \delta}(\delta, \hat{p}, x)\Big|_{\delta=0} = 0,$$

so that this diversion of funds has a neutral effect on the expected utility. Since the derivative of Q need not exist, we have the following

Definition 10.1 For a given $x > 0$, we call $\hat{p}$ a *weak solution* of (10.2) if, for every function $\varphi : (-x, x) \mapsto \mathbb{R}$ of class C^1 which satisfies

$$\varphi(\delta) \geq Q(\delta, \hat{p}, x), \forall \delta \in (-x, x), \quad \varphi(0) = Q(0, p, x) = V(x),$$

we have $\varphi'(0) = 0$. If it is unique, then we call it the *utility based price* of B.

Theorem 10.1 *Under the conditions of Theorem 8.2, the utility based price of B is given as in* (10.1).

11 The transaction costs model

In the remaining sections we consider a financial market with proportional transaction costs. More precisely, the market consists of one riskless asset, *bank account* with price $B(\cdot)$ given by

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1$$

and of one risky asset, *stock*, with price-per-share $S(\cdot)$ governed by the stochastic equation

$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)] \ , \quad S(0) = s \in (0, \infty),$$

for $t \in [0,T]$. Here, $W = \{W(t), 0 \le t \le T\}$ is a standard, one-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, endowed with a filtration $\{\mathcal{F}_t\}$, the augmentation of the filtration generated by $W(\cdot)$. The coefficients of the model $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot) > 0$ are assumed to be bounded and $\mathbf{F}-$progressively measurable processes; furthermore, $\sigma(\cdot)$ is also assumed to be bounded away from zero (uniformly in (t, ω)).

Now, a *trading strategy* is a pair (L, M) of $\mathbf{F}-$adapted processes on $[0,T]$, with left-continuous, nondecreasing paths and $L(0) = M(0) = 0$; $L(t)$ (respectively, $M(t)$) represents the total amount of funds transferred from bank-account to stock (respectively, from stock to bank-account) by time t. Given *proportional transaction costs* $0 < \lambda, \mu < 1$ for such transfers, and initial holdings x, y in bank and stock, respectively, the portfolio holdings $X(\cdot) = X^{x,L,M}(\cdot), Y(\cdot) = Y^{y,L,M}(\cdot)$ corresponding to a given trading strategy (L, M), evolve according to the equations:

$$X(t) = x - (1+\lambda)L(t) + (1-\mu)M(t) + \int_0^t X(u)r(u)du, \ 0 \le t \le T \quad (11.1)$$

$$Y(t) = y + L(t) - M(t) + \int_0^t Y(u)[b(u)du + \sigma(u)dW(u)], \ 0 \le t \le T. \quad (11.2)$$

Definition 11.1 A *contingent claim* is a pair (C_0, C_1) of $\mathcal{F}_T-$measurable random variables. We say that a trading strategy (L, M) *hedges the claim* (C_0, C_1) *starting with* (x, y) as initial holdings, if $X(\cdot), Y(\cdot)$ of (11.1), (11.2) satisfy

$$X(T) + (1-\mu)Y(T) \geq C_0 + (1-\mu)C_1 \tag{11.3}$$

$$X(T) + (1+\lambda)Y(T) \geq C_0 + (1+\lambda)C_1. \tag{11.4}$$

Interpretation: Here C_0 (respectively, C_1) is understood as a target-position in the bank-account (resp., the stock) at the terminal time $t = T$: for example

$$C_0 = -k\mathbf{1}_{\{S(T)>k\}}, \ C_1 = S(T)\mathbf{1}_{\{S(T)>k\}}$$

in the case of a European call-option; and

$$C_0 = k\mathbf{1}_{\{S(T)<k\}}, \ C_1 = -S(T)\mathbf{1}_{\{S(T)<k\}}$$

for a European put-option (both with exercise price $k \geq 0$).

"Hedging", in the sense of (11.3) and (11.4), simply means that one is able to cover these positions at $t = T$. Indeed, assume that we have both $Y(T) \geq C_1$ and (11.3), in the form

$$X(T) + (1-\mu)[Y(T) - C_1] \geq C_0 \ ;$$

then (11.4) holds too, and the agent can cover the position in the bank-account as well, by transferring the amount $Y(T) - C_1 \geq 0$ to it. Similarly for the case $Y(T) < C_1$.

The equations (11.1), (11.2) can be written in the equivalent form

$$d\left(\frac{X(t)}{B(t)}\right) = \left(\frac{1}{B(t)}\right)[(1-\mu)dM(t) - (1+\lambda)dL(t)] \ , \quad X(0) = x \tag{11.5}$$

$$d\left(\frac{Y(t)}{S(t)}\right) = \left(\frac{1}{S(t)}\right)[dL(t) - dM(t)] \ , \quad Y(0) = y \tag{11.6}$$

in terms of "number-of-shares" (rather than amounts) held.

12 State-price densities

Consider the class $\mathcal{D}$ of pairs of strictly positive $\mathbf{F}-$martingales $(Z_0(\cdot), Z_1(\cdot))$ with

$$Z_0(0) = 1 \ , \quad z := Z_1(0) \in [s(1-\mu), s(1+\lambda)])$$

and

$$1 - \mu \leq R(t) := \frac{Z_1(t)}{Z_0(t)P(t)} \leq 1 + \lambda, \quad \forall\ 0 \leq t \leq T, \tag{12.1}$$

where

$$P(t) := \frac{S(t)}{B(t)} = s + \int_0^t P(u)[(b(u) - r(u))du + \sigma(u)dW(u)] \ , \quad 0 \leq t \leq T \tag{12.2}$$

is the *discounted stock price.*

The martingales $Z_0(\cdot), Z_1(\cdot)$ are the feasible *state-price densities* for holdings in bank and stock, respectively, in this market with transaction costs; as such, they reflect the "constraints" or "frictions" inherent in this market, in the form of condition (12.1). From the martingale representation theorem there exist $\mathbf{F}-$progressively measurable processes $\theta_0(\cdot), \theta_1(\cdot)$ with $\int_0^T (\theta_0^2(t) + \theta_1^2(t))dt < \infty$ a.s. and

$$Z_i(t) = Z_i(0) \exp\left\{ \int_0^t \theta_i(s)dW(s) - \frac{1}{2}\int_0^t \theta_i^2(s)ds \right\} \ , \quad i = 0, 1; \tag{12.3}$$

thus, the process $R(\cdot)$ of (12.1) has the dynamics

$$\begin{aligned} dR(t) &= R(t)[\sigma^2(t) + r(t) - b(t) - (\theta_1(t) - \theta_0(t))(\sigma(t) + \theta_0(t))]dt \\ &\quad + R(t)(\theta_1(t) - \sigma(t) - \theta_0(t))dW(t), \quad R(0) = z/s. \end{aligned} \tag{12.4}$$

Remark 12.1 A rather "special" pair $(Z_0^*(\cdot), Z_1^*(\cdot)) \in \mathcal{D}$ is obtained, if we take in (12.3) the processes $(\theta_0(\cdot), \theta_1(\cdot))$ to be given as

$$\theta_0^*(t) := \frac{r(t) - b(t)}{\sigma(t)} \ , \quad \theta_1^*(t) := \sigma(t) + \theta_0^*(t) \ , \quad 0 \leq t \leq T, \tag{12.5}$$

and let $Z_0^*(0) = 1,\ s(1-\mu) \le Z_1^*(0) = z \le s(1+\lambda)$. Because then, from (12.4), $R^*(\cdot) := \frac{Z_1^*(\cdot)}{Z_0^*(\cdot)P(\cdot)} \equiv \frac{z}{s}$; in fact, the pair of (12.5) and $z = s$ provide the *only* member $(Z_0^*(\cdot), Z_1^*(\cdot))$ of $\mathcal{D}$, if $\lambda = \mu = 0$. Notice that the processes $\theta_0^*(\cdot)$, $\theta_1^*(\cdot)$ of (12.5) are *bounded.*

Let us observe also that

$$\begin{aligned} Z_0(t)\frac{X(t)}{B(t)} \quad &+ \quad Z_1(t)\frac{Y(t)}{S(t)} + \int_0^t \frac{Z_0(s)}{B(s)}[(1+\lambda) - R(s)]dL(s) \\ &\qquad + \int_0^t \frac{Z_0(s)}{B(s)}[R(s) - (1-\mu)]dM(s) \qquad (12.6) \\ &= \quad x + \frac{yz}{s} + \int_0^t \frac{Z_0(s)}{B(s)}[X(s)\theta_0(s) + R(s)Y(s)\theta_1(s)]dW(s),\ t \in [0,T] \end{aligned}$$

is a $\mathbf{P}-$local martingale, for any $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ and any trading strategy (L, M); this follows directly from (11.5), (11.6), (12.3) and the product rule. Equivalently, (12.6) can be re-written as

$$\begin{aligned} &\frac{X(t) + R(t)Y(t)}{B(t)} + \int_0^t \frac{(1+\lambda) - R(s)}{B(s)}dL(s) + \int_0^t \frac{R(s) - (1-\mu)}{B(s)}dM(s) \\ = \quad & x + \frac{yz}{s} + \int_0^t \frac{R(s)Y(s)}{B(s)}(\theta_1(s) - \theta_0(s))dW_0(s), \qquad (12.7) \end{aligned}$$

where

$$W_0(t) := W(t) - \int_0^t \theta_0(s)ds, \quad 0 \le t \le T \qquad (12.8)$$

is a Brownian motion under the equivalent probability measure

$$\mathbf{P_0}(A) := E[Z_0(T)\mathbf{1}_A],\ A \in \mathcal{F}_T. \qquad (12.9)$$

We shall denote by $Z_0^*(\cdot), W_0^*(\cdot)$ and $\mathbf{P_0^*}$ the processes and probability measure, respectively, corresponding to the process $\theta_0^*(\cdot)$ of (12.5), via the equations (12.3) (with $Z_0^*(0) = 1$), (12.8) and (12.9). With this notation, (12.2) becomes $dP(t) = P(t)\sigma(t)dW_0^*(t)$, $P(0) = s$.

Definition 12.1 Let $\mathcal{D}_\infty$ be the class of positive martingales $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, for which the random variable

$$\frac{Z_0(T)}{Z_0^*(T)}, \qquad \text{and thus also } \frac{Z_1(T)}{Z_0^*(T)P(T)},$$

is essentially bounded.

Definition 12.2 We say that a given trading strategy (L, M) is *admissible* for (x, y), and write $(L, M) \in \mathcal{A}(x, y)$, if

$$\frac{X(\cdot) + R(\cdot)Y(\cdot)}{B(\cdot)} \text{ is a } \mathbf{P_0}-\text{supermartingale}, \ \forall\, (Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty. \quad (12.10)$$

Consider, for example, a trading strategy (L, M) that satisfies the no-bankruptcy conditions

$$X(t) + (1+\lambda)Y(t) \geq 0 \text{ and } X(t) + (1-\mu)Y(t) \geq 0, \quad \forall\, 0 \leq t \leq T.$$

Then $X(\cdot) + R(\cdot)Y(\cdot) \geq 0$ for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ (recall (12.1), and note Remark 12.2 below); this means that the $\mathbf{P_0}-$local martingale of (12.7) is nonnegative, hence a $\mathbf{P_0}-$supermartingale. But the second and the third terms

$$\int_0^\cdot \frac{1+\lambda-R(s)}{B(s)}dL(s), \quad \int_0^\cdot \frac{R(s)-(1-\mu)}{B(s)}dM(s)$$

in (12.7) are increasing processes, thus the first term $\frac{X(\cdot)+R(\cdot)Y(\cdot)}{B(\cdot)}$ is also a $\mathbf{P_0}-$supermartingale, for every pair $(Z_0(\cdot), Z_1(\cdot))$ in $\mathcal{D}$. The condition (12.10) is actually weaker, in that it requires this property only for pairs in $\mathcal{D}_\infty$. This provides a motivation for Definition 12.2, namely, *to allow for as wide a class of trading strategies as possible, and still exclude arbitrage opportunities.* This is usually done by imposing a lower bound on the wealth process; however, that excludes simple strategies of the form "trade only once, by buying a fixed number of shares of the stock at a specified time t", which may require (unbounded) borrowing. We will need to use such strategies in the sequel.

Remark 12.2 Here is a trivial (but useful) observation: if $x + (1-\mu)y \geq a + (1-\mu)b$ and $x + (1+\lambda)y \geq a + (1+\lambda)b$, then $x + ry \geq a + rb$, $\forall\ 1-\mu \leq r \leq 1+\lambda$.

13 The minimal superreplication price

Suppose that we are given an initial holding $y \in \mathbb{R}$ in the stock, and want to hedge a given contingent claim (C_0, C_1) with strategies which are admissible (in the sense of Definitions 11.1, 12.1. What is the smallest amount of holdings in the bank

$$h(C_0, C_1; y) := \inf\{x \in \mathbb{R} /\ \exists (L, M) \in \mathcal{A}(x, y) \text{ and } \ (L, M) \text{ hedges } (C_0, C_1)\} \tag{13.1}$$

that allows to do this? We call $h(C_0, C_1; y)$ the *superreplication price* of the contingent claim (C_0, C_1) for initial holding y in the stock, and with the convention that $h(C_0, C_1; y) = \infty$ if the set in (13.1) is empty.

Suppose this is not the case, and let $x \in \mathbb{R}$ belong to the set of (13.1); then for any $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$ we have from (12.10), the Definition 11.1 of hedging, and Remark 12.2:

$$\begin{aligned} x + \frac{y}{s} E Z_1(T) &= x + \frac{y}{s} z \geq E_0\left[\frac{X(T) + R(T)Y(T)}{B(T)}\right] \\ &\geq E_0\left[\frac{C_0 + R(T)C_1}{B(T)}\right] = E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1)\right], \end{aligned}$$

so that $x \geq E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{s} Z_1(T)\right]$. Therefore

$$h(C_0, C_1; y) \geq \sup_{\mathcal{D}_\infty} E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{s} Z_1(T)\right], \tag{13.2}$$

and this inequality is clearly also valid if $h(C_0, C_1; y) = \infty$.

Lemma 13.1 *If the contingent claim (C_0, C_1) is bounded from below, in the sense*

$$C_0+(1+\lambda)C_1 \geq -K \text{ and } C_0+(1-\mu)C_1 \geq -K, \text{ for some } 0 \leq K < \infty \quad (13.3)$$

then

$$\begin{aligned}\sup_{\mathcal{D}_\infty} E\Big[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) \quad &- \quad \frac{y}{s}Z_1(T)\Big] \\ &= \quad \sup_{\mathcal{D}} E\Big[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{s}Z_1(T)\Big].\end{aligned}$$

Proof: Start with arbitrary $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ and define the sequence of stopping times $\{\tau_n\} \uparrow T$ by

$$\tau_n := \inf\{t \in [0,T] \;/\; \frac{Z_0(t)}{Z_0^*(t)} \geq n\} \wedge T, \;\; n \in \mathbb{N}.$$

Consider also, for $i = 0, 1$ and in the notation of (12.5):

$$\theta_i^{(n)}(t) := \left\{ \begin{array}{ll} \theta_i(t), & 0 \leq t < \tau_n \\ \theta_i^*(t), & \tau_n \leq t \leq T \end{array} \right\}$$

and

$$Z_i^{(n)}(t) = z_i \exp\{\int_0^t \theta_i^{(n)}(s)dW(s) - \frac{1}{2}\int_0^t (\theta_i^{(n)}(s))^2 ds\}$$

with $z_0 = 1, \;\; z_1 = Z_1(0) = EZ_1(T)$. Then, for every $n \in \mathbb{N}$, both $Z_0^{(n)}(\cdot)$ and $Z_1^{(n)}(\cdot)$ are positive martingales, $R^{(n)}(\cdot) = \frac{Z_1^{(n)}(\cdot)}{Z_0^{(n)}(\cdot)P(\cdot)} = R(\cdot \wedge \tau_n)$ takes values in $[1-\mu, 1+\lambda]$ (by (12.1) and Remark 12.1), and $Z_0^{(n)}(\cdot)/Z_0^*(\cdot)$ is bounded by n (in fact, constant on $[\tau_n, T]$). Therefore, $(Z_0^{(n)}(\cdot), Z_1^{(n)}(\cdot)) \in \mathcal{D}_\infty$. Now let κ denote an upper bound on $K/B(T)$, and observe, from Remark 12.2, (13.3) and Fatou's lemma:

$$\begin{aligned} & E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{s}Z_1(T)\right] + \frac{y}{s}Z_1(0) + \kappa \\ = \quad & E\left[Z_0(T)\left\{\frac{C_0 + R(T)C_1}{B(T)} + \kappa\right\}\right] \end{aligned}$$

$$
\begin{aligned}
&= E\left[\lim_n Z_0^{(n)}(T)\left\{\frac{C_0+R^{(n)}(T)C_1}{B(T)}+\kappa\right\}\right]\\
&\le \underline{\lim}_n E\left[Z_0^{(n)}(T)\left\{\frac{C_0+R^{(n)}(T)C_1}{B(T)}+\kappa\right\}\right]\\
&= \underline{\lim}_n E\left[\frac{Z_0^{(n)}(T)}{B(T)}(C_0+R^{(n)}(T)C_1)-\frac{y}{s}Z_1^{(n)}(T)\right]+\frac{y}{s}Z_1(0)+\kappa.
\end{aligned}
$$

This shows that the left-hand-side dominates the right-hand-side in the statement of the lemma; the reverse inequality is obvious. $\diamond$

Remark: Formally taking $y = 0$ in the above, we deduce

$$
E_0\left(\frac{C_0+R(T)C_1}{B(T)}\right) \le \underline{\lim}_{n\to\infty} E_0^{(n)}\left(\frac{C_0+R^{(n)}(T)C_1}{B(T)}\right), \qquad (13.4)
$$

where $E_0, E_0^{(n)}$ denote expectations with respect to the probability measures $\mathbf{P_0}$ of (12.9) and $\mathbf{P_0^{(n)}}(\cdot) = E[Z_0^{(n)}(T)\mathbf{1}.]$, respectively.

Here is the main result of this section.

Theorem 13.1 *Under the conditions* (13.3) *and*

$$
E_0^*(C_0^2 + C_1^2) < \infty \ , \qquad (13.5)
$$

we have

$$
h(C_0, C_1; y) = \sup_{\mathcal{D}} E\left[\frac{Z_0(T)}{B(T)}(C_0+R(T)C_1)-\frac{y}{s}Z_1(T)\right].
$$

In (13.5), E_0^* denotes expectation with respect to the probability measure $\mathbf{P}_0^*$. The conditions (13.3), (13.5) are both easily verified for a European call or put. In fact, one can show that if a pair of admissible terminal holdings $(X(T), Y(T))$ hedges a pair $(\tilde{C}_0, \tilde{C}_1)$ satisfying (13.5) (for example, $(\tilde{C}_0, \tilde{C}_1) \equiv (0,0)$), then necessarily the pair $(X(T), Y(T))$ also satisfies (13.5) – and so does any other pair of random variables (C_0, C_1) which are bounded from below and are hedged by $(X(T), Y(T))$. In particular, any strategy which satisfies the "no-bankruptcy" condition of hedging $(0,0)$, necessarily results in a square-integrable final wealth. In this sense, the condition (13.5) is consistent with the standard "no-bankruptcy" condition, hence not very restrictive (this, however, is not necessarily the case if there are no transaction costs).

Proof: In view of Lemma 13.1 and the inequality (13.2), it suffices to show

$$h(C_0, C_1; y) \leq \sup_{\mathcal{D}} E\left[Z_0(T)\frac{C_0}{B(T)} + Z_1(T)\left(\frac{C_1}{S(T)} - \frac{y}{s}\right)\right] =: R. \qquad (13.6)$$

For simplicity we take $s = 1$, $r(\cdot) \equiv 0$, thus $B(\cdot) \equiv 1$, for the remainder of the section; the reader will verify easily that this entails no loss of generality.

We start by taking an arbitrary $b < h(C_0, C_1; y)$ and considering the sets

$$A_0 := \{(U, V) \in (\mathbf{L}_2^*)^2 : \exists (L, M) \in \mathcal{A}(0,0) \text{ that hedges } (U, V) \text{ starting with } x = 0, y = 0\} \qquad (13.7)$$

$$A_1 := \{(C_0 - b, C_1 - yS(T))\},$$

where $\mathbf{L}_2^* = \mathbf{L}_2(\Omega, \mathcal{F}_T, \mathbf{P}_0^*)$. It is not hard to prove (see below) that

$$A_0 \textit{ is a convex cone, and contains the origin } (0,0), \textit{ in } (\mathbf{L}_2^*)^2, \qquad (13.8)$$

$$A_0 \cap A_1 = \emptyset. \qquad (13.9)$$

It is, however, considerably harder to establish that

$$A_0 \textit{ is closed in } (\mathbf{L}_2^*)^2. \qquad (13.10)$$

The proof can be found in the appendix of Cvitanić & Karatzas (1996). From (13.8)-(13.10) and the Hahn-Banach theorem there exists a pair of random variables $(\rho_0^*, \rho_1^*) \in (\mathbf{L}_2^*)^2$, not equal to $(0,0)$, such that

$$E_0^*[\rho_0^* V_0 + \rho_1^* V_1] = E[\rho_0 V_0 + \rho_1 V_1] \leq 0, \quad \forall\ (V_0, V_1) \in A_0 \qquad (13.11)$$

$$E_0^*[\rho_0^*(C_0 - b) + \rho_1^*(C_1 - yS(T))] = E[\rho_0(C_0 - b) + \rho_1(C_1 - yS(T))] \geq 0, \quad (13.12)$$

where $\rho_i := \rho_i^* Z_0^*(T),\ i=0,1$. It is also not hard to check (see below) that

$$(1-\mu)E[\rho_0|\mathcal{F}_t] \le \frac{E[\rho_1 S(T)|\mathcal{F}_t]}{S(t)} \le (1+\lambda)E[\rho_0|\mathcal{F}_t],\ \forall\ 0\le t\le T \quad (13.13)$$

$$\rho_1 \ge 0,\ \rho_0 \ge 0 \text{ and } E[\rho_0]>0,\ E[\rho_1 S(T)]>0. \quad (13.14)$$

In view of (13.14), we may take $E[\rho_0]=1$, and then (13.12) gives

$$b \le E[\rho_0 C_0 + \rho_1(C_1 - yS(T))]. \quad (13.15)$$

Consider now arbitrary $0<\varepsilon<1,\ (Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, and define

$$\tilde{Z}_0(t) := \varepsilon Z_0(t) + (1-\varepsilon)E[\rho_0|\mathcal{F}_t],\ \tilde{Z}_1(t) := \varepsilon Z_1(t) + (1-\varepsilon)E[\rho_1 S(T)|\mathcal{F}_t],$$

for $0\le t\le T$. Clearly these are positive martingales, and $\tilde{Z}_0(0)=1$; on the other hand, multiplying in (13.13) by $1-\varepsilon$, and in $(1-\mu)Z_0(t)\le Z_1(t)/S(t)\le (1+\lambda)Z_0(t),\ 0\le t\le T$ by ε, and adding up, we obtain $(\tilde{Z}_0(\cdot),\tilde{Z}_1(\cdot))\in\mathcal{D}$. Thus, in the notation of (13.6),

$$\begin{aligned} R &\ge E\left[\tilde{Z}_0(T)C_0 + \tilde{Z}_1(T)\left(\frac{C_1}{S(T)} - y\right)\right] \\ &= (1-\varepsilon)E[\rho_0 C_0 + \rho_1(C_1 - yS(T))] + \varepsilon E\left[Z_0(T)C_0 + Z_1(T)\left(\frac{C_1}{S(T)} - y\right)\right] \\ &\ge b(1-\varepsilon) + \varepsilon E\left[Z_0(T)C_0 + Z_1(T)\left(\frac{C_1}{S(T)} - y\right)\right] \end{aligned}$$

from (13.15); letting $\varepsilon \downarrow 0$ and then $b \uparrow h(C_0, C_1; y)$, we obtain (13.6), as required to complete the proof of Theorem 13.1.

Proof of (13.9)**:** Suppose that $A_0 \cap A_1$ is not empty, i.e., that there exists $(L,M)\in\mathcal{A}(0,0)$ such that, with $X(\cdot)=X^{0,L,M}(\cdot)$ and $Y(\cdot)=Y^{0,L,M}(\cdot)$, the process $X(\cdot)+R(\cdot)Y(\cdot)$ is a $\mathbf{P_0}-$supermartingale for every $(Z_0(\cdot),Z_1(\cdot))\in\mathcal{D}_\infty$, and we have:

$$X(T)+(1-\mu)Y(T) \ge (C_0-b)+(1-\mu)(C_1-yS(T)),$$

$$X(T)+(1+\lambda)Y(T) \ge (C_0-b)+(1+\lambda)(C_1-yS(T)).$$

But then, with

$$\tilde{X}(\cdot) := X^{b,L,M}(\cdot) = b + X(\cdot),\ \tilde{Y}(\cdot) := Y^{y,L,M}(\cdot) = Y(\cdot) + yS(\cdot)$$

we have, from above, that $\tilde{X}(\cdot)+R(\cdot)\tilde{Y}(\cdot) = X(\cdot)+R(\cdot)Y(\cdot)+b+yZ_1(\cdot)/Z_0(\cdot)$ is a $\mathbf{P_0}-$supermartingale for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$, and that

$$\tilde{X}(T) + (1-\mu)\tilde{Y}(T) \geq C_0 + (1-\mu)C_1,$$

$$\tilde{X}(T) + (1+\lambda)\tilde{Y}(T) \geq C_0 + (1+\lambda)C_1.$$

In other words, (L, M) belongs to $\mathcal{A}(b, y)$ and hedges (C_0, C_1) starting with (b, y) – a contradiction to the definition (13.1), and to the fact $h(C_0, C_1; y) > b$.

Proof of (13.13)**,** (13.14)**:** Fix $t \in [0, T)$ and let ξ be an arbitrary bounded, nonnegative, $\mathcal{F}_t-$measurable random variable. Consider the strategy of starting with $(x, y) = (0, 0)$ and buying ξ shares of stock at time $s = t$, otherwise doing nothing ("buy-and-hold strategy"); more explicitly, $M^\xi(\cdot) \equiv 0$, $L^\xi(s) = \xi S(t)\mathbf{1}_{(t,T]}(s)$ and thus

$$\begin{array}{rcl} X^\xi(s) & := & X^{0,L^\xi,M^\xi}(\cdot) = -\xi(1+\lambda)S(t)\mathbf{1}_{(t,T]}(s), \\ Y^\xi(s) & := & Y^{0,L^\xi,M^\xi}(s) = \xi S(s)\mathbf{1}_{(t,T]}(s), \end{array}$$

for $0 \leq s \leq T$. Consequently, $Z_0(s)[X^\xi(s) + R(s)Y^\xi(s)] = \xi[Z_1(s) - (1 + \lambda)S(t)Z_0(s)]\mathbf{1}_{(t,T]}(s)$ is a $\mathbf{P}-$supermartingale for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, since, for instance with $t < s \leq T$:

$$\begin{array}{rcl} E[Z_0(s)(X^\xi_s & + & R_sY^\xi_s)|\mathcal{F}_t] = \xi\left(E[Z_1(s)|\mathcal{F}_t] - (1+\lambda)S_tE[Z_0(s)|\mathcal{F}_t]\right) \\ & = & \xi[Z_1(t) - (1+\lambda)S(t)Z_0(t)] = \xi S(t)Z_0(t)[R(t) - (1+\lambda)] \\ & \leq & 0 = Z_0(t)[X^\xi(t) + R(t)Y^\xi(t)]. \end{array}$$

Therefore, $(L^\xi, M^\xi) \in \mathcal{A}(0, 0)$, thus $(X^\xi(T), Y^\xi(T))$ belongs to the set A_0 of (13.7), and, from (13.11):

$$0 \quad \geq \quad E[\rho_0X^\xi(T) + \rho_1Y^\xi(T)] = E[\xi(\rho_1S(T) - (1+\lambda)\rho_0S(t))]$$

$$= \quad E[\xi(E[\rho_1 S(T)|\mathcal{F}_t] - (1+\lambda)S(t)E[\rho_0|\mathcal{F}_t])].$$

From the arbitrariness of $\xi \geq 0$, we deduce the inequality of the right-hand side in (13.13), and a dual argument gives the inequality of the left-hand side, for given $t \in [0,T)$. Now all three processes in (13.13) have continuous paths; consequently, (13.13) is valid for *all* $t \in [0,T]$.

Next, we notice that (13.13) with $t = T$ implies $(1-\mu)\rho_0 \leq \rho_1 \leq (1+\lambda)\rho_0$, so that ρ_0, hence also ρ_1, is nonnegative. Similarly, (13.13) with $t = 0$ implies $(1-\mu)E[\rho_0] \leq E[\rho_1 S(T)] \leq (1+\lambda)E[\rho_0]$, and therefore, since (ρ_0, ρ_1) is not equal to $(0,0)$, $E[\rho_0] > 0$, hence also $E[\rho_1 S(T)] > 0$. This proves (13.14). $\diamond$

Remark 13.1 For the European call option with $y = 0$, we have

$$h(C_0, C_1; 0) = \sup_{\mathcal{D}} E\left[Z_1(T)\mathbf{1}_{\{S(T)>k\}} - k\frac{Z_0(T)}{B(T)}\mathbf{1}_{\{S(T)>k\}}\right],$$

and therefore, $h(C_0, C_1, 0) \leq \sup_{\mathcal{D}} E[Z_1(T)] = \sup_{\mathcal{D}} Z_1(0) \leq (1+\lambda)s$. The number $(1+\lambda)s$ corresponds to the cost of the "buy-and-hold strategy", of acquiring one share of the stock at $t = 0$, and holding on to it until $t = T$. Davis & Clark (1994) conjectured that this hedging strategy is actually the least expensive superreplication strategy:

$$h(C_0, C_1, 0) = (1+\lambda)s.$$

The conjecture was proved by Soner, Shreve & Cvitanić (1995) by analytic methods. Moreover, the following analogous result has been obtained in more general continuous-time models and for more general contingent claims by Levental & Skorohod (1997) (using probabilistic methods) and Cvitanić, Pham & Touzi (1998) (using Theorem 13.1): "the cheapest buy-and-hold strategy which dominates a given claim in a market with transaction costs is equal to its least expensive superreplication strategy". However, the result is not always true, and, in particular, it does not hold for discrete-time models.

14 Utility maximization under transaction costs

Consider now a small investor who starts with initial capital $(x, 0)$, $x > 0$, and derives utility $U(X(T+))$ from his *terminal wealth*

$$X(T+) := X(T) + f(Y(T)) \geq 0, \text{ where } f(u) := \left\{ \begin{array}{l} (1+\lambda)u \; ; \; u \leq 0 \\ (1-\mu)u \; ; \; u > 0 \end{array} \right\}.$$

In other words, this agent liquidates at time T his position in the stock, incurs the appropriate transaction cost, and collects all the money in the bank-account. Denote by $\mathcal{A}^+(x)$ the set of terminal holdings $(X(T), Y(T))$ that hedge $(0, 0)$, so that, in particular, $X(T+) \geq 0$. The agent's optimization problem is to find an admissible pair $(\hat{L}, \hat{M}) \in \mathcal{A}^+(x)$ that *maximizes expected utility from terminal wealth*, i.e., attains the supremum

$$V(x) := \sup_{\mathcal{A}^+(x)} EU(X(T+)). \tag{14.1}$$

Here, $U : (0, \infty) \to \mathbb{R}$ is a strictly concave, strictly increasing, continuously differentiable utility function which satisfies $U'(0+) = \infty$, $U'(\infty) = 0$ and

Assumption 14.1 The utility function $U(x)$ has asymptotic elasticity strictly less than 1, i.e.

$$AE(U) := \overline{\lim_{x \to \infty}} \frac{xU'(x)}{U(x)} < 1. \tag{14.2}$$

It is shown in Kramkov & Schachermayer (1998) (henceforth [KS98]) that this condition is basically necessary and sufficient to ensure nice properties of value function $V(x)$ and the existence of an optimal solution.

We are again going to consider the dual problem. However, unlike the case of portfolio constraints, we have to go beyond the set of state-price densities for the dual problem, and we introduce the set

$$\mathcal{H} := \left\{ Z \in \mathbf{L}^0_+ \; / \; E\left[\frac{Z}{B(T)} (X(T) + f(Y(T))) \right] \leq x, \; \forall \; (X(T), Y(T)) \in \mathcal{A}^+(x) \right\}. \tag{14.3}$$

(Here, $\mathbf{L}^0$ is the set of all random variables on $(\Omega, \mathcal{F}, \mathbf{P})$.) In particular, if $(Z_0(T), Z_1(T)) \in \mathcal{D}$, then $Z_0(T) \in \mathcal{H}$. For a given $z > 0$, the *auxiliary dual problem* associated with (14.1) is given by

$$\tilde{V}(z) := \inf_{Z \in \mathcal{H}} E\tilde{U}(zZ/B(T)). \tag{14.4}$$

More precisely, similarly as in Cvitanić & Karatzas (1996) (henceforth [CK96]), for every $z > 0$, $Z \in \mathcal{H}$ and $(X(T), Y(T)) \in \mathcal{A}^+(x)$ we have

$$EU(X(T+)) \leq E[\tilde{U}(zZ/B(T)) + X(T+)Z/B(T)] \leq E\tilde{U}(zZ/B(T)) + zx. \tag{14.5}$$

Consequently, we have

$$V(x) \leq \inf_{z>0}[\tilde{V}(z) + zx] =: \inf_{z>0} \gamma(z). \tag{14.6}$$

Remark 14.1 The duality approach used in the market with portfolio constraints suggests that we should look for pairs $(\hat{z}, \hat{Z}) \in (0, \infty) \times \mathcal{H}$ and $(\hat{X}(T+), 0) \in \mathcal{A}^+(x)$ such the inequalities in (14.5) and (14.6) become equalities. The pair $(\hat{X}(T+), 0)$ is then optimal for (14.1). It is easily seen that this is the case (namely that those inequalities become equalities) if and only if

$$(\hat{X}(T+), 0) = I(\hat{z}\hat{Z}/B(T)), 0) \in \mathcal{A}^+(x), \quad E\left[\hat{Z} I\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)\right] = x.$$

We first state our results and then provide the proofs.

Proposition 14.1 *For every $z > 0$ there exists $\hat{Z}_z \in \mathcal{H}$ that attains the infimum in* (14.4).

Proposition 14.2 *For every $x \in (0, \infty)$ there exists $\hat{z} \in (0, \infty)$ that attains the infimum of $\gamma(z)$ in* (14.6).

Denote $\hat{Z} := \hat{Z}_{\hat{z}}$ the optimal solution to (14.4) with $z = \hat{z}$ denoting the optimal solution to $\inf_{z>0} \gamma(z)$ of (14.6). The main result of this section is the following

Theorem 14.1 *The pair $(\hat{C}_0, 0) := (I(\hat{z}\hat{Z}/B(T)), 0)$ belongs to the set $\mathcal{A}^+(x)$ of (nonnegative) terminal holdings that can be hedged starting with initial wealth $x > 0$ in the bank account. Furthermore,*

$$E\left[U\left(I\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)\right)\right] = V(x) = \inf_{z>0}[\tilde{V}(z) + zx] = \tilde{V}(\hat{z}) + x\hat{z}.$$

In particular, the strategy that hedges $(\hat{C}_0, 0)$ is optimal for the utility maximization problem (14.1).

Remark 14.2 Under Assumption 14.1, there exist $z_0 > 0$, $0 < \gamma, \mu < 1$ and $0 < c < \infty$ such that

$$zI(z) < \frac{\gamma}{1-\gamma}\tilde{U}(z) \qquad \text{and} \qquad \tilde{U}(\mu z) < c\tilde{U}(z), \qquad \forall\, 0 < z < z_0; \tag{14.7}$$

see [KS98] Lemma 6.3 and Corollary 6.1 for details.

Proof of Proposition 14.1: We first observe that $\mathcal{H}$ is convex, closed under a.s-convergence by Fatou's lemma, and bounded in $\mathbf{L}^1(\mathbf{P})$; the latter is seen by setting $(X(T), Y(T)) = (xB(T), 0)$ in (14.3), implying $E[Z] \leq 1$ for $Z \in \mathcal{H}$. Fix $z > 0$ and let $\{Z_n\}$ be a minimizing sequence for (14.4). By Komlós theorem (see Schwartz 1986), there exists a subsequence Z'_k such that

$$\tilde{Z}_k := \frac{1}{k}\sum_{i=1}^{k} Z'_i \to \hat{Z}_z \in \mathcal{H}$$

as $k \to \infty$, almost surely. As in Lemma 3.4 of [KS98], Fatou's lemma is applicable here, so that $\liminf_{k\to\infty} E\tilde{U}(z\tilde{Z}_k) \geq E\tilde{U}(z\hat{Z}_z)$. In conjunction with convexity of $\tilde{U}$ this easily implies that $\hat{Z}_z$ is optimal for (14.4). $\diamond$

For a given progressively measurable process $\theta(\cdot)$ introduce the local martingale

$$Z_\theta(t) := \exp\{\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\}\ , \quad 0 \leq t \leq T. \tag{14.8}$$

In this section we will use the notation $Z_0 := Z_{\theta_0^*}(T)$ for the risk-neutral density for the market without transaction costs, where, as before, $\theta^*(t) := (r(t) - b(t))/\sigma(t)$. We have $Z_0 \in \mathcal{H}$.

Lemma 14.1 *Value function $V(\cdot) : (0, \infty) \to \mathbb{R}$ is finite, decreasing and strictly convex.*

Proof: It is straightforward to check that $\tilde{V}(\cdot)$ is decreasing and strictly convex. Next, since $r(\cdot)$ is bounded, we have $k^{-1} \leq B(T) \leq k$ for some $k > 0$. In conjunction with Jensen's inequality, we obtain

$$E\tilde{U}(zZ/B(T)) \geq \tilde{U}(zkE[Z]) \geq \tilde{U}(zk), \tag{14.9}$$

hence $\tilde{V}(z) \geq \tilde{U}(zk) > -\infty$. On the other hand, Assumption 14.1 ensures the existence of $0 < \alpha < 1$, $z_1 > 0$ such that

$$\tilde{U}(\mu z_1) < \mu^{\frac{\alpha}{\alpha-1}} \tilde{U}(z_1) \qquad \text{for all} \quad 0 < \mu < 1;$$

see [KS98] Lemma 6.3 for the proof. We get, since $Z_0 \in \mathcal{H}$,

$$\begin{aligned}
\tilde{V}(z) &\leq E\tilde{U}(zZ_0/B(T)) \\
&= E[\tilde{U}(zZ_0/B(T))\mathbf{1}_{\{zZ_0/B(T) > z_1\}}] + E[\tilde{U}(zZ_0/B(T))\mathbf{1}_{\{zZ_0/B(T) \leq z_1]\}} \\
&\leq |\tilde{U}(z_1)| + (z/z_1)^{\frac{\alpha}{\alpha-1}} |\tilde{U}(z_1)| \cdot E\left[(Z_0/B(T))^{\frac{\alpha}{\alpha-1}}\right] < \infty.
\end{aligned}$$

Proof of Proposition 14.2: We have $\tilde{V}(0+) = \tilde{U}(0+)$, so $\lim_{z \downarrow 0} \gamma(z) = \tilde{U}(0+) = U(\infty)$. Therefore, if $U(\infty) = \infty$, the infimum $\gamma(z)$ on $[0, \infty)$ cannot be attained at $z = 0$. Suppose now that $U(\infty) < \infty$ and that the infimum is attained at $\hat{z} = 0$, namely that $\inf_{z>0} \gamma(z) = \tilde{U}(0+)$. Then we have

$$x \geq \frac{\tilde{U}(0+) - \tilde{U}(zH)}{z} \geq E[HI(zH)]$$

for all $H \in \mathcal{H}$ and $z > 0$. Leting $z \to 0$ we get $x \geq \infty$, a contradiction. Therefore, either the infimum of $\gamma(z)$ is attained at a (unique) number $\hat{z} =$

$\hat{z}_x \in (0, \infty)$ or it is attained at $\hat{z} = \infty$. If the latter is the case, then there exists a sequence $z_n \to \infty$ such that for z_n large enough and a fixed $z < z_n$, we have (by (14.9))

$$x \leq \frac{\tilde{V}(z) - \tilde{V}(z_n)}{z_n - z} \leq \frac{\tilde{V}(z) - \tilde{U}(z_n k)}{z_n - z}.$$

Letting $z_n \to \infty$ we get $x \leq 0$ by de l'Hospital's Rule, a contradiction. $\diamond$

Lemma 14.2

$$\tilde{V}'(\hat{z}) = -E\left[\frac{\hat{Z}}{B(T)} I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right] = -x.$$

Proof: Let $h(z) := E[\tilde{U}(z\hat{Z}/B(T))]$. Then $h(\cdot)$ is convex, $h(\cdot) \geq \tilde{V}(\cdot)$ and $h(\hat{z}) = \tilde{V}(\hat{z})$. These three facts easily imply $\Delta^- h(\hat{z}) \leq \Delta^- \tilde{V}(\hat{z}) \leq \Delta^+ \tilde{V}(\hat{z}) \leq \Delta^+ h(\hat{z})$, where $\Delta^\pm$ denotes the left and the right derivatives. Because of this, it is sufficient to prove the lemma with $\tilde{V}$ replaced by h. It is easy to show, by monotone convergence theorem, that

$$\Delta^+ h(\hat{z}) \leq -E\left[\frac{\hat{Z}}{B(T)} I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right]. \tag{14.10}$$

On the other hand,

$$\Delta^- h(\hat{z}) \geq \overline{\lim_{\varepsilon \to 0+}}\, E\left[-\frac{\hat{Z}}{B(T)} I\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right)\right]$$

We claim that

$$\begin{aligned} \frac{\hat{Z}}{B(T)} I\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right) &= \frac{\hat{Z}}{B(T)} I\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right) \mathbf{1}_{\{\hat{z}\frac{\hat{Z}}{B(T)} \geq z_0\}} \\ &\quad + \frac{\hat{Z}}{B(T)} I\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right) \mathbf{1}_{\{\hat{z}\frac{\hat{Z}}{B(T)} < z_0\}} \end{aligned}$$

is uniformly integrable when ϵ is small enough, where z_0 is the number from (14.7). Indeed, the first term is dominated by $\frac{\hat{Z}}{B(T)} I\left(\frac{\hat{z}-\varepsilon}{\hat{z}} z_0\right)$, which is uniformly integrable when ε is sufficiently small since $E[\frac{\hat{Z}}{B(T)}] \leq k \cdot E[\hat{Z}] \leq k$.

It follows from (14.7) that the second term is dominated by

$$\frac{1}{\hat{z}-\varepsilon}\frac{\gamma}{1-\gamma}\tilde{U}\left((\hat{z}-\varepsilon)\frac{\hat{Z}}{B(T)}\right),$$

which is in turn dominated by

$$\frac{1}{\hat{z}-\varepsilon}\frac{\gamma c}{1-\gamma}\tilde{U}\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)$$

when ε is small. The uniform integrability follows from $E\left|\tilde{U}\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)\right| < \infty$.

Therefore, we can use the Mean Convergence Criterion to get the inequality

$$\Delta^{-}h(\hat{z}) \geq -E\left[\frac{\hat{Z}}{B(T)}I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right].$$

Together with (14.10) we establish $h'(\hat{z}) = -E[\frac{\hat{Z}}{B(T)}I(\hat{z}\frac{\hat{Z}}{B(T)})] = -x$. The latter equality follows from the fact that $\hat{z}$ attains $\inf_{z>0}[\tilde{V}(z) + xz]$. $\diamond$

Lemma 14.3 *We have*

$$\sup_{Z\in\mathcal{H}} E\left[\frac{Z}{B(T)}I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right] = E\left[\frac{\hat{Z}}{B(T)}I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right] = x.$$

Proof: For a given $Z \in \mathcal{H}$, $\varepsilon \in (0,1)$, let $Z_\varepsilon := (1-\varepsilon)\hat{Z} + \varepsilon Z \in \mathcal{H}$. By optimality of $\hat{Z}$ we get

$$\begin{aligned} 0 &\geq \frac{1}{\varepsilon}E\left[\tilde{U}\left(\frac{\hat{z}\hat{Z}}{B(T)}\right) - \tilde{U}\left(\frac{\hat{z}Z_\varepsilon}{B(T)}\right)\right] \geq -\frac{1}{\varepsilon}E\left[\frac{\hat{z}(\hat{Z}-Z_\varepsilon)}{B(T)}I\left(\frac{\hat{z}Z_\varepsilon}{B(T)}\right)\right] \\ &= E\left[\frac{\hat{z}(Z-\hat{Z})}{B(T)}I\left(\frac{\hat{z}Z_\varepsilon}{B(T)}\right)\right]. \end{aligned} \tag{14.11}$$

However, it follows that, as in the proof of Lemma 14.2,

$$\left(\frac{Z-\hat{Z}}{B(T)}I\left(\frac{\hat{z}Z_\varepsilon}{B(T)}\right)\right)^{-} \leq \frac{\hat{Z}}{B(T)}I\left(\frac{\hat{z}(1-\varepsilon)\hat{Z}}{B(T)}\right)$$

is uniformly integrable. We can now use Fatou's lemma in (14.11), to get

$$E\left[\frac{Z-\hat{Z}}{B(T)}I\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)\right] \leq 0,$$

which completes the proof. $\diamond$

Proof of Theorem 14.1: For fixed $x > 0$ define

$\mathcal{C} := \{\xi \in \mathbf{L}^0_+ \ / \ xB(T)\xi \leq X(T)+f(Y(T)),$ for some $(X(T),Y(T)) \in \mathcal{A}^+(x)\}$.

Denote by

$$\mathcal{C}^0 := \{Z \in \mathbf{L}^0_+ \ / \ E[Z\xi] \leq 1, \ \forall \ \xi \in \mathcal{C}\}$$

the polar of set $\mathcal{C}$. It is clear then that $\mathcal{H} = \mathcal{C}^0$. We also want to show $\mathcal{C} = \mathcal{H}^0 = \mathcal{C}^{00}$. By the bipolar theorem of Brannath & Schachermayer (1998), it is sufficient to show that $\mathcal{C}$ is convex, solid and closed under a.s.-convergence (a subset $\mathcal{C}$ of $\mathbf{L}^0_+$ is solid if $f \in \mathcal{C}$ and $0 \leq g \leq f$ imply $g \in \mathcal{C}$). It is obvious that $\mathcal{C}$ is convex and solid. On the other hand, from Theorem 13.1 we know that $\xi \in \mathcal{C}$ *if and only if*

$$E^*_0[(\xi B(T))^2] < \infty \quad \text{and} \quad \sup_{Z\in\mathcal{H}} E[Z\xi] \leq 1.$$

This implies (by Fatou's lemma) that $\mathcal{C}$ is closed under a.s-convergence, because the set $\{\xi B(T)\}_{\xi\in\mathcal{C}}$ is bounded in $\mathbf{L}^2(\mathbf{P}^*_0)$. Indeed, the latter follows from [CK96] (as remarked in Appendix B of that paper, this can be shown by setting $U_n = V_n = 0$ in the arguments of its Appendix A; see (A.8)-(A.11) on p156). We conclude that $\mathcal{C} = \mathcal{H}^0$. Now, Lemma 14.3 implies $\frac{I(\hat{z}\hat{Z}/B(T))}{xB(T)} \in \mathcal{H}^0 = \mathcal{C}$, hence $(I(\hat{z}\hat{Z}/B(T)), 0) \in \mathcal{A}^+(x)$. This, in conjunction with Lemma 14.3 and Remark 14.1, implies the remaining statements of the theorem.

$\diamond$

Notice that, if $r(\cdot)$ is deterministic, then Jensen's inequality gives

$$E\left[\tilde{U}\left(z\frac{Z}{B(T)}\right)\right] \quad \geq \quad \tilde{U}\left(\frac{z}{B(T)}E[Z]\right)$$

$$\geq \tilde{U}\left(\frac{z}{B(T)}\right), \tag{14.12}$$

for all $Z \in \mathcal{H}$. We will use this observation to find examples, in which the optimal strategy $(\hat{L}, \hat{M})$ never trades.

Example 14.9 Let us assume that $r(\cdot)$ is deterministic. In this case we see from (14.12) that

$$\tilde{V}(z) \geq \tilde{U}(z/B(T)),$$

and the infimum is attained by taking $\hat{Z} \equiv 1$, if $1 \in \mathcal{H}$. A sufficient condition for this is $(1, \hat{Z}_1(\cdot)) \in \mathcal{D}$ for some positive martingale $\hat{Z}_1(\cdot)$ such that $1 - \mu \leq \hat{R}(\cdot) = \hat{Z}_1(\cdot)/P(\cdot) \leq 1 + \lambda$. In particular, one can set $\hat{Z}_1(0) = (1 + \lambda)s$ and $\hat{Z}_1(\cdot) = Z_{\hat{\theta}_1}(\cdot)$, where $\hat{\theta}_1(\cdot) \equiv \sigma(\cdot)$, in which case $(1, \hat{Z}_1(\cdot)) \in \mathcal{D}$ if and only if

$$0 \leq \int_0^t (b(s) - r(s))ds \leq \log \frac{1+\lambda}{1-\mu}\ , \forall\ 0 \leq t \leq T. \tag{14.13}$$

Furthermore,

$$\hat{X}(T+) = I(\hat{z}/B(T)) = xB(T).$$

This means that the *no-trading-strategy* $\hat{L} \equiv 0$, $\hat{M} \equiv 0$ is optimal. Condition (14.13) is satisfied, for instance, if

$$r(\cdot) \leq b(\cdot) \leq r(\cdot) + \rho\ ,\ \text{for some } 0 \leq \rho \leq \frac{1}{T} \log \frac{1+\lambda}{1-\mu}. \tag{14.14}$$

If $b(\cdot) = r(\cdot)$ the result is not surprising – even without transaction costs, it is then optimal not to trade. However, if there are no transaction costs, in the case $b(\cdot) > r(\cdot)$ the optimal portfolio always invests a positive amount in the stock; the same is true even in the presence of transaction costs, if one is maximizing expected discounted utility from consumption over an infinite time-horizon, and if the market coefficients are constant – see Shreve & Soner (1994), Theorem 11.6. The situation here, on the finite time-horizon $[0, T]$, is quite different: if the excess rate of return $b(\cdot) - r(\cdot)$ is positive but small relative to the transaction costs, and/or if the time-horizon is small, in the sense of (14.14), then it is optimal not to trade.

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