Moral Hazard in Dynamic Risk Management*

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Abstract. We consider a contracting problem in which a principal hires an agent to manage a risky project. When the agent chooses volatility components of the output process and the principal observes the output continuously, the principal can compute the quadratic variation of the output, but not the individual components. This leads to moral hazard with respect to the risk choices of the agent. To find the optimal contract, we develop a novel approach to solving principal-agent problems: first, we identify a family of admissible contracts for which the optimal agent's action is explicitly characterized; then, we show that we do not lose on generality when finding the optimal contract inside this family, up to integrability conditions. To do this, we use the recent theory of singular changes of measures for Itô processes. We solve the problem in the case of CARA preferences, and show that the optimal contract is linear in these factors: the contractible sources of risk, including the output, the quadratic variation of the output and the cross-variations between the output and the contractible risk sources. Thus, like sample Sharpe ratios used in practice, path-dependent contracts naturally arise when there is moral hazard with respect to risk management. In a numerical example, we show that the loss of efficiency can be significant if the principal does not use the quadratic variation component of the optimal contract.

Keywords: principal–agent problem, moral hazard, risk-management, volatility/portfolio selection.

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1 Introduction

In many cases managers are in charge of managing exposures to many different types of risk, and they do that dynamically. A well-known example is the management of a portfolio of many risky assets. Nevertheless, virtually all existing continuous-time principal-agent models with moral hazard and continuous output value process suppose that the agent controls the drift of the output process, and not its volatility components. The drift is what the agent controls in the seminal models of Holmstrom and Milgrom (1987), henceforth HM (1987), in which utility is drawn from terminal payoff, and of Sannikov (2008), in which utility is drawn from inter-temporal payments. In fact, in those papers the moral hazard cannot arise from volatility choice anyway when there is only one source of risk (one Brownian motion), because if the principal observes the output process continuously, there is no moral hazard with respect to volatility choice: the volatility can be deduced from the output's quadratic variation process. However, when the agent manages many non-contractible sources of risk, his choices of exposures to the individual risk sources cannot be deduced from the output observations, even continuous.

One reason this problem has not been studied is the previous lack of a workable mathematical methodology to tackle it. When the drift of an Itô process is picked by the agent, this can be formulated as a Girsanov change of the underlying probability measure to an equivalent probability measure, and there is an extensive mathematical theory behind it. However, changing volatility components requires singular changes of measures, a problem that, until recently, has not been successfully studied. We take advantage of recent progress in this regard, and use the new theory to analyze our principal-agent problem. However, we depart from the usual modeling assumption that the agent's effort consists in changing the distribution of the output (i.e., the underlying probability measure), and, instead, we employ a novel approach to solving principal agent problems, as follows. We use the standard stochastic control framework in which the agent actually changes the values of the controlled process, while the probability measure stays the same. The reason why all the existing literature uses the so-called "weak formulation" of changing the distribution, is that the agent's problem becomes tractable *for any given contract*. Instead, we make the agent's problem tractable by first restricting the family of admissible contracts to a natural subset of feasible contracts. Essentially, we restrict the admissible contracts to those for which the agent's problem is mathematically solvable. We then show first, that in the classical Holmstrom-Milgrom model the restricted family is not actually restricted at all; and second, that even in the general case our restriction of admissible contracts does not change the principal's value function. Thus, in addition to solving the new agency problem with volatility control, we offer an alternative new method for studying principal-agent problems.

The above is our main contribution on the methodological side. For economic insights, we specialize to CARA utility functions, as in HM (1987), that being the most tractable case. The main economic insight of the paper is as follows. The optimal contract is linear (in the CARA case), but not only in the output process as in HM (1987), rather, also in these factors: the output, its quadratic variation, the contractible sources of risk (if any), and the cross-variations between the output and the contractible risk sources. Thus, the use of path dependent contracts naturally arises when there is moral hazard with respect to risk management. In particular, our model is consistent with the use of the sample Sharpe ratio when compensating portfolio managers. However, unlike the typical use of Sharpe ratios, there are parameter values for which the principal rewards the agent for higher values of quadratic variation, thus, for taking higher risk.

In case there are two sources of risk, and at least one is observable and contractible, the first best is attained, because there are two risk factors and at least two contractible variables, the output and at least one risk source; however, to attain the first best, the optimal contract makes use of the quadratic and cross-variation factors. In case of two non-contractible sources of risk, we solve numerically a CARA example with a quadratic cost function. In this case, first best is generically not attainable. Numerical computations show that the loss in expected utility can be significant if the principal does not use the path-dependent components of the optimal contract.

Literature review. An early continuous-time paper on volatility moral hazard is Sung (1995). However, in that paper moral hazard is a result of the output being observable only at the terminal time, and not because of multiple sources of risk. Consequently, the optimal contract is still a linear function of the terminal output value only. The paper Ou-Yang (2003) shows that the optimal contract depends on the final value of the output and a "benchmark" portfolio, in an economy in which all the sources of risk (all the risky assets available for investment) are observable, but the output is observable at final time only. Some of his results are extended in Cadenillas, Cvitanić and Zapatero (2007), who show that, if the market is complete, first-best is attainable by contracts that depend only on the final value of the output. Thus, second best may be different from first best only if the market is not complete. In our model, the principal observes the whole path of the output, but not all the exogenous sources of risk, and also there is a non-zero cost of volatility choice, which makes the market incomplete. Thus, first best and second best are indeed different for generic parameter configurations.

More recently, Wong (2013) considers the moral hazard of risk-taking in a model different from ours: the horizon is infinite, as in Sannikov (2008), and, while the volatility is fixed, the agent's effort influences the arrival rate of Poisson shocks to the output process. Lioui and Poncet (2013), like us, consider a principal-agent problem in which the volatility is chosen by the agent. Theirs is the first-best framework; however, unlike the above mentioned papers, they assume that the agent has enough bargaining power to require that the contract be linear in the output and in a benchmark factor. Working paper Leung (2014) proposes a model in which volatility moral hazard arises because there is an exogenous factor multiplying the (one-dimensional) volatility choice of the agent, and that factor is not observed by the principal.

In terms of methodological contributions, a number of papers in the mathematics literature have been developing tools for comparing stochastic differential systems corresponding to differing volatility structures. We cite them in the main body of the paper, as we use a lot of their results. Here, we only mention two papers that not only contribute to the development of those tools, but also apply them to problems in financial economics: Epstein and Ji (2013) and (2014). Theirs is not the principal-agent problem, but the ambiguity problem, that is, a model in which the decision-maker has multiple priors on the drift and the volatility of market factors. There is no ambiguity in our model, it is the agent who controls the drift and the volatility of the output process. Sung (2014) studies a combined problem, ambiguity in a principal-agent setting, and also finds that the optimal contract uses a quadratic variation component. There is only one Brownian motion in his model, so, without ambiguity, the volatility would be observed.

We start by Section 2 presenting the simplest possible example in our context, with CARA utility functions and quadratic penalty; we describe the general model in Section 3; in Section 4 we present the contracting problem and our approach to solving it; we consider the case with no exogenous contractible factors in Section 5, and conclude with Section 6. The mathematical intuition for the definition of admissible contracts and the longer proofs are provided in Appendix.

2 Example: Portfolio Management with CARA Utilities and quadratic cost

As an illustrative tractable example, we present here the Merton's portfolio selection problem with two risky assets, S_1 and S_2, and a risk-free asset with the continuously compounded rate set equal to zero. Holding amount v_i in asset i, the portfolio value process X_t follows the dynamics

$$dX = \frac{v_1}{S_1} dS_1 + \frac{v_2}{S_2} dS_2.$$

Suppose

$$dS_{i,t}/S_{i,t} = b_i dt + dB_t^i,$$

where B^i are independent Brownian motions and b_i are constants. We have then

$$dX_t = [v_{1,t} b_1 + v_{2,t} b_2]\, dt + v_{1,t} dB_t^1 + v_{2,t} dB_t^2.$$

The principal hires an agent to manage the portfolio, that is, to choose the values of $v_t = (v_{1,t}, v_{2,t})$, continuously in time. We assume that the agent is paid only at the final time T in the amount ξ_T. The utility of the principal is $U_P(X_T - \xi_T)$ and the utility of the agent is $U_A(\xi_T - K_T^v)$ where $K_T^v := \int_0^T k(v_s) ds$ and $k : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative convex cost function.

If the principal observes only the path of X, then she can deduce the value of $v_{1,t}^2 + v_{2,t}^2$, but not the values of $v_{1,t}$ and $v_{2,t}$ separately, which leads to moral hazard. On the other hand, if she also observes the price path of one of the assets, say S_1, then she can deduce the (absolute) values of $v_{1,t}$ and of $v_{2,t}$.

We assume that the principal and the agent have exponential utility functions,

$$U_P(x) = -e^{-R_P x}\ ,\ \ U_A(x) = -e^{-R_A x}.$$

and that the agent's running cost of portfolio (v_1, v_2) is of the form

$$k(v_1, v_2) = \frac{1}{2}\beta_1(v_1 - \alpha_1)^2 + \frac{1}{2}\beta_2(v_2 - \alpha_2)^2.$$

Thus, it is costly to move the volatility v_i away from α_i, and the cost intensity is β_i. An interpretation is that α_i are the initial risk exposures of the principal at the time the manager starts his contract.

2.1 First-best contract

Given a "bargaining-power" parameter $\rho > 0$, the principal's first-best problem is defined as

$$\sup_v \sup_{\xi_T} \mathbb{E}\left[U_P(X_T - \xi_T) + \rho U_A(\xi_T - K_T^v)\right].$$

The first order condition for ξ_T is then

$$U_P'(X_T - \xi_T) = \rho U_A'(\xi_T - K_T^v).$$

With CARA utilities, we obtain

$$\xi_T = \frac{1}{R_A + R_P}\left(R_P X_T + R_A K_T^v + \log\left(\frac{\rho R_A}{R_P}\right)\right).$$

Thus, the optimal first best contract is linear in the final value X_T of the output. Plugging back into the optimization problem, we get that it is equivalent to

$$\begin{aligned}
&-C_\rho \inf_v \mathbb{E}\left[\exp\left(-\frac{R_A R_P}{R_A + R_P}(X_T - K_T^v)\right)\right] \\
&= -C_\rho \inf_v \mathbb{E}\left[\mathscr{E}\left(-\frac{R_A R_P}{R_A + R_P}\int_0^T v_s \cdot dX_s\right)\exp\left(-\frac{R_A R_P}{R_A + R_P}\int_0^T f(v_s)ds\right)\right],
\end{aligned}$$

where $x \cdot y$ denotes the inner product, C_ρ is a constant, $\mathscr{E}$ denotes the Doléans-Dade stochastic exponential*, and

$$f(v) := b \cdot v - k(v) - \frac{1}{2}\frac{R_A R_P}{R_A + R_P}\|v\|^2.$$

*Stochastic exponential is defined by

$$\mathscr{E}\left(\int_t^u X_s dB_s\right) = e^{-\frac{1}{2}\int_t^u |X_s|^2 ds + \int_t^u X_s dB_s}.$$

Under technical conditions, Girsanov theorem can be applied and the above can be written as

$$-C_\rho \inf_v \mathbb{E}^{\widehat{\mathbb{P}}}\left[\exp\left(-\frac{R_A R_P}{R_A+R_P}\int_0^T f(v_s)ds\right)\right].$$

for an appropriate probability measure $\widehat{\mathbb{P}}$. Thus, first best optimal $v = v^{FB}$ is deterministic, found by the pointwise maximization of the function $f(\cdot)$, and given by[†]

$$v_i^{FB} := \frac{b_i+\beta_i\alpha_i}{\beta_i+\bar{R}}, \quad \text{where} \quad \frac{1}{\bar{R}} := \frac{1}{R_A}+\frac{1}{R_P}. \tag{2.1}$$

2.2 Second best contracts

We now take into account that it is not the principal who controls v, but the agent. We consider linear contracts based on the path of the observable portfolio value X, on the observable quadratic variation of X, and, if S_1 is contractible, on S_1 via B^1, and on the co-variation of X and B_1. More precisely, denoting by $\mathbf{1}_O$ the indicator function that is equal to one if the path of S_1 is observable and contractible, and zero otherwise, let

$$\xi_T = \xi_0 + \int_0^T \left[Z_s^X dX_s + Y_s^X d\langle X\rangle_s + \mathbf{1}_O\left(Z_s^1 dB_s^1 + Y_s^1 d\langle X, B_1\rangle_s\right) + H_s ds\right], \tag{2.2}$$

for some constant ξ_0, and some adapted processes Z^X, Z^1, Y^X, Y^1 and H. To be consistent with the notation in the general theory that follows later, we work instead with arbitrary adapted processes $Z^X, Z^1, \Gamma^X, \Gamma^1$ and G such that

$$\begin{aligned} Y^X &= \frac{1}{2}\left(\Gamma^X + R_A(Z^X)^2\right), \\ Y^1 &= \Gamma^1 + R_A Z^X Z^1, \\ H &= -G + \frac{1}{2}R_A(Z^1)^2. \end{aligned}$$

Solving the agent's problem if those processes were deterministic would be easy, but not for arbitrary choices of those processes. We will allow them to be stochastic, but we will restrict the choice of process G_t, motivated by a stochastic control analysis of the agent's problem, discussed in a later section. We will see then that the natural choice for G_t is $G_t := G(Z_t^X, Z_t^1, \Gamma_t^X, \Gamma_t^1)$, where

$$\begin{aligned} G(Z^X, Z^1, \Gamma^X, \Gamma^1) &:= \sup_{v_1,v_2} g(v_1, v_2, Z^X, Z^1, \Gamma^X, \Gamma^1) \\ &:= \sup_{v_1,v_2}\left\{-k(v_1,v_2) + \frac{1}{2}\Gamma^X(v_1^2+v_2^2) + Z^X b\cdot v + \mathbf{1}_O\Gamma^1 v_1\right\}. \end{aligned} \tag{2.3}$$

One of our main results will be that considering only the contracts of the above form, with G_t as defined here, leads to no loss of generality, under mild technical conditions.

[†]In the case the cost β_i is zero, the first best volatility v_i^{FB} is simply a product of the risk premium b_i and the aggregate prudence $1/\bar{R}$, a well-known result.

Given such a contract, the agent is maximizing $-\mathbb{E}[-\exp(-R_A(\xi_T - K_T^v)]$ with ξ_T as in (2.2). This turns out to be an easy optimization problem: by Girsanov theorem (assuming appropriate technical conditions), similarly as in the first best problem, the agent's objective can be written as

$$-\mathbb{E}_t^{\mathbb{P}^*}\left[e^{-R_A\int_0^T[g_s-G_s]ds}\right],$$

for an appropriate probability $\mathbb{P}^*$. With our definition of G, we see that this is never larger than minus one, and it is equal to minus one for any pair $(v_1^*(s), v_2^*(s))$ (if it exists) that maximizes $g_s := g(Z_s^X, Z_s^1, \Gamma_s^X, \Gamma_s^1)$, s by s, and outcome by outcome. Thus, the agent would choose one of such pairs.

Notice that this implies that the principal can always make the agent indifferent about one of the portfolio positions. For example, if b_2 is not zero, she can set $Z^X = -\alpha_2\beta_2/b_2$, and $\Gamma^X \equiv \beta_2$, to make g independent of v_2, hence the agent indifferent with respect to v_2. This is also possible if there is only one stock, say S_2, so that in that case the first best is attained with such a contract, if we assume that the agent will choose what is best for the principal, when indifferent. Moreover, when the cost function is zero, the principal can pay a fixed cash payment, setting $Z^X, Z^1, \Gamma^X, \Gamma^1$ equal to zero, in which case the agent will be indifferent with respect to which v_1, v_2 to choose, hence first best is attained. (This is not necessarily the unique way of attaining first best.)

Next, we look for the principal's optimal choice in the above family of admissible contracts.

2.2.1 Contractible S_1: first best is attained

With two risky assets, if S_1 is observed, then also the covariation between S^1 and X is observed, which means that v_1 is observed. Since also the quadratic variation is observed, then $|v_2|$ is observed, and, if the observed processes are contractible, we would expect first best to be attainable. Indeed, (v_1^*, v_2^*) is obtained by maximizing

$$g = -\frac{1}{2}\beta_1(v_1-\alpha_1)^2 - \frac{1}{2}\beta_2(v_2-\alpha_2)^2 + Z^X b\cdot v + \Gamma^1 v_1 + \frac{1}{2}\Gamma^X\|v\|^2 + Z^1 b_1 + (Z^1)^2 + 2Z^1 Z^X v_1.$$

Assume, for example, that $b_2 \neq 0$, $\beta_2 \leq \beta_1$. Suppose the principal sets

$$\begin{aligned}
\Gamma_t^X &\equiv \beta_2,\\
Z_t^X &\equiv -\alpha_2\beta_2/b_2,\\
\Gamma_t^1 &= -\alpha_1\beta_1 - Z_t^X b_1 + (\beta_1-\beta_2)v_1^{FB},\\
Z_1 &\equiv 0.
\end{aligned}$$

Then,

$$g = (\beta_2-\beta_1)\left[\frac{1}{2}v_1^2 - v_1 v_1^{FB}\right] + const.$$

We see that the agent is indifferent with respect to which v_2 he applies, and he would choose $v_1^* = v_1^{FB}$. Thus, if, when indifferent, the agent will choose what is best for the principal, he will choose the first best actions. It can also be verified that the principal will attain the first best expected utility with this contract.

2.2.2 Non-contractible S_1

If S_1 is not contractible, the optimal (v_1^*, v_2^*) is obtained by maximizing

$$g(v_1, v_2) = -\frac{1}{2}\beta_1(v_1 - \alpha_1)^2 - \frac{1}{2}\beta_2(v_2 - \alpha_2)^2 + Z^X b \cdot v + \frac{1}{2}\Gamma^X \|v\|^2.$$

Assume, for example, $\beta_2 \leq \beta_1$. If $\Gamma^X > \beta_2$, then the agent will optimally choose $|v_2^*| = \infty$. It is straightforward to verify that this cannot be optimal for the principal. The same is true if $\Gamma^X = \beta_2$ and Z^X is not equal to $(-\alpha_2\beta_2)$. If $\Gamma^X = \beta_2$ and $Z^X = -\alpha_2\beta_2$, then the agent is indifferent with respect to which v_2 to choose. If $\Gamma^X < \beta_2 \leq \beta_1$, the optimal positions are

$$v_i^* = \frac{Z^X b_i + \alpha_i \beta_i}{\beta_i - \Gamma^X}.$$

The principal's utility is proportional to

$$-\mathbb{E}\left[e^{-R_P\left(\int_0^T\left[(1-Z_s^X)dX_s - \frac{1}{2}\Gamma_s^X d\langle X\rangle_s + G_s ds\right]\right)}\right].$$

With the above choice of v_i^*, by a similar Girsanov change of probability measure as before, it is straightforward to verify that maximizing this is the same as maximizing, over $Z = Z^X$ and $\Gamma = \Gamma^X$,

$$b \cdot v^*(Z, \Gamma) - \frac{1}{2}[R_A Z^2 + R_P(1-Z)^2]\|v^*\|^2 - k(v^*(Z, \Gamma)).$$

This is a problem that can be solved numerically.[‡] The optimal $Z^X, Z^1, \Gamma^X, \Gamma^1$ are constant, and thus the optimal contract is linear in the underlying factors, of the form, for some constant $\tilde{C}$,

$$\xi_T = \tilde{C} + Z^X X_T + Y^X \langle X\rangle_T + \mathbf{1}_O\left(Z^1 B_T^1 + Y^1 \langle X, B_1\rangle_T\right), \tag{2.4}$$

Based on the above analysis, we present numerical examples that will show us first, that first best is not attained, second, that the optimal contract contains a non-zero quadratic variation component and that ignoring it can lead to substantial loss in expected utility, and third, that there are parameter values for which the principal rewards the agent for taking high risk (unlike the typical use of portfolio Sharpe ratios in practice).

In Figure 1 we plot the percentage loss in the principal's second best utility certainty equivalent relative to the first best, when varying the parameter α_2, and keeping everything else fixed. The loss can be significant for extreme values of initial exposure α_2. That is, when the initial risk

[‡]In Appendix, we provide sufficient conditions for existence of optimizers. Numerically, we found optimizers for all parametric choices we tried.

exposure is far from desirable, the moral hazard cost of providing incentives to the agent to modify the exposure is high.

In Figure 2, we compare the principal's second best certainty equivalent to the one she would obtain if offering the contract that is optimal among those that are linear in the output, but do not depend on its quadratic variation. Again we see that the corresponding relative percentage loss can be large.

Figure 3 plots the values of the coefficient (the sensitivity) multiplying the quadratic variation in the optimal contract. We see that the principal uses quadratic variation as an incentive tool: for low values of the initial risk exposure α_2 she wants to increase the risk exposure by rewarding higher variation (the sensitivity is positive), and for its high values she wants to decrease it by penalizing high variation (the sensitivity is negative). This is because when the initial risk exposure α_2 is not at the desired value v_2^*, incentives are needed to make the agent apply costly effort to modify the exposure.

In the rest of the paper, we generalize this example and we identify (mild) technical conditions under which there is no loss of generality in admitting only the contract payoffs that can be represented as in (2.2), with G_t defined analogously to (2.3) .

3 The General Model

We consider the following general model for the output process X^u:

$$X_t^u = \int_0^t \sigma_s(v_s) \cdot (b_s(a_s)ds + dB_s), \tag{3.1}$$

where $u := (v,a)$ represents the control pair of the agent, allowing also for separate control a of the drift, where v and a are adapted processes taking values in some subset $\mathscr{V} \times \mathscr{A}$ of $\mathbb{R}^m \times \mathbb{R}^n$, for some $(m,n) \in \mathbb{N}^* \times \mathbb{N}^*$. Moreover, $b : [0,T] \times \mathscr{A} \longrightarrow \mathbb{R}^d$ and $\sigma : \mathscr{V} \longrightarrow \mathbb{R}^d$ are given deterministic functions such that

$$\|b(a)\| + \|\sigma(v)\| \leq C\left(1 + \|v\| + \|a\|\right), \tag{3.2}$$

for some constant $C > 0$, and $B = (B^1, \ldots, B^d)$ is a d-dimensional Brownian motion, and the products are inner products of vectors, or a matrix acting on a vector.

The example of delegated portfolio management corresponds to the case in which $m = n = d$, and

$$\sigma_t(v) := v^T \sigma, \ b_t(a) := b, \tag{3.3}$$

for some fixed $b \in \mathbb{R}^d$ and some invertible $d \times d$ matrix σ. In that case, the interpretation of the process X^u is that of the portfolio value process dependent on the agent's choice of the vector v of portfolio dollar-holdings in d risky assets with volatility matrix σ, and the vector b of risk premia. Another special case, when $d = 1$, v is fixed and the agent controls a only, is the original continuous-time principal-agent model of Holmstrom and Milgrom (1987).

In addition to the output process X^u, we may want to allow contracts based on additional observable and contractible risk factors $B^1, \ldots, B^{d_0}$, for some $1 \leq d_0 < d$. For example, $S_t = S_0 + \mu_S t + \sigma_S B_t^1$ might be a model for a contractible stock index.

Usually in contract theory, for sake of tractability, the model is considered in its so-called weak formulation, in which the agent changes the output process not by changing directly the controls (v, a), but by changing the probability measure over the underlying probability space. When, as in standard continuous-time contract models, the effort is present only in the drift, changing measures is done by the means of the Girsanov theorem. Until recently, though, such a tool had not been available for singular changes of measure that are needed when changing volatility, as is the case here. The mathematics to formulate rigorously the weak formulation of our problem is now available. However, we take a different, novel approach: instead of assuming the weak formulation, we will adopt the standard strong formulation of stochastic control (no changes of measure). We will still be able to explicitly characterize the solution to the agent's problem, by first working within a restricted family of admissible contracts. [§] We will then provide sufficient conditions under which working with the restricted family of contracts results in no loss of generality, in the sense that the value function of the principal with that family of contracts is the same as with the general one. As a by-product, we provide a new alternative way of solving the Holmstrom-Milgrom (1987) problem.

We first need to introduce some notation and the framework. We work on the canonical space Ω of continuous functions on $[0, T]$, with its Borel σ-algebra $\mathscr{F}$. The $d-$dimensional canonical process is denoted B, and $\mathbb{F} := (\mathscr{F}_t)_{0 \leq t \leq T}$ is its natural filtration. Let $\mathbb{P}_0$ denote the $d-$dimensional Wiener measure on Ω. Thus, B is a $d-$dimensional Brownian motion under $\mathbb{P}_0$. We denote by $\mathbb{E}$ the expectation operator under $\mathbb{P}_0$.

A pair (v, a) of $\mathbb{F}$-predictable processes taking values in $\mathscr{V} \times \mathscr{A}$ is said to be admissible if

$$\int_0^T |\sigma_s(v_s) \cdot b_s(a_s)| < +\infty, \ \mathbb{P}_0 - a.s., \ \mathbb{E}\left[\exp\left(p \int_0^T ||\sigma_s(v_s)||^2 ds\right)\right] < +\infty, \text{ for all } p > 0, \quad (3.4)$$

and

$$\text{Doléans-Dade exponential } \mathscr{E}\left(\int_0^\cdot b_s(a_s) \cdot dB_s\right) \text{ is a } \mathbb{P}_0\text{-martingale in } L^{1+\eta}, \text{ for some } \eta > 0. \quad (3.5)$$

[§] We used the weak formulation in an earlier version of this paper. Interestingly, to derive our results, even though our model is stated in the strong formulation, we need to use the weak formulation in the proofs. In fact, for the admissible contracts in our restricted family, the weak and the strong formulation for the agent's problem are equivalent.

Moreover, we assume that the first d_0 entries of vector b do not depend on the control process a (because they will correspond to exogenous contractible factors). ¶

As in the example of the previous section, we assume that the agent is paid only at the final time T in the amount ξ_T. When the agent chooses the controls $u \in \mathscr{U}$, the utility of the principal is $U_P(X_T^u - \xi_T)$, and the utility of the agent is $-e^{-R_A(\xi_T - K_{0,T}^u)}$ where

$$K_{t,T} = \int_t^T k(u_s)ds.$$

and $k : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a non-negative convex cost function.

4 Second Best with Contractible Risks

In this section, we assume there is exactly one exogenous contractible risk factor, that is, we set $d_0 = 1$.‖ Thus, we interpret B^1 as the observable and contractible systemic risk.

4.1 Admissible contracts

We allow the contract payoff to depend both on the output X^u and B^1. That is, given a pair $u := (v,a)$ chosen by the agent, the principal can offer contract payoffs measurable with respect to $\mathscr{F}_T^{\text{obs}}$, a σ-field contained in the filtration $\mathbb{F}^{\text{obs}} := \mathbb{F}^{X^{v,a}} \vee \mathbb{F}^{B^1}$ generated by (X^u, B^1), where $\mathbb{F}^{X^u} := \{\mathscr{F}_t^{X^u}\}_{0 \le t \le T}$ is the (completed) filtration generated by the output process X^u. Recall that our assumptions imply that b^1 does not depend on a, and introduce the following compact vector notation for the two contractible factors:

$$B^{\text{obs},u} := (X^u, B^1)^T = \int_0^{\cdot} \mu_s(v_s, a_s)ds + \int_0^{\cdot} \Sigma_s(v_s)dB_s,$$

In this subsection we will define the set of admissible contract payoffs. A detailed motivation for the definition is given in Appendix. For the reader not familiar with Hamilton-Jacobi-Bellman approach to stochastic optimal control, we summarize the motivation here as follows:

Consider the agent's value function at time t, for a given choice of control $u := (v,a) \in \mathscr{U}$

$$V_t^{A,u} := \underset{u' \in \mathscr{U}_t(u)}{\text{essup}} \; \mathbb{E}_t \left[U_A(\xi_T - K_{t,T}^{u'}) \right], \tag{4.1}$$

¶The above integrability conditions are suitable for the case of CARA utilities, and they may have to be modified for other utility functions.

‖We could equally have any subset of $\{B^1, \dots, B^d\}$ contractible, and the rest non-contractible. We assume that there is only one contractible risk source for simplicity of notation, and because it is also consistent with the "systemic risk – stock index" interpretation. We consider the case in which none of the risk sources is contractible in the following section.

where the set $\mathscr{U}_t(u)$ denotes the subset of elements of $\mathscr{U}$ which coincide with u, $dt \times \mathbb{P}_0-$a.e. on $[0,t]$. Note that we have the following explicit relationship between the payoff and the terminal value of the value function:

$$\xi_T = U_A^{-1}(V_T^{A,u}). \tag{4.2}$$

The idea is to consider the most general representation of the value function $V^{A,u}$ that we can reasonably expect to have, and then define the admissible contract payoffs via (4.2). When U_A is a CARA utility function, we may guess from (4.2) that contracts ξ_T are such that ξ_T can be written as a linear combination of various integrals. For example, in the special case $d_0 = 0$ with no outside contractible factors, with $U_A(x) = -e^{-R_A x}$, we might expect the contracts to satisfy

$$U_A(\xi_T) = C \exp\left[-R_A\left(\int_0^T Z_s dX_s^u + \int_0^T Y_s d\langle X^u\rangle_s + \int_0^T H_s ds\right)\right],$$

for some constant $C < 0$ and some adapted processes H, Y, Z. This would give

$$\xi_T = \tilde{C} + \int_0^T Z_s dX_s^u + \int_0^T Y_s d\langle X^u\rangle_s + \int_0^T H_s ds, \tag{4.3}$$

for some constant $\tilde{C}$. We will show below that it is easy to solve the agent's problem under such contracts when H has a particular form, and that considering only such contracts represents no loss of generality.

The precise definition is as follows. The term H needs to be related to the Hamiltonian of the agent's problem, and it is done in the following way.

For z_{obs} taking values in $\mathbb{R}^2$, and γ^{obs} taking values in the space $\mathbb{S}_2$ of symmetric matrices (thinking of z_{obs} as the gradient vector of the agent's value function, and of γ^{obs} as its Hessian matrix) denote

$$G_1^{\text{obs}}(t, z^{\text{obs}}, \gamma^{\text{obs}}) \quad := \sup_{(v,a)\in\mathscr{V}\times\mathscr{A}} g_1^{\text{obs}}(t, z^{\text{obs}}, \gamma^{\text{obs}}, v, a), \tag{4.4}$$

where, with Tr denoting the trace of a matrix,

$$g_1^{\text{obs}}(t, z^{\text{obs}}, \gamma^{\text{obs}}, v, a) := -k(v,a) + z^{\text{obs}} \cdot \mu_t(v,a) + \frac{1}{2}\text{Tr}\left[\gamma^{\text{obs}}\Sigma_t(v)\Sigma_t^T(v)\right].$$

Definition 4.1. *An admissible contract payoff $\xi_T = \xi_T(Z,\Gamma)$ is a $\mathscr{F}_T^{obs}-$measurable random variable that satisfies*

$$\xi_T(Z,\Gamma) = \tilde{C} + \int_0^T \left\{ Z_t \cdot dB_t^{obs,u} - G_1^{obs}(t, Z_t, \Gamma_t)dt + \frac{1}{2}\text{Tr}\left[\left(\Gamma_t + R_A Z_t Z_t^T\right) d\langle B^{obs,u}\rangle_t\right]\right\}. \tag{4.5}$$

for some constant $\tilde{C}$, and some pair (Z,Γ) of bounded $\mathbb{F}^{obs}-$predictable processes with values in $\mathbb{R}^2$ and $\mathbb{S}^2$, respectively, that are such that there is a maximizer $u^* := (v^*(Z,\Gamma), a^*(Z,\Gamma)) \in \mathscr{U}$ of $g_1^{obs}(\cdot, Z, \Gamma)$, $dt \times d\mathbb{P}_0$-a.e. We denote by $\mathfrak{C}$ the set of all admissible contracts, and by $\mathfrak{U}$ the set of the corresponding (Z,Γ).

The assumption of bounded (Z,Γ) is technical, assumed to simplify the proofs, and it can be relaxed. If, as in the example section, the optimal Z and Γ are constant processes, then the assumption is satisfied. The assumption that $g_1^{\text{obs}}(\cdot,Z,\Gamma)$ has a maximizer is needed to prove the incentive compatibility of contract ξ_T, and to solve the principal's problem.

Remark 4.1. *In addition to a constant term and the "dt" integral term, with U_A a CARA utility function, an admissible contract is linear (in the integration sense) in the following factors: the contractible variables, that is, the output and the contractible sources of risk; and the quadratic variation and cross-variation processes of the contractible variables. As seen in the numerical example, the optimal contract generally makes use of all of these components. This is to be contrasted with the first best contract, and with the case of controlling the drift only as in Holmstrom-Milgrom (1987), in which only the output is used in the optimal contract.*

Remark 4.2. *We argue now that, up to technical smoothness conditions, an "option-like" contract of the form $\xi_T = F(X_T^u, B_T^1)$, for a given function F is an admissible contract. For notational simplicity, assume $b=0$ and $a=0$, and consider the following PDE, with subscripts denoting partial derivatives, and with $G_1 = G_1(z_1,z_2,\gamma_1,\gamma_2,\gamma_3)$ where γ_1 and γ_2 are the diagonal entries of a symmetric matrix γ and γ_3 is the value of the off-diagonal entries,*

$$u_t + G_1(u_x,u_y,u_{xx}-R_Au_x^2,u_{yy}-R_Au_y^2,u_{xy}-R_Au_xu_y) = 0,\ (t,x)\in[0,T)\times\mathbb{R},\ u(T,x,y) = F(x,y).$$

Then, assuming that the PDE has a smooth solution, it follows from Itô's formula applied to $u(t,X_t^u,B_t^1)$ that

$$\begin{aligned}
F(X_T^u,B_T^1) &= u(0,0,0) + \int_0^T u_x(s,X_s^u,B_s^1)dX_s^{v,0} + \int_0^T u_y(s,X_s^u,B_s^1)dB_s^1 \\
&+ \int_0^T \frac{1}{2}\left(u_{xx}(s,X_s^u,B_s^1)d\langle X^u\rangle_s + u_{yy}(s,X_s^u,B_s^1)d\langle B^1\rangle_s\right) + \int_0^T u_{xy}(s,X_s^u,B_s^1)d\langle X^u,B^1\rangle_s \\
&- \int_0^T G_1^{obs}\left(u_x,u_y,u_{xx}-R_Au_x^2,u_{yy}-R_Au_y^2,u_{xy}-R_Au_xu_y\right)(s,X_s^u,B_s^1)ds.
\end{aligned}$$

Thus, $F(X_T^u,B_T^1)$ is of the form $\xi_T(Z,\Gamma)$, where the vector Z_t has entries given by $u_x(t,X_t^u,B_t^1)$ and $u_y(t,X_t^u,B_t^1)$, and where the matrix Γ_t has diagonal entries given by $(u_{xx}-R_Au_x^2)(t,X_t^u,B_t^1)$, $(u_{yy}-R_Au_y^2)(t,X_t^u,B_t^1)$, and off-diagonal entries given by $(u_{xy}-R_Au_xu_y)(t,X_t^u,B_t^1)$.

4.2 Solving the agent's problem: incentive compatibility

We now show that under the above definition of admissible contracts, one can easily characterize the agent's optimal action. Introduce the set of the controls that are optimal for maximizing g_1^{obs}, given Z,Γ:

$$\mathfrak{V}_1(Z,\Gamma) = \left\{u^*(Z,\Gamma) := (v^*,a^*)(Z,\Gamma) \text{ maximizers of (4.4), satisfying Definition 4.1}\right\}.$$

The next proposition states that for a given contract $\xi_T(Z,\Gamma) \in \mathfrak{C}$, any control $u^*(Z,\Gamma) \in \mathfrak{V}_1(Z,\Gamma)$ is incentive compatible, that is, optimal for the agent.

Proposition 4.1. *An admissible contract $\xi_T(Z,\Gamma) \in \mathfrak{C}$ as defined in* (4.5) *is incentive compatible with $\mathfrak{V}_1(Z,\Gamma)$. That is, given the contract $\xi_T(Z,\Gamma)$, any control $u^*(Z,\Gamma) \in \mathfrak{V}_1(Z,\Gamma)$ is optimal for the agent. Moreover, the corresponding agent's value function satisfies equation* (7.5) *in Appendix, with $(Z^{obs}, Z^{\cancel{obs}}) = (Z,0)$, and $(\Gamma^{obs}, \Gamma^{\cancel{obs}}) = (\Gamma,0)$.*

Proof: Let (Z,Γ) be an arbitrary pair process in $\mathfrak{U}$, and consider the agent's problem with contract $\xi_T(Z,\Gamma)$:

$$V_t^{A,u}\big(\xi_T(Z,\Gamma)\big) := \underset{u' \in \mathscr{U}_t(u)}{\operatorname{essup}}\, \mathbb{E}_t\left[U_A\big(\xi_T(Z,\Gamma) - K_{t,T}^{u'}\big)\right],\ \mathbb{P}_0 - a.s.$$

We first compute, for all $u' := (a',v') \in \mathscr{U}_t(u)$,

$$\begin{aligned} U_A\left(\xi_T(Z,\Gamma)\right) e^{R_A K_{t,T}^{u'}} &= U_A\left(\xi_t(Z,\Gamma)\right) \mathscr{E}\Big(-R_A \int_t^{\cdot} Z_r \cdot \Sigma_r(v_r') dB_r\Big)_T \\ &\quad \times \exp\left(R_A \int_t^T \left[G_1^{\text{obs}}(r,Z_r,\Gamma_r) - g_1^{\text{obs}}(r,Z_r,\Gamma_r,v_r',a_r')\right] dr\right),\ \mathbb{P}_0 - a.s., \end{aligned}$$

where $\xi_t(Z,\Gamma)$ has the same form (4.5) as $\xi_T(Z,\Gamma)$, when we substitute t for T.

Since Z is bounded by definition and σ satisfies the linear growth condition (3.2), we have by definition of $\mathscr{U}$ (see (3.4) in particular) that

$$\mathbb{E}\left[\exp\left(\frac{R_A^2}{2}\int_0^T \left\|\Sigma_r^T(v_r')Z_r\right\|^2 dr\right)\right] < +\infty.$$

Hence, by Novikov criterion, we may define a probability measure $\bar{\mathbb{P}}$ equivalent to $\mathbb{P}_0$ via the density

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}_0}\Big|_{\mathscr{F}_T} = \mathscr{E}\left(-R_A \int_t^{\cdot} Z_r \cdot \Sigma_r(v_r') dB_r\right)_T.$$

Then,

$$\mathbb{E}_t\left[U_A\big(\xi_T(Z,\Gamma)\big)e^{R_A K_{t,T}^{u'}}\right] = U_A\big(\xi_t(Z,\Gamma)\big)\mathbb{E}_t^{\bar{\mathbb{P}}}\left[e^{R_A \int_t^T G_1^{\text{obs}}(r,Z_r,\Gamma_r) - g_1^{\text{obs}}(r,Z_r,\Gamma_r,v_r',a_r')dr}\right].$$

Since $G_1^{\text{obs}} - g_1^{\text{obs}} \geq 0$, we see that

$$\mathbb{E}_t\left[U_A\big(\xi_T(Z,\Gamma)\big)e^{R_A K_{t,T}^{u'}}\right] \leq U_A\big(\xi_t(Z,\Gamma)\big),$$

and by the arbitrariness of $u' \in \mathscr{U}_t(u)$, it follows that $V_t^{A,u}(\xi(Z,\Gamma)) \leq U_A\big(\xi_t(Z,\Gamma)\big)$.

Thus, any control $u^* := (a^*,v^*)$ for which $G_1^{\text{obs}}(r,Z_r,\Gamma_r) = g_1^{\text{obs}}(Z_r,\Gamma_r,v_r^*,a_r^*)$, attains the upper bound. Hence,

$$V_t^{A,u}\big(\xi_T(Z,\Gamma)\big) = U_A\big(\xi_t(Z,\Gamma)\big),$$

and the dynamics of $V_t^{A,u}$ are as stated. $\square$

Remark 4.3. *When U_A is not a CARA utility function, the same approach would work if the agent draws utility/disutility of the form $U_A(\xi_T) - K^u_T$, that is, if the cost of effort is separable from the agent's utility function.* [**] *For example, in the case in which only $X := X^u$ is contractible and $\sigma = I_d$, we would define admissible contracts $\xi_T = \xi_T(Z,\Gamma)$ to be those that satisfy*

$$U_A(\xi_T(Z,\Gamma)) = \tilde{C} + \int_0^T Z_u dX_u + \int_0^T \frac{1}{2}\Gamma_u d\langle X\rangle_u - \int_0^T \widetilde{G}_0(Z_u,\Gamma_u)du,$$

for some constant $\tilde{C}$, some $\mathbb{F}^{X^u}-$predictable processes Z and Γ with values in $\mathbb{R}$, and a process process $\widetilde{G}_0(Z_t,\Gamma_t)$ defined similarly as above. Thus, $U_A(\xi_T)$, rather than ξ_T, would be required to be linear (in the integration sense). However, while the agent's problem would be tractable, the difficulty is that, in general, it would be hard to solve the principal's maximization problem. This framework is considered in more detail in Cvitanić, Possamaï and Touzi (2015).

4.3 Solving the principal's problem

We again assume that the utilities are exponential for both the principal and the agent, that is we have $U_I(x) = -e^{-R_I x}$, $I = R_A, R_P$. We first solve the principal's problem when the contracts are restricted to the class $\mathfrak{C}$.

Fix an admissible $\xi_T(Z,\Gamma) \in \mathfrak{C}$ and introduce the notation

$$Z = \begin{pmatrix} Z^X \\ Z^{B^1} \end{pmatrix} \text{ and } \Gamma = \begin{pmatrix} \Gamma^X & \Gamma^{XB^1} \\ \Gamma^{XB^1} & \Gamma^{B^1} \end{pmatrix}.$$

The principal maximizes the expected utility of her terminal payoff $X^u_T - \xi_T(Z,\Gamma)$. Since the contract $\xi_T(Z,\Gamma)$ is incentive compatible in the sense of Proposition 4.1, the optimal volatility and drift choices by the agent correspond to any $u^*(Z,\Gamma) := (v^*(Z,\Gamma), a^*(Z,\Gamma)) \in \mathfrak{V}_1(Z,\Gamma)$. Then, assuming that the agent lets the principal choose among the control choices that are optimal for the agent, the principal's problem is:

$$U^P_0 := \sup_{(Z,\Gamma)\in\mathfrak{U}\ ,\ u^*(Z,\Gamma)\in\mathfrak{V}_1(Z,\Gamma)} \mathbb{E}\big[U_P\big(X_T^{u^*(Z,\Gamma)} - \xi_T(Z,\Gamma)\big)\big].$$

Denoting $u^* := (v^*, a^*) := (v^*(Z,\Gamma), a^*(Z,\Gamma))$, and substituting the expression for $\xi_T(Z,\Gamma)$, we

[**] A possible interpretation is that the function k represents, in a stylized way, joint effects of the risk aversion to the choice of v and a and of their cost.

get

$$
\begin{aligned}
X_T^{u^*} - \xi_T(Z,\Gamma) = & -C + \int_0^T \Big(\sigma_s^T(v_s^*) b_s(a_s^*)(1-Z_s^X) - \frac{1}{2}\big\|\sigma_s(v_s^*)\big\|^2 \big(\Gamma_s^X + R_A\big|Z_s^X\big|^2\big)\Big) ds \\
& + \int_0^T \Big(G_1^{\mathrm{obs}}(s,Z_s,\Gamma_s) - \frac{1}{2}\big(\Gamma_s^{B^1} + R_A\big|Z_s^{B^1}\big|^2\big)\Big) ds \\
& - \int_0^T \big(\Gamma_s^{XB^1} + R_A Z_s^X Z_s^{B^1}\big) d\langle X, B^1\rangle_s \\
& - \int_0^T Z_s^{B^1} d(B^{\mathrm{obs},u^*})_s^1 + \int_0^T \big(1-Z_s^X\big) v_s^* \cdot \sigma dB_s^{u^*},\ \mathbb{P}_0 - \text{a.s.}
\end{aligned}
$$

Introduce a vector $\theta^* := \sigma_s(v_s^*)$ and denote its first entry $\theta_1^*(s)$, and denote by $\theta_{-1}^*(s)$ the $(d-1)$-dimensional vector without the first entry. Arguing exactly as in the proof of Proposition 4.1, in particular, by isolating the appropriate stochastic exponential, it follows that the principal problem reduces to maximizing

$$
\begin{aligned}
& \theta_s^*(Z,\Gamma)\cdot b_s(a_s^*(Z,\Gamma))\left(1-Z^X\right) - \frac{1}{2}\left\|\theta_s^*(Z,\Gamma)\right\|^2\left(\Gamma^X + R_A\big|Z^X\big|^2\right) + G_1^{\mathrm{obs}}(s,Z,\Gamma) \\
& \quad - \frac{1}{2}\big(\Gamma^{B^1} + R_A\big|Z^{B^1}\big|^2\big) - \big(\Gamma^{XB^1} + R_A Z^X Z^{B^1}\big)\theta_1(s,Z,\Gamma) \\
& \quad - \frac{R_P}{2}\left[\big\|\theta_{-1}^*(s,Z,\Gamma)\big\|^2\left(1-Z^X\right)^2 + \big(\theta_1^*(s,Z,\Gamma)\big(1-Z^X\big) - Z^{B^1}\big)^2\right].
\end{aligned} \tag{4.6}
$$

Since the supremum in the definition of $G_1^{\mathrm{obs}}(s,Z,\Gamma)$ is attained at (v^*,a^*), this is equivalent to the minimization problem

$$
\begin{aligned}
& \inf_{Z,\Gamma,v^*(Z,\Gamma),a^*(Z,\Gamma)} \Big\{ -\theta^*(s,Z,\Gamma)\cdot b_s(a_s^*(Z,\Gamma)) + \frac{1}{2}\left\|\theta^*(Z,\Gamma)\right\|^2 R_A\left(Z^X\right)^2 + k(v^*,a^*) + \frac{R_A}{2}\left(Z^{B^1}\right)^2 \\
& + R_A Z^X Z^{B^1}\theta_1^*(s,Z,\Gamma) + \frac{R_P}{2}\Big[\left\|\theta_s^*(Z,\Gamma)\right\|^2\left(1-Z^X\right)^2 - 2\theta_1^*(s,Z,\Gamma)\left(1-Z^X\right)Z^{B^1} + \left(Z^{B^1}\right)^2\Big]\Big\}.
\end{aligned} \tag{4.7}
$$

Note that if minimizers Z^*, Γ^*, v^* and a^* exist, they are then necessarily deterministic, since b, σ and k are non-random. By Proposition 4.1, the contract $\xi_T(Z^*,\Gamma^*)$ is incentive compatible for $(v^*(Z^*,\Gamma^*), a^*(Z^*,\Gamma^*))$, if $\xi_T(Z^*,\Gamma^*) \in \mathfrak{C}$. Moreover, as we have just shown, it is also optimal for the principal's problem. We have thus proved

Theorem 4.1. *Consider the set of admissible contracts $\xi_T(Z,\Gamma) \in \mathfrak{C}$. Then, the contract that is optimal in that set and provides the agent with expected utility $V_0^A < 0$ is $\xi_T(Z^*,\Gamma^*)$ corresponding to Z^*,Γ^*,v^*,a^* which are the minimizers in* (4.7)*, provided such minimizers exist. Moreover, the contract cash constant $\tilde{C}$ is given by $\tilde{C} := -\frac{1}{R_A}\log(-V_0^A)$.*

Proof. The only thing to check here is the admissibility of the contract $\xi_T(Z^*,\Gamma^*)$, but this is just a consequence of the optimizers being deterministic.

Next, we provide here sufficient conditions for existence of at least one minimizer of (4.7) when, for simplicity, there is no optimization with respect to a, and when the cost function k is super-quadratic. The case of a quadratic k is actually harder and is treated in Appendix in Proposition 7.1.

Proposition 4.2. *Assume that the agent does not control the drift, i.e. $a = 0$, and consider the setting of* (3.3) *with $\mathscr{V} = \mathbb{R}^d$. Assume moreover that the cost function $k(v) := k(v,0)$ is at least C^1 and satisfies for some constant $C > 0$*

$$\|\nabla k(v)\| \leq C\left(1 + \|v\|^{1+\varepsilon}\right), \textit{ for some } \varepsilon > 0, \textit{ and } \varliminf_{\|v\|\to+\infty} \frac{k(v)}{\|v\|^2} = +\infty.$$

Then, the infimum in (4.7) *is attained.*

4.4 How general is the class of restricted contracts $\mathfrak{C}$?

The question arises whether the principal could be better off using a feasible contract payoff which does not belong to $\mathfrak{C}$. We say that a contract payoff ξ_T is feasible if the value function of the agent is well-defined and admits at least one optimal control u^*. We call this class $\overline{\mathfrak{C}}$. It contains the set $\mathfrak{C}$ of restricted contracts.

We state here two results, proved in Appendix, showing that our approach presents no loss of generality. Namely, under technical conditions, we show: (i) when the volatility is not controlled by the agent, but fixed, as in Homstrom-Milgrom (1987), all feasible contracts ξ_T belong to our family of restricted contracts; (ii) in the general case in which the volatility may be controlled by the agent, the value function of the principal when maximizing over $\mathfrak{C}$ is the same when maximizing over $\overline{\mathfrak{C}}$. [††]

More precisely, we have

Theorem 4.2. (i) *If v is uncontrolled and fixed to $v_0 \in \mathscr{V}$, and if there is a constant $C > 0$ and $\varepsilon \in [1, +\infty)$ such that*

$$\varlimsup_{\|a\|\to+\infty} \frac{k(v_0,a)}{\|a\|} = +\infty,\ \|D_a k(v_0,a)\| \leq C\left(1 + \|a\|^{\varepsilon}\right),$$

(a *condition satisfied in the case of quadratic cost*)*, then any $\mathscr{F}_T^{obs}$-measurable random variable ξ_T for which the agent's value function is well-defined can be represented as in* (4.5)[‡‡].

(ii) *In the general case with volatility control, a feasible contract payoff can be written in the form* (4.5) *if the non-martingale component of the agent's value function has a certain "smoothness" property, as in Assumption 7.1 in Appendix.*

[††] For simplicity, we state and prove the latter result in the case in which the agent does not control the drift a.

[‡‡] With a possibly unbounded process Z, but see the discussion after Definition 4.1.

(iii) *However, even if not all feasible contract payoffs can be written in the form* (4.5)*, under the assumptions of Proposition* 4.2 *we have*

$$U_0^P = U_0^{P,w},$$

where $U_0^{P,w}$ is the value function of the principal (defined in (7.17) *in Appendix) when she maximizes over contracts in $\overline{\mathfrak{C}}$.*

The proofs of these statements, provided in Appendix, rely on the possibility of the HJB characterization of the value function in the non-Markovian case, which has been approached by introducing and studying so-called second-order BSDEs; see, e.g., Soner, Touzi and Zhang (2012) and more recently Possamaï, Tan and Zhou (2015). However, identification (or even existence) of the optimal control is a very hard task, which may require strong regularity assumptions. Instead, using results of that recent theory, it is shown that any feasible contract payoff can be approximated (in some sense) by contract payoffs of the form (4.5), thus showing that the restriction to the contract payoffs in $\mathfrak{C}$ is without loss of generality. [§§]

5 Second-best with non-contractible risks

Consider now the case in which the only contractible process is $X^{v,a}$. In that case, we need to modify our approach by adopting the following changes, as can be verified using similar arguments. First of all, the principal can now only offer contract payoffs measurable with respect to $\mathscr{F}_T^{X^u}$, a sigma-field contained in the filtration $\mathbb{F}^{X^u} := \{\mathscr{F}_t^{X^u}\}_{0\leq t\leq T}$ generated by the output process X^u.

Similarly as in the previous section, we introduce the function $G_0^{\rm obs}$, the counterpart of the function $G_1^{\rm obs}$ above, defined for any $(s,z,\gamma)\in[0,T]\times\mathbb{R}\times\mathbb{R}$ by

$$\begin{aligned}G_0^{\rm obs}(s,z,\gamma) &:= \sup_{(v,a)\in\mathscr{V}\times\mathscr{A}} g_0^{\rm obs}(s,z,\gamma,v,a)\\ &:= \sup_{(v,a)\in\mathscr{V}\times\mathscr{A}} \left\{\sigma_s^T(v)b_s(a)z+\frac{1}{2}\left\|\sigma_s(v)\right\|^2\gamma-k(v,a)\right\},\ \ z,\gamma\in\mathbb{R}.\end{aligned}$$

Once again, if a maximizer exists, we denote it by $(v^*(z,\gamma),a^*(z,\gamma))$. We now introduce the set of admissible contracts in this case.

Definition 5.1. *An admissible contract payoff $\xi_T=\xi_T(Z,\Gamma)$ is an $\mathscr{F}_T^{X^u}-$measurable random variable that satisfies*

$$\xi_T^0(Z,\Gamma) := \tilde{C}+\int_0^T\left\{Z_t dX_t^u - G_0^{obs}(t,Z_t,\Gamma_t)dt+\frac{1}{2}\big(\Gamma_t+R_A Z_t^2\big)d\langle X^u\rangle_t\right\}. \tag{5.1}$$

[§§]This result is the main focus of the follow-up paper Cvitanić, Possamaï, Touzi (2015), which considers a more general framework, but with non-CARA utility functions and with the cost separable from utility.

for some constant $\tilde{C}$, and some pair (Z,Γ) of bounded $\mathbb{F}^{X^u}$−predictable processes with values in $\mathbb{R}$, and such that there is a maximizer $(v^*(Z,\Gamma),a^*(Z,\Gamma)) \in \mathscr{U}$ of $g_0^{obs}(\cdot,Z,\Gamma)$, $dt \times d\mathbb{P}_0$-a.e..

We denote by $\mathfrak{C}_0$ the set of all admissible contracts, and by $\mathfrak{U}_0$ the set of the corresponding (Z,Γ).

Similarly as before, we introduce the set of controls that are optimal for maximizing g_0^{obs}, given Z,Γ:

$$\mathfrak{V}_0(Z,\Gamma) = \{(v^*(Z,\Gamma),a^*(Z,\Gamma)), \text{ such that the conditions of Definition 5.1 are satisfied}\}.$$

The following proposition is the analogue of Proposition 4.1 in this setting, and can be proved by the same argument.

Proposition 5.1. *An admissible contract $\xi_T^0(Z,\Gamma)$, as defined in* (5.1)*, is incentive compatible with $\mathfrak{V}_0(Z,\Gamma)$. That is, given the contract $\xi_T^0(Z,\Gamma)$, any control in $\mathfrak{V}_0(Z,\Gamma)$ is optimal for the agent's problem.*

Accordingly, the principal's problem is modified as follows, denoting again for notational simplicity $u^* := (v^*,a^*) := (v^*(Z,\Gamma),a^*(Z,\Gamma))$, and assuming $U_P(x) = -e^{-R_P x}$:

$$\inf_{(Z,\Gamma,v^*,a^*)\in\mathfrak{U}_0\times\mathfrak{V}_0(Z,\Gamma)} \mathbb{E}\Bigg[\exp\Bigg\{R_P\Bigg(\int_0^T \Bigg(\sigma_s^T(v_s^*)b_s(a_s^*)(Z_s-1)+\frac{1}{2}\|\sigma_s(v_s^*)\|^2\left(\Gamma_s+R_A Z_s^2\right)\Bigg)ds$$
$$-\int_0^T G_0^{\text{obs}}(u,Z_u,\Gamma_u)du+\int_0^T (Z_s-1)dX_s^{u^*}\Bigg)\Bigg\}\Bigg].$$

Similarly, as above, denote by θ_s^* the vector $\sigma_s(v_s^*)$. The principal's problem then becomes

$$\inf_{Z,\Gamma,v^*,a^*}\Big\{-\theta_s^*(Z,\Gamma)b_s(a_s^*)+\frac{R_A}{2}\|\theta_s^*(Z,\Gamma)\|^2 Z^2+k(v^*,a^*)+\frac{R_P}{2}\|\theta_s^*(Z,\Gamma)\|^2(1-Z)^2\Big\}. \quad (5.2)$$

We then have, similarly as before,

Theorem 5.1. *Consider the set of admissible contracts $\xi_T^0(Z,\Gamma) \in \mathfrak{C}_0$. Then, the optimal contract that provides the agent with optimal expected utility V_0^A is the one corresponding to Z^*,Γ^*,v^*,a^* which are the minimizers in* (5.2)*, provided such minimizers exist with $v^* \neq 0$. Moreover, the contract cash term in the contract is given by $-\frac{1}{R_A}\log(-V_0^A)$.*

The analogue of Proposition 4.2 still holds in this context, with the same statement and the same proof. Similarly for Theorem 4.2.

6 Conclusions

We build a framework for studying moral hazard in dynamic risk management, using recently developed mathematical techniques. While those allow us to solve the problem in which utility is drawn solely from terminal payoff, we leave for future search a similar problem on infinite horizon with inter-temporal payments, a la Sannikov (2008). In the case of terminal payoff, we find that the optimal contract is implemented by compensation based on the output, its quadratic variation (corresponding in practice to the sample variance used when computing Sharpe ratios), the contractible sources of risk, and the cross-variations between the output and the risk sources. (Or, it could be implemented by derivatives that provide such payoffs.) We apply a novel approach, in which we restrict the family of possible contract payoffs to those that would make the agent's value function well-behaved. We then prove that there is essentially no loss of generality in doing so. In our framework it is assumed that the principal knows the model parameters (for example, the mean return rates and the variance-covariance matrix of the returns of the assets the hedge fund manager is investing in). In practice, the principal may not have that information, and it would be of interest to extend the model to include this adverse selection problem.

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7 Appendix

7.0.1 Motivating the definition of admissibility

Here, we motivate our definition of admissibility of contracts. The motivation is conceptually relatively simple, based on the dynamic programming representation of the agent's value function, but it requires using heavy notation.

It is convenient to introduce an auxiliary non-contractible factor of the form

$$B^{\mathrm{obs},u} := \int_0^{\cdot} \Sigma_s^{\perp}(v_s)dB_s$$

where $\Sigma_s^{\perp}$ is a $(d-2)\times d$ matrix orthogonal to Σ. ¶¶ Notice that the agent's value function $V_t^{A,u}$

¶¶That is, we have, for any $v \in \mathscr{V}$ and any $s \in [0,T]$,

$$\Sigma_s^{\perp}(v)\Sigma_s^T(v) = 0_{d-2,2} \text{ and } \Sigma_s^{\perp}(v)\left(\Sigma_s^{\perp}\right)^T(v) = I_{d-2}.$$

is an $\mathscr{F}_t-$measurable function and can thus be written in a functional form as

$$V_t^{A,u} \;=\; V^A\big(t, B_\cdot^{\text{obs},u}, B_\cdot^{\cancel{\text{obs}},u}\big) \;=\; V^A\big(t, (B_s^{\text{obs},u})_{s\le t}, (B_s^{\cancel{\text{obs}},u})_{s\le t}\big).$$

If the value function is smooth in the sense of the Dupire (2009) functional differentiation (see also Cont and Fournié (2013) for more details), then, the Dupire time derivative $\partial_t V^{A,u}$ exists, and one can find predictable processes

$$\tilde{Z}^u = \begin{pmatrix} \tilde{Z}^{\text{obs},u} \\ \tilde{Z}^{\cancel{\text{obs}},u} \end{pmatrix} \quad \text{and} \quad \tilde{\Gamma}^u = \begin{pmatrix} \tilde{\Gamma}^{\text{obs},u} & 0 \\ 0 & \tilde{\Gamma}^{\cancel{\text{obs}},u} \end{pmatrix},$$

with $\tilde{Z}^{\text{obs},u}, \tilde{Z}^{\cancel{\text{obs}},u}$ taking values in $\mathbb{R}^2$ and $\mathbb{R}^{d-2}$, respectively, $\tilde{\Gamma}^{\text{obs},u}, \tilde{\Gamma}^{\cancel{\text{obs}},u}$ taking values in the spaces $\mathbb{S}_2$ and $\mathbb{S}_{d-2}$ of symmetric matrices, respectively, such that we have the following generalization of Itô's rule (see Theorem 1 in Dupire (2009) or Theorem 4.1 in Cont and Fournié (2013)),

$$\begin{aligned} dV_t^{A,u} &= \partial_t V_t^{A,u} + \frac{1}{2}\mathrm{Tr}\Big[\tilde{\Gamma}_t^{\text{obs},u} d\langle B^{\text{obs},u}\rangle_t\Big] + \frac{1}{2}\mathrm{Tr}\Big[\tilde{\Gamma}_t^{\cancel{\text{obs}},u} d\langle B^{\cancel{\text{obs}},u}\rangle_t\Big] + \tilde{Z}_t^{\text{obs},u}\cdot dB_t^{\text{obs},u} \\ &\quad + \tilde{Z}_t^{\cancel{\text{obs}},u}\cdot dB_t^{\cancel{\text{obs}},u} \\ &= \Big(\partial_t V_t^{A,u} + \frac{1}{2}\mathrm{Tr}\big[\tilde{\Gamma}_t^{\text{obs},u}\Sigma_t(v_t)\Sigma_t^T(v_t)\big] + \frac{1}{2}\mathrm{Tr}\big[\tilde{\Gamma}_t^{\cancel{\text{obs}},u}\big] + \tilde{Z}_t^{\text{obs},u}\cdot\mu(v_t,a_t)\Big)dt \\ &\quad + \Big(\Sigma_t^T(v_t)\tilde{Z}_t^{\text{obs},u} + \Big(\Sigma_t^{\perp}\Big)^T(v_t)\tilde{Z}_t^{\cancel{\text{obs}},u}\Big)\cdot dB_t,\ \mathbb{P}_0 - a.s. \end{aligned} \tag{7.2}$$

Here the matrix $\tilde{\Gamma}^u$ is bloc-diagonal without loss of generality because $\langle B^{\text{obs},u}, B^{\cancel{\text{obs}},u}\rangle_t = 0$. The previous decomposition is motivated by the Markov case, in which $V_t^{A,u} = f(t, B_t^{\text{obs},u}, B_t^{\cancel{\text{obs}},u})$ for some smooth function $f(t,x,y)$, and it follows from Itô's rule that the processes $\tilde{Z}^{\text{obs},u}, \tilde{Z}^{\cancel{\text{obs}},u}$ and $\tilde{\Gamma}^{\text{obs},u}, \tilde{\Gamma}^{\cancel{\text{obs}},u}$ are given by

$$\begin{aligned} \tilde{Z}_t^{\text{obs},u} &= \partial_x f(t, B_t^{\text{obs},u}, B_t^{\cancel{\text{obs}},u}), &\quad \tilde{Z}_t^{\cancel{\text{obs}},u} &= \partial_y f(t, B_t^{\text{obs},u}, B_t^{\cancel{\text{obs}},u}), \\ \tilde{\Gamma}_t^{\text{obs},u} &= \partial_{xx} f(t, B_t^{\text{obs},u}, B_t^{\cancel{\text{obs}},u}), &\quad \tilde{\Gamma}_t^{\cancel{\text{obs}},u} &= \partial_{yy} f(t, B_t^{\text{obs},u}, B_t^{\cancel{\text{obs}},u}), \end{aligned}$$

with $\partial_x, \partial_{xx}, \partial_y, \partial_{yy}$ denoting the partial derivatives with respect to the corresponding variables.

Next, from the martingale optimality principle of the classical stochastic control theory, the dynamic programming principle suggests that the process $V_t^{A,u} e^{R_A K_{0,t}^u}$ should be a supermartingale

Note that vector $\Big(B^{\text{obs},u} \quad B^{\cancel{\text{obs}},u}\Big)^T$ generates the same filtration $\mathbb{F}$ as B if and only if the density of its quadratic variation is invertible. It can be shown that the necessary and sufficient condition for it is

$$\|\sigma_s(v_s)\|^2 - \sum_{i=1}^{d_0}\big|\sigma_s^i(v_s)\big|^2 \neq 0. \tag{7.1}$$

In the case of portfolio management problem with $a=0$ and $\sigma = I_d$, this condition means that the investor has to invest in at least one of the non-contractible sources of risk. When there are no contractible source of risk (that is when $d_0 = 0$), this condition reduces to $\sigma_s(v_s) \neq 0_{1,d}$.

for all admissible controls u, and that it should be a martingale for any optimal control u^*, provided such exists. Writing formally that the drift coefficient of the supermartingale $V_t^{A,u}e^{R_A K_{0,t}^u}$ is non-positive, and that of the martingale corresponding to u^* must vanish, we obtain the following path-dependent HJB (Hamilton-Jacobi-Bellman) equation:

$$-\partial_t V_t^{A,u} + R_A V_t^{A,u} G_1(t, Z_t^u, \Gamma_t^u) = 0, \text{ where } (Z_t^u, \Gamma_t^u) := -\frac{1}{R_A V_t^{A,u}} \left(\tilde{Z}_t^u, \tilde{\Gamma}_t^u\right), \tag{7.3}$$

and where

$$G_1(t,z,\gamma) := \sup_{(v,a)\in\mathscr{V}\times\mathscr{A}} g_1(t,z,\gamma,v,a), \tag{7.4}$$

with

$$g_1(t,z,\gamma,v,a) \quad := \quad -k(v,a) + z^{\mathrm{obs}} \cdot \mu_t(v,a) + \frac{1}{2}\mathrm{Tr}\big[\gamma^{\mathrm{obs}}\Sigma_t(v)\Sigma_t^T(v)\big] + \frac{1}{2}\mathrm{Tr}\big[\gamma^{\cancel{\mathrm{obs}}}\big].$$

Substituting the above into (7.2), it follows that, $\mathbb{P}_0 - a.s.$,

$$\begin{aligned} d(V_t^{A,u} e^{R_A K_{0,t}^u}) = & -R_A V_t^{A,u} e^{R_A K_{0,t}^u} \big(g_1(t, Z_t^u, \Gamma_t^u, v_t, a_t) - G_1(t, Z_t^u, \Gamma_t^u)\big) dt \\ & -R_A V_t^{A,u} e^{R_A K_{0,t}^u} \Big(\Sigma_t(v_t)^T Z_t^{\mathrm{obs},u} + \big(\Sigma_t^{\perp}\big)^T (v_t) Z_t^{\cancel{\mathrm{obs}},u}\Big) \cdot dB_t. \end{aligned} \tag{7.5}$$

We then see by directly solving the latter stochastic differential equation that

$$\begin{aligned} V_T^{A,u} = V_0^A \exp & \left[R_A \left(\int_0^T \left(G_1(t, Z_t^u, \Gamma_t^u) - \frac{1}{2}\mathrm{Tr}\big[\Gamma_t^{\mathrm{obs},u} d\langle B^{\mathrm{obs},u}\rangle_t\big] - \frac{1}{2}\mathrm{Tr}\big[\Gamma_t^{\cancel{\mathrm{obs}},u}\big]\right) dt\right)\right] \\ \times \exp & \left[-R_A \left(\int_0^T Z_t^{\mathrm{obs},u} \cdot dB_t^{\mathrm{obs},u} + \frac{R_A}{2}\int_0^T Z_t^{\mathrm{obs},u} \cdot d\langle B^{\mathrm{obs},u}\rangle_t Z_t^{\mathrm{obs},u}\right)\right] \\ \times \exp & \left[-R_A \left(\int_0^T Z_t^{\cancel{\mathrm{obs}},u} \cdot dB_t^{\cancel{\mathrm{obs}},u} + \frac{R_A}{2}\int_0^T \left\| Z_t^{\cancel{\mathrm{obs}},u}\right\|^2 dt\right)\right], \ \mathbb{P}_0 - a.s. \end{aligned}$$

Next, we recall that the principal must offer a contract based on the information set $\mathbb{F}^{\mathrm{obs}}$ only. From the definition of G_1 we can check that the expression for $\xi_T = -\frac{1}{R_A}\log(V_T^{A,u})$ does not depend on $\Gamma^{\cancel{\mathrm{obs}},u}$, and we expect it also not to depend on $Z^{\cancel{\mathrm{obs}},u}$, that is, to have $Z^{\cancel{\mathrm{obs}},u} \equiv 0$. In that case, from (4.2), denoting

$$G_1^{\mathrm{obs}}(t, z^{\mathrm{obs}}, \gamma^{\mathrm{obs}}) \quad := \quad \sup_{(v,a)\in\mathscr{V}\times\mathscr{A}} g_1^{\mathrm{obs}}(t, z^{\mathrm{obs}}, \gamma^{\mathrm{obs}}, v, a), \tag{7.6}$$

and

$$g_1^{\mathrm{obs}}(t, z^{\mathrm{obs}}, \gamma^{\mathrm{obs}}, v, a) := -k(v,a) + z^{\mathrm{obs}} \cdot \mu_t(v,a) + \frac{1}{2}\mathrm{Tr}\big[\gamma^{\mathrm{obs}}\Sigma_t(v)\Sigma_t^T(v)\big].$$

the contract payoff $\xi_T = -\frac{1}{R_A}\log(V_T^{A,u})$ would be as in Definition 4.1.

7.1 Technical proofs

Proof of Proposition 4.2. Notice first that the assumption on k implies that it has a super quadratic growth at infinity, which means that for every $(z,\gamma)\in\mathbb{R}^2\times\mathbb{S}^2$, the infimum in the definition of $G_1^{\mathrm{obs}}(z,\gamma)$ is always attained for at least one $v^*(z,\gamma)$. Moreover, as it is then an interior maximizer, it necessarily satisfies the first-order conditions, which can be rewritten here as

$$Mz+\gamma\tilde{M}v^*(z,\gamma)-\nabla k(v^*(z,\gamma))=0, \tag{7.7}$$

for some matrices M and $\tilde{M}$, independent of (z,γ).

In particular, the above shows that $v^*(z,\gamma)$ cannot remain bounded as $\|z\|$ and $\|\gamma\|$ go to $+\infty$. Let us also verify that we have

$$\|v^*(z,\gamma)\|\leq C_0\left(1+\|z\|^{\frac{1}{1+\varepsilon}}+\|\gamma\|^{\frac{1}{\varepsilon}}\right),$$

for some constant $C_0>0$.

Indeed assume first that $\|v^*(z,\gamma)\|/\|z\|^{\frac{1}{1+\varepsilon}}$ does not remain bounded when $\|z\|$ goes to $+\infty$. Then, we deduce from (7.7) that

$$M\frac{z}{\|z\|^{\frac{1}{1+\varepsilon}}}+\gamma\tilde{M}\frac{v^*(z,\gamma)}{\|z\|^{\frac{1}{1+\varepsilon}}}-\frac{\nabla k(v^*(z,\gamma))}{\|v^*(z,\gamma)\|^{1+\varepsilon}}\frac{\|v^*(z,\gamma)\|^{1+\varepsilon}}{\|z\|}\|z\|^{\frac{\varepsilon}{1+\varepsilon}}=0.$$

As $\|z\|$ goes to $+\infty$, the third term above then clearly dominates the other two, which contradicts the fact that their sum should be 0.

Similarly, if we assume that $\|v^*(z,\gamma)\|/\|\gamma\|^{\frac{1}{\varepsilon}}$ does not remain bounded when $\|\gamma\|$ goes to $+\infty$, we deduce that

$$M\frac{z}{|\gamma|^{\frac{1}{\varepsilon}}}+\gamma\tilde{M}\frac{v^*(z,\gamma)}{\|\gamma\|^{\frac{1}{\varepsilon}}}-\frac{\nabla k(v^*(z,\gamma))}{\|v^*(z,\gamma)\|^{1+\varepsilon}}\frac{\|v^*(z,\gamma)\|^{1+\varepsilon}}{\|\gamma\|^{\frac{1+\varepsilon}{\varepsilon}}}\|\gamma\|=0.$$

Again, the third term dominates the others as $\|\gamma\|$ goes to $+\infty$, which contradicts the equality.

We next deduce that for every $\eta>0$

$$\frac{\|v^*(z,\gamma)\|}{1+\|z\|^{\frac{1}{1+\varepsilon}-\eta}+\|\gamma\|^{\frac{1}{\varepsilon}-\eta}}\text{ is not bounded as }\|z\|\text{ and }\|\gamma\|\text{ go to }+\infty.$$

Indeed, if we assume that $\|v^*(z,\gamma)\|/\|z\|^{\frac{1}{1+\varepsilon}-\eta}$ remains bounded near infinity, then dividing (7.7) by $\|z\|^{1/(1+\varepsilon)-\eta}$, we obtain that the first term behaves, as $\|z\|$ goes to $+\infty$, like $\|z\|^{\varepsilon/(1+\varepsilon)+\eta}$, while the second one is bounded and the third one behaves like $\|z\|^{\varepsilon/(1+\varepsilon)-\varepsilon\eta}$. Hence, the first term dominates and we again have a contradiction. The result for the growth with respect to γ is proved in the same manner.

From the above growth for $v^*(z,\gamma)$, it is clear that the dominating terms at infinity in (4.7) are

$$\frac{1}{2}\|\theta^*(Z,\Gamma)\|^2R_A\left(Z^X\right)^2,\ \frac{R_P}{2}\|\theta^*(Z,\Gamma)\|^2\left(1-Z^X\right)^2,\text{ and }k(v^*(z,\gamma)),$$

which are all non-negative. In particular, (4.7) goes to $+\infty$ as $\|z\|$ and $\|\gamma\|$ go to $+\infty$, and the minimum is therefore attained. $\square$

Proof of Theorem 4.2. We adapt arguments of Soner, Touzi & Zhang (2011, 2012, 2013). In Step 1 we transform our problem to the weak formulation used in those papers. The agent's problem is then analyzed in Step 2, which relates to the statement (ii) of the theorem. Step 3 specializes to the case of uncontrolled volatility, proving statement (i). Finally, in Step 4, we prove statement (iii) of the theorem.

Step 1: An alternative formulation of the agent's problem

Let us consider the following family of processes, indexed by admissible processes v:

$$M^v_\cdot := \begin{pmatrix} \int_0^\cdot \sigma_s(v_s)\cdot dB_s \\ B^1_\cdot \\ \vdots \\ B^{d_0}_\cdot \\ \int_0^\cdot \Sigma_s^\perp(v_s)dB_s \end{pmatrix} = \begin{pmatrix} \int_0^\cdot \Sigma_s(v_s)dB_s \\ \int_0^\cdot \Sigma_s^\perp(v_s)dB_s \end{pmatrix},\ \mathbb{P}_0 - a.s., \tag{7.8}$$

where, similarly as in Section 4.1, for any $s\in[0,T]$ and any $v\in\mathscr{V}$ the $(d_0+1)\times d$ matrix $\Sigma(v)$ is defined by

$$\Sigma_s(v) := \begin{pmatrix} \sigma_s^T(v) \\ I_{d_0,d} \end{pmatrix}, \text{with } I_{d_0,d} := \begin{pmatrix} I_{d_0} & 0_{d_0,d-d_0} \end{pmatrix}.$$

Furthermore, $\Sigma_s^\perp$ is now a $(d-d_0-1)\times d$ matrix satisfying, for any $s\in[0,T]$ and any $v\in\mathscr{V}$,

$$\Sigma_s^\perp(v)\Sigma_s(v)^T = 0_{d-d_0-1,d_0+1} \text{ and } \Sigma_s^\perp(v)(\Sigma_s^\perp)^T(v) = I_{d-d_0-1}.$$

We then set $\mathscr{P}_m$ to be the set of probability measures $\mathbb{P}^v$ on $(\Omega,\mathscr{F})$ of the form

$$\mathbb{P}^v := \mathbb{P}_0\circ(M^v_\cdot)^{-1},\ \text{for any admissible } v. \tag{7.9}$$

We recall that by Bichteler (1981), we can also define a pathwise version of the quadratic variation process $\langle B\rangle$ and of its density process with respect to the Lebesgue measure, a positive symmetric matrix $\widehat{\alpha}$:

$$\widehat{\alpha}_t := \frac{d\langle B\rangle_t}{dt}.$$

The elements of family $\mathscr{P}_m$ correspond to possible choices of the volatility vector $\sigma_s(v)$ by the agent.*** We remark that by our definitions, we have the following weak formulation:

$$\text{The law of } (B,\widehat{\alpha}) \text{ under } \mathbb{P}^v = \text{the law of } \left(M^v, \begin{pmatrix} \Sigma_s(v)\Sigma_s(v)^T & 0_{d_0+1,d-d_0-1} \\ 0_{d-d_0-1,d_0+1} & I_{d-d_0-1} \end{pmatrix}\right) \text{ under } \mathbb{P}_0.$$

***This family could also be characterized by considering all the choices of control u for which there exists at least one strong solution to the SDE for M (see Soner, Touzi and Zhang (2011)), or, equivalently, to a certain martingale problem (see Kazi-Tani, Possamaï and Zhou (2013) and Neufeld and Nutz (2014)).

Moreover, exactly as in Section 4.1, the density of the quadratic variation of B is invertible under $\mathbb{P}^\nu$, if and only if $\Sigma_s(\nu)\Sigma_s(\nu)^T$ is invertible, which can be shown to be equivalent to

$$\|\sigma_s(\nu_s)\|^2 - \sum_{i=1}^{d_0} \left|\sigma_s^i(\nu_s)\right|^2 \neq 0,$$

which is the same as (7.1).

Then, according to Lemma 2.2 in Soner, Touzi, Zhang (2013), for every admissible ν, there exists a $\mathbb{F}$-progressively measurable mapping $\beta_\nu : [0,T]\times\Omega \longrightarrow \mathbb{R}^d$ such that

$$B = \beta_\nu(M^\nu),\ \mathbb{P}_0 - a.s.,\ \widehat{\alpha}_s(B) = \begin{pmatrix} \Sigma_s(\beta_\nu(B))\Sigma_s^T(\beta_\nu(B)) & 0_{d_0+1,d-d_0-1} \\ 0_{d-d_0-1,d_0+1} & I_{d-d_0-1} \end{pmatrix},\ \mathbb{P}^\nu - a.s.$$

This implies in particular, that the process

$$W_t := \int_0^t \alpha_s^{-1/2} dB_s,\ \mathbb{P} - a.s, \tag{7.10}$$

is a $\mathbb{R}^d$-valued, $\mathbb{P}$-Brownian motion, for every $\mathbb{P} \in \mathscr{P}_m$ [†††]. In particular, this implies that the canonical process B admits the following dynamics, for every admissible ν,

$$\begin{aligned} B_t^1 &= B_0^1 + \int_0^t \sigma_s(\nu_s(W_\cdot))\cdot dW_s,\ \mathbb{P}^\nu - a.s., \\ B_t^j &= B_0^j + W_t^{j-1},\ \mathbb{P}^\nu - a.s.,\ j = 2\dots d_0+1,\ \text{if } d_0>0, \\ \begin{pmatrix} B_t^{d_0+2} \\ \vdots \\ B_t^d \end{pmatrix} &= \begin{pmatrix} B_0^{d_0+2} \\ \vdots \\ B_0^d \end{pmatrix} + \int_0^t \Sigma_s^\perp(\nu_s(W_\cdot))\, dW_s,\ \mathbb{P}^\nu - a.s. \end{aligned} \tag{7.11}$$

Thus, the first coordinate of the canonical process is the desired output process, observed by both the principal and the agent, the following d_0 coordinates represent the contractible sources of risk, while the remaining ones represent the factors that are not contractible.

The introduction of the controlled drift $b_s(a_s)$ can now be done by using Girsanov transformations [‡‡‡]. We define for any $(\nu,a)\in\mathscr{U}$ and any $\mathbb{P}^\nu\in\mathscr{P}_m$, the equivalent probability measure $\mathbb{P}^{\nu,a}$ by

$$\frac{d\mathbb{P}^{\nu,a}}{d\mathbb{P}^\nu} := \mathscr{E}\left(\int_0^\cdot b_s(a_s)\cdot dW_s\right)_T,$$

[†††]Notice that we should actually have considered a family $(W^{\mathbb{P}})_{\mathbb{P}\in\mathscr{P}_m}$, since the stochastic integral in (7.10) is, a priori, only defined $\mathbb{P}$-a.s. However, we can use results of Nutz (2012) to provide an aggregated version of this family, which is the process we denote by W. That result holds under a "good" choice of set theoretic axioms that we do not specify here.

[‡‡‡]Note that by assumption that the first d_0 entries of vector b do not depend on a, the choice of control a does not modify the distribution of the exogenous sources of risk $(B^2,\dots,B^{d_0+1})$.

and we denote by $\mathscr{P} := (\mathbb{P}^{v,a})_{(v,a)\in\mathscr{U}}$.

Next, by Girsanov theorem, the following process W^a is a $\mathbb{P}^{v,a}$-Brownian motion

$$W_t^a := W_t - \int_0^t b_s(a_s)ds,\ \mathbb{P}^{v,a} - a.s.\ \text{(thus also } \mathbb{P}^v - a.s.\text{)}$$

Then, by (7.11), we have

$$\begin{aligned}
B_t^1 &= B_0^1 + \int_0^t \sigma_s(v_s(W_\cdot))\cdot(b_s(a_s)ds + dW_s^a),\ \mathbb{P}^{v,a} - a.s.,\\
B_t^j &= B_0^j + W_t^{j-1},\ \mathbb{P}^{v,a} - a.s.,\ j = 2\ldots d_0+1,\\
\begin{pmatrix} B_t^{d_0+2} \\ \vdots \\ B_t^d \end{pmatrix} &= \begin{pmatrix} B_0^{d_0+2} \\ \vdots \\ B_0^d \end{pmatrix} + \int_0^t \Sigma_s^{\perp}\left(v_s(W_\cdot)\right)dW_s,\ \mathbb{P}^{v,a} - a.s.,
\end{aligned} \tag{7.12}$$

which can then be rewritten as

$$\begin{aligned}
B^{\mathrm{obs}} &:= \begin{pmatrix} B_t^1 \\ \vdots \\ B_t^{d_0+1} \end{pmatrix} = \begin{pmatrix} B_0^1 \\ \vdots \\ B_0^{d_0+1} \end{pmatrix} + \int_0^t \mu_s(v_s(W_\cdot), a_s)ds + \int_0^t \Sigma_s(v_s(W_\cdot))dW_s^a,\ \mathbb{P}^{v,a} - a.s.,\\
B^{\cancel{\mathrm{obs}}} &:= \begin{pmatrix} B_t^{d_0+2} \\ \vdots \\ B_t^d \end{pmatrix} = \begin{pmatrix} B_0^{d_0+2} \\ \vdots \\ B_0^d \end{pmatrix} + \int_0^t \Sigma_s^{\perp}\left(v_s(W_\cdot)\right)dW_s,\ \mathbb{P}^{v,a} - a.s.,
\end{aligned} \tag{7.13}$$

where for any $s \in [0,T]$ and any $(v,a) \in \mathscr{U}$, $\mu_s(v,a)$ is a $\mathbb{R}^{d_0+1}$ vector defined by

$$\mu_s(v,a) := \begin{pmatrix} \sigma_s^T(v)b_s(a) \\ \begin{pmatrix} I_{d_0} & 0_{d_0,d_0-d} \end{pmatrix} \end{pmatrix}.$$

Notice then that, for a given measure $\mathbb{P} \in \mathscr{P}$, according to (7.13), we can always find two m-dimensional and $n-$dimensional vectors $v^{\mathbb{P}}$ and $a^{\mathbb{P}}$ such that

$$B_t^{\mathrm{obs}} = B_0^{\mathrm{obs}} + \int_0^t \mu_s\left(v_s^{\mathbb{P}}, a_s^{\mathbb{P}}\right)ds + \int_0^t \Sigma_s\left(v_s^{\mathbb{P}}\right)dW_s^{a^{\mathbb{P}}},\ \mathbb{P} - a.s.,$$

which gives us the following correspondence

$$v^{\mathbb{P}^{v,a}}(B_\cdot) = v(W_\cdot),\ a^{\mathbb{P}^{v,a}}(B_\cdot) = a(B_\cdot),\ dt \times \mathbb{P}^{v,a} - a.e.$$

In particular, this implies that $v^{\mathbb{P}^{a,v}} = v^{\mathbb{P}^v}$, for any admissible control a. We will therefore always write $v^{\mathbb{P}^v}$ from now on.

Using the above notation, we re-write the value function of the agent as:

$$V_t^{A,w} := \underset{\mathbb{P}'\in\mathscr{P}(t,\mathbb{P})}{\mathrm{essup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[U_A(\xi_T - K_{t,T}^{\mathbb{P}'})\right],\ \mathbb{P} - a.s.,\ \text{for all } \mathbb{P} \in \mathscr{P}, \tag{7.14}$$

where for any $\mathbb{P} \in \mathscr{P}$, the set $\mathscr{P}(t,\mathbb{P})$ is the set of probability measures in $\mathscr{P}$ which agree with $\mathbb{P}$ on $\mathscr{F}_t$, and where we have

$$K_{t,T}^{\mathbb{P}} := \int_t^T k(v_s^{\mathbb{P}}, a_s^{\mathbb{P}})ds.$$

The definition of the value function depends *a priori* explicitly on the measure $\mathbb{P}$, and we should instead have defined a family $(V_t^{A,w,\mathbb{P}})_{\mathbb{P}\in\mathscr{P}}$. Indeed, it is not immediately clear whether this family can be aggregated into a universal process $V^{A,w}$ or not. Such problems are inherent to the weak formulation of stochastic control problems involving volatility control of the diffusion, see Soner, Touzi and Zhang (2011), Nutz and Soner (2012), Nutz and van Handel (2013), Possamaï, Royer and Touzi (2013), Epstein and Ji (2013). In our context, it suffices to remark that following similar arguments as in Section 5 of Possamaï, Royer and Touzi (2013), one can show that family $\mathscr{P}$ satisfies their condition 5.4, and to remark that their approach can be extended to non-martingale measures (as in Nutz and van Handel (2013)). This allows us to define properly the value function of the agent[§§§].

Step 2: Solving the agent's problem

Let us start by fixing some $(v,a) \in \mathscr{U}$. Then, under sufficient integrability conditions for ξ_T, the process $(e^{R_A K_{0,t}^{\mathbb{P}^{v,a}}} V_t^{A,w})_{0\le t\le T}$ is a càdlàg, $\mathbb{P}^{v,a}$-supermartingale for the filtration $\mathbb{F}$, which is equal to $\mathbb{F}^{\mathrm{obs}} \vee \mathbb{F}^{\cancel{\mathrm{obs}}}$. Using results of Soner, Touzi and Zhang (2012), we know that the martingale representation property still holds under $\mathbb{P}^{v,a}$, so that, by Doob-Meyer's decomposition, there exists a pair of processes $\tilde{Z}^{v,a,\mathrm{obs}}$ and $\tilde{Z}^{v,a,\cancel{\mathrm{obs}}}$, which are respectively $\overline{\mathbb{F}^{\mathrm{obs}}}^{\mathbb{P}^{v,a}}$ and $\overline{\mathbb{F}^{\cancel{\mathrm{obs}}}}^{\mathbb{P}^{v,a}}$-predictable process, and an $\overline{\mathbb{F}}^{\mathbb{P}^{v,a}}$-adapted process $\tilde{A}^{v,a}$, which is non-decreasing $\mathbb{P}^{v,a}-a.s.$, such that, after applying Itô's formula, we have the decomposition

$$\begin{aligned} V_t^{A,w} = U_A(\xi_T) &+ \int_t^T R_A V_s^{A,w} k(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}})ds - \int_t^T e^{-R_A K_{0,s}^{\mathbb{P}^{v,a}}} \tilde{Z}_s^{v,a,\mathrm{obs}} \cdot \Sigma_s(v_s^{\mathbb{P}^v}) dW_s^{a^{\mathbb{P}^{v,a}}} \\ &- \int_t^T e^{-R_A K_{0,s}^{\mathbb{P}^{v,a}}} \tilde{Z}_s^{v,a,\cancel{\mathrm{obs}}} \cdot \Sigma_s^{\perp}(v_s^{\mathbb{P}^v}) dW_s^{a^{\mathbb{P}^{v,a}}} + \int_t^T e^{-R_A K_{0,s}^{\mathbb{P}^{v,a}}} d\tilde{A}_s^{v,a},\ \mathbb{P}^{v,a}-a.s. \end{aligned}$$

By definition of W^a, we deduce that, $\mathbb{P}^{v,a}-a.s.$,

$$\begin{aligned} V_t^{A,w} = U_A(\xi_T) &- \int_t^T R_A V_s^{A,w} \Big(\mu_s(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) \cdot Z_s^{v,a,\mathrm{obs}} + \ell_s(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) \cdot Z_s^{v,a,\cancel{\mathrm{obs}}} - k(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) \Big) ds \\ &+ \int_t^T R_A V_s^{A,w} Z_s^{v,a,\mathrm{obs}} \cdot \Sigma_s(v_s^{\mathbb{P}^v}) dW_s + \int_t^T R_A V_s^{A,w} Z_s^{v,a,\cancel{\mathrm{obs}}} \cdot \Sigma_s^{\perp}(v_s^{\mathbb{P}^v}) dW_s - \int_t^T R_A V_s^{A,w} dA_s^{v,a}, \end{aligned}$$

where we defined

$$Z_t^{v,a,\mathrm{obs}} := -\frac{e^{-R_A K_{0,t}^{\mathbb{P}^{v,a}}}}{R_A V_t^{A,w}} \tilde{Z}_t^{v,a,\mathrm{obs}},\ Z_t^{v,a,\cancel{\mathrm{obs}}} := -\frac{e^{-R_A K_{0,t}^{\mathbb{P}^{v,a}}}}{R_A V_t^{A,w}} \tilde{Z}_t^{v,a,\cancel{\mathrm{obs}}},\ A_t^{v,a} := -\int_0^t \frac{e^{-R_A K_{0,s}^{\mathbb{P}^{v,a}}}}{R_A V_s^{A,w}} d\tilde{A}_s^{v,a},$$

[§§§]Notice that to be completely rigorous, we should then introduce the so-called universal filtration on Ω, completed by the polar sets generated by $\mathscr{P}$, to which the process $V^{A,w}$ would then be adapted; see the references mentioned above.

and for any $s \in [0,T]$ and any $(v,a) \in \mathscr{V} \times \mathscr{A}$

$$\ell_s(v,a) := \Sigma_s^{\perp}(v) b_s(a).$$

Next, using the pathwise construction of the quadratic co-variation of Bichteler (1981), it is actually possible to aggregate the families $(Z^{v,a,\mathrm{obs}})_{(v,a)\in\mathscr{U}}$, $(Z^{v,a,\cancel{\mathrm{obs}}})_{(v,a)\in\mathscr{U}}$ into universal processes Z^{obs} and $Z^{\cancel{\mathrm{obs}}}$ (see for instance Nutz & van Handel (2013)), so that we obtain for all $(v,a) \in \mathscr{U}$, $\mathbb{P}^{v,a} - a.s.$,

$$\begin{aligned} V_t^{A,w} = &- U_A(\xi_T) - \int_t^T R_A V_s^{A,w} \left(\mu_s(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) \cdot Z_s^{\mathrm{obs}} + \ell_s(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) \cdot Z_s^{\cancel{\mathrm{obs}}} - k(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) \right) ds \\ &+ \int_t^T R_A V_s^{A,w} Z_s^{\mathrm{obs}} \cdot \Sigma_s(v_s^{\mathbb{P}^v}) dW_s + \int_t^T R_A V_s^{A,w} Z_s^{\cancel{\mathrm{obs}}} \cdot \Sigma_s^{\perp}(v_s^{\mathbb{P}^v}) dW_s - \int_t^T R_A V_s^{A,w} dA_s^{v,a}. \end{aligned}$$

Under this form, we see that the triplet $(V^{A,w}, (Z^{\mathrm{obs}}, Z^{\cancel{\mathrm{obs}}}), (\overline{A}^{v,a})_{(v,a)\in\mathscr{U}})$, with

$$\overline{A}_t^{v,a} := -R_A \int_0^t V_s^{A,w} dA_s^{v,a},$$

is a solution to the (linear) second-order backward stochastic differential equation (2BSDE for short), as introduced by Soner, Touzi & Zhang (2012), with terminal condition $U_A(\xi_T)$ and generator $F : [0,T] \times \mathbb{R} \times \mathbb{R}^{d_0+1} \times \mathbb{R}^{d-d_0-1}, \times \mathscr{V} \times \mathscr{A} \longrightarrow \mathbb{R}$, defined by

$$F(t, y, z^{\mathrm{obs}}, z^{\cancel{\mathrm{obs}}}, v, a) := R_A y \left(\mu_t(v,a) \cdot z^{\mathrm{obs}} + \ell_t(v,a) \cdot z^{\cancel{\mathrm{obs}}} - k(v,a) \right).$$

Indeed, according to Definition 3.1 in Soner, Touzi & Zhang (2012), the only thing that we have to check is that the family of non-decreasing processes $(\overline{A}^{v,a})_{(v,a)\in\mathscr{U}})$ satisfies the so-called *minimality* condition

$$\overline{A}_t^{v,a} = \underset{(v',a')\in\mathscr{U}_t((v,a))}{\mathrm{essinf}^{\mathbb{P}^{v,a}}} \mathbb{E}_t^{\mathbb{P}^{v',a'}} \left[\overline{A}_T^{v',a'} \right], \text{ for any } (v,a) \in \mathscr{U}.$$

However, this property can be immediately deduced from the definition of the value function $V^{A,w}$ as an essup (see Step (ii) of the proof of Theorem 4.6 in Soner, Touzi & Zhang (2012) for similar arguments). Moreover, solving the above equation, we get

$$\begin{aligned} V_t^{A,w} = V_0^{A,w} &\exp\left[-R_A \int_0^t \left(k(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) - \mu_s(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) \cdot Z_s^{\mathrm{obs}} - \ell_s(v_s^{\mathbb{P}^v}, a_s^{\mathbb{P}^{v,a}}) \cdot Z_s^{\cancel{\mathrm{obs}}} \right) ds \right] \\ &\times \exp\left[-\frac{R_A^2}{2} \int_0^t \left(\left\| \Sigma_s^T(v_s^{\mathbb{P}^v}) Z_s^{\mathrm{obs}} \right\|^2 + \| Z_s^{\cancel{\mathrm{obs}}} \|^2 \right) ds + R_A A_t^{v,a} \right] \\ &\times \exp\left[-R_A \left(\int_0^t Z_s^{\mathrm{obs}} \cdot \Sigma_s(v_s^{\mathbb{P}^v}) dW_s + \int_0^t Z_s^{\cancel{\mathrm{obs}}} \cdot \Sigma_s^{\perp}(v_s^{\mathbb{P}^v}) dW_s \right) \right], \ \mathbb{P}^{v,a} - a.s. \end{aligned} \tag{7.15}$$

The difficulty is that, a priori, we do not know anything about the non-decreasing processes $A^{v,a}$, and thus, it is not in general possible to characterize further the optimal choice of the agent. We will show below that this difficulty disappears when the volatility is not controlled. When it is controlled, there are still cases for which it is possible to obtain further information about $A^{v,a}$, which correspond basically to the situations where the contract ξ_T is smooth enough. So far, the most general result in this direction has been obtained by Peng, Song and Zhang (2014), and can be written in our context as the following assumption, denoting,

$$\begin{aligned} G^{\text{obs}}_{d_0}(s,z,\gamma) &:= \sup_{(v,a)\in\mathscr{V}\times\mathscr{A}} g^{\text{obs}}_0(s,z,\gamma,v,a) \\ &:= \sup_{(v,a)\in\mathscr{V}\times\mathscr{A}} \left\{ \frac{1}{2}\text{Tr}\left[\gamma\Sigma_s(v)\Sigma^T_s(v)\right] + \mu_s(v,a)\cdot z - k(v,a) \right\}. \end{aligned}$$

Assumption 7.1. *The contract ξ_T is such that there exists a $\mathbb{F}^{obs}$-predictable process Γ^{obs} taking values in the set of real $(d_0+1)\times(d_0+1)$ matrices, such that the non-decreasing process $A^{v,a}$ in the decomposition* (7.15) *is of the form*

$$A^{v,a}_t = \int_0^t \left(G^{obs}_{d_0}(s,Z^{obs}_s,\Gamma^{obs}_s) - g^{obs}_{d_0}(s,Z^{obs}_s,\Gamma^{obs}_s,v^{\mathbb{P}^v}_s,a^{\mathbb{P}^{v,a}}_s) \right) ds,\ \mathbb{P}^{v,a}-a.s.$$

We now recall that ξ_T is assumed to be $\mathbb{F}^{\text{obs}}$-measurable. Hence, under Assumption 7.1, plugging the explicit expression of $A^{v,a}$ into (7.15), we see that we necessarily have $Z^{\cancel{\text{obs}}} = 0$ and $\Gamma^{\cancel{\text{obs}}} = 0$. We deduce from the representation of $A^{v,a}$ and (7.15) that for any $\mathbb{P}\in\mathscr{P}$

$$\begin{aligned} U_A(\xi_T) = V^{A,w}_0 \exp&\left[-R_A\int_0^T \left(\frac{1}{2}\text{Tr}\left[\left(\Gamma^{\text{obs}}_s + R_A Z^{\text{obs}}_s (Z^{\text{obs}}_s)^T\right) d\langle B^{\text{obs}}\rangle_s\right] - G^{\text{obs}}_{d_0}(s,Z^{\text{obs}}_s,\Gamma^{\text{obs}}_s)\right) ds\right] \\ &\times\exp\left[-R_A\int_0^T Z^{\text{obs}}_s\cdot dB^{\text{obs}}_s\right],\ \mathbb{P}^{v,a}-a.s. \end{aligned}$$

Now, let us define for any $(v,a)\in\mathscr{U}$

$$Z^{\text{obs},v,a}(B_\cdot) := Z^{\text{obs}}(B^{\text{obs},v,a}(B_\cdot)),\ \Gamma^{\text{obs},v,a}(B_\cdot) := \Gamma^{\text{obs}}(B^{\text{obs},v,a}(B_\cdot)).$$

Then, we deduce by definition of $\mathbb{P}^{v,a}$ that

$$\begin{aligned} U_A(\xi_T) = &-V^{A,w}_0 \exp\left[-R_A\int_0^T \frac{1}{2}\text{Tr}\left[\left(\Gamma^{\text{obs},v,a}_s + R_A Z^{\text{obs},v,a}_s (Z^{\text{obs},v,a}_s)^T\right) d\langle B^{\text{obs},v,a}\rangle_s\right] ds\right] \\ &\times\exp\left[-R_A\left(-\int_0^T G^{\text{obs}}_{d_0}(s,Z^{\text{obs},v,a}_s,\Gamma^{\text{obs},v,a}_s)ds + \int_0^T Z^{\text{obs},v,a}_s\cdot dB^{\text{obs},v,a}_s\right)\right],\ \mathbb{P}_0-a.s., \end{aligned}$$

which is exactly the form given in (4.5).

Step 3: The case of uncontrolled volatility (Holmstrom-Milgrom 1987).

Let us assume now that the set $\mathscr{V}$ is reduced to the singleton $\{v_0\}\subset\mathbb{R}^m$. In this case, the value function of the agent can be rewritten, for any admissible a, as

$$V^{A,w,a}_t := \underset{(v_0,a')\in\mathscr{U}(t,(v_0,a))}{\text{essup}} \mathbb{E}^{\mathbb{P}^{v_0,a'}}_t\left[U_A(\xi_T)e^{R_A K^{v_0,a'}_{t,T}}\right],\ \mathbb{P}^{v_0}-a.s. \tag{7.16}$$

For any a' such that $(v_0,a') \in \mathscr{U}(t,(v_0,a))$, define now

$$V_t^{A,w,a,a'} := \mathbb{E}_t^{\mathbb{P}^{v_0,a'}}\left[U_A(\xi_T)e^{R_A K_{t,T}^{v_0,a'}}\right],\ \mathbb{P}^{v_0}-a.s.$$

Then, $V_t^{A,w,a,a'}$ is an $\mathbb{F}$-martingale under $\mathbb{P}^{v_0,a'}$, so that by the martingale representation, there are $\mathbb{R}^{d_0}$ and $\mathbb{R}^{d-d_0-1}$-valued predictable processes $\tilde{Z}^{\mathrm{obs},a,a'}$ and $\tilde{Z}^{\cancel{\mathrm{obs}},a,a'}$ such that

$$\begin{aligned}V_t^{A,w,a,a'} &= U_A(\xi_T) + \int_t^T R_A V_s^{A,w,a,a'} k(v_0,a'_s)ds - \int_t^T e^{-R_A K_{0,s}^{v_0,a'}} \tilde{Z}_s^{\mathrm{obs},a,a'} \cdot \Sigma_s(v_0)dW_s^{a'} \\ &\quad - \int_t^T e^{-R_A K_{0,s}^{v_0,a'}} \tilde{Z}_s^{\cancel{\mathrm{obs}},a,a'} \cdot \Sigma_s^{\perp}(v_0)dW_s^{a'},\ \mathbb{P}^{v_0}-a.s.\end{aligned}$$

By definition of $W^{a'}$, we deduce that, $\mathbb{P}^{v_0}-a.s.$,

$$\begin{aligned}V_t^{A,w,a,a'} &= U_A(\xi_T) - \int_t^T R_A V_s^{A,w,a,a'}\Big(\mu_s(v_0,a'_s)\cdot Z_s^{\mathrm{obs},a,a'} + \ell_s(v_0,a'_s)\cdot Z_s^{\cancel{\mathrm{obs}},a,a'} - k(v_0,a'_s)\Big)ds \\ &\quad + \int_t^T R_A V_s^{A,w,a,a'} Z_s^{\mathrm{obs},a,a'}\cdot \Sigma_s(v_0)dW_s + \int_t^T R_A V_s^{A,w,a,a'} Z_s^{\cancel{\mathrm{obs}},a,a'}\cdot \Sigma_s^{\perp}(v_0)dW_s,\end{aligned}$$

where we defined

$$Z_t^{\mathrm{obs},a,a'} := -\frac{e^{-R_A K_{0,t}^{v_0,a'}}}{R_A V_t^{A,w,a,a'}}\tilde{Z}_t^{\mathrm{obs},a,a'},\ Z_t^{\cancel{\mathrm{obs}},a,a'} := -\frac{e^{-R_A K_{0,t}^{v_0,a'}}}{R_A V_t^{A,w,a,a'}}\tilde{Z}_t^{\cancel{\mathrm{obs}},a,a'}.$$

For simplify we now set

$$Y_t^{a,a'} := -\frac{\ln\left(-V_t^{A,w,a,a'}\right)}{R_A}.$$

where we recall that $V_t^{A,w,a,a'}$ is negative by definition. Then by Itô's formula, we deduce that the following holds, $\mathbb{P}^{v_0}-a.s.$,

$$\begin{aligned}Y_t^{a,a'} &= \frac{\log(-U_A(\xi_T))}{R_A} + \int_t^T \Big(-\frac{R_A}{2}\left\|\Sigma_s^T(v_0)Z_s^{\mathrm{obs},a,a'}\right\|^2 + \mu_s(v_0,a'_s)\cdot Z_s^{\mathrm{obs},a,a'} + \ell_s(v_0,a'_s)\cdot Z_s^{\cancel{\mathrm{obs}},a,a'} \\ &\quad - k(v_0,a'_s)\Big)ds - \int_t^T Z_s^{\mathrm{obs},a,a'}\cdot \Sigma_s(v_0)dW_s - \int_t^T Z_s^{\cancel{\mathrm{obs}},a,a'}\cdot \Sigma_s^{\perp}(v_0)dW_s.\end{aligned}$$

The above equation can be identified as a linear-quadratic backward SDE with terminal condition $\log(-U_A(\xi_T))/R_A$. Therefore, using the theory of BSDEs with quadratic growth, and in particular the corresponding comparison theorem (see for instance El Karoui, Peng, Quenez (1997), Kobylanski (2000) and Briand and Hu (2008)), we deduce that if $U_A(\xi_T)$ has second order moments under $\mathbb{P}^{v_0}$ and if we define

$$Y_t^a := \operatorname*{essup}_{(v_0,a')\in\mathscr{U}(t,(v_0,a))} Y_t^{a,a'},$$

then there exists a $\mathbb{F}^{\mathrm{obs}}$-predictable process Z^a and a $\mathbb{F}^{\cancel{\mathrm{obs}}}$-predictable process $Z^{\cancel{\mathrm{obs}},a}$ such that, $\mathbb{P}^{\nu_0}-a.s.$,

$$
\begin{aligned}
Y_t^a = \frac{\log(-U_A(\xi_T))}{R_A} &+ \int_t^T \sup_{a\in\mathscr{A}} \left\{\mu_s(v_0,a)\cdot Z_s^{\mathrm{obs},a} + \ell_s(v_0,a)\cdot Z_s^{\cancel{\mathrm{obs}},a} - k(v_0,a)\right\} ds \\
&- \int_t^T \frac{R_A}{2}\left(\left\|\Sigma_s^T(v_0) Z_s^{\mathrm{obs},a}\right\|^2 + \left\|Z_s^{\cancel{\mathrm{obs}},a}\right\|^2\right) ds - \int_t^T Z_s^{\mathrm{obs},a}\cdot\Sigma_s(v_0)dW_s - Z_s^{\cancel{\mathrm{obs}},a}\cdot\Sigma_s^{\perp}(v_0)dW_s,
\end{aligned}
$$

that is $(Y^a, Z^{\mathrm{obs},a}, Z^{\cancel{\mathrm{obs}},a})$ solves the backward SDE with terminal condition $\log(-U_A(\xi_T))/R_A$ and generator f, where

$$
f(s,z^1,z^1) := -\frac{R_A}{2}\left(\left\|\Sigma_s^T(v_0)z^1\right\|^2 + \left\|z^2\right\|^2\right) + \sup_{a\in\mathscr{A}}\left\{\mu_s(v_0,a)\cdot z^1 + \ell_s(v_0,a)\cdot z^2 - k(v_0,a)\right\}.
$$

By the assumptions of the theorem on the cost function k, it can be easily shown, using the linear growth of μ and ℓ in a, that the sup in the definition of f is always attained for some $a^*(s,z^1,z^2)$ that satisfies

$$
\left\|a^*(s,z^1,z^2)\right\| \leq C_0\left(1 + \left\|z^1\right\|^{\frac{1}{\varepsilon}} + \left\|z^2\right\|^{\frac{1}{\varepsilon}}\right).
$$

In particular this implies that the above BSDE is quadratic in z, and is therefore indeed well-posed. Finally, we deduce that

$$
\begin{aligned}
V_t^{A,w,a} = V_0^{A,w} &\exp\left[-R_A\int_0^t \left(k(v_0,a_s) - \mu_s(v_0,a_s)\cdot Z_s^{\mathrm{obs},a} - \ell_s(v_0,a_s)\cdot Z_s^{\cancel{\mathrm{obs}},a}\right) ds\right] \\
&\times \exp\left[-\frac{R_A^2}{2}\int_0^t \left(\left\|\Sigma_s^T(v_0) Z^{\mathrm{obs},a}\right\|^2 + \left\|Z_s^{\cancel{\mathrm{obs}},a}\right\|^2\right) ds + R_A A_t^a\right] \\
&\times \exp\left[-R_A\left(\int_0^t Z_s^{\mathrm{obs},a}\cdot\Sigma_s(v_0)dW_s + \int_0^t Z_s^{\cancel{\mathrm{obs}},a}\cdot\Sigma_s^{\perp}(v_0)dW_s\right)\right],\ \mathbb{P}^{\nu_0}-a.s.,
\end{aligned}
$$

where the non-decreasing process A^a is defined by

$$
\begin{aligned}
A_t^a = \int_0^t \sup_{a\in\mathscr{A}} \Big\{ &\mu_s(v_0,a)\cdot Z_s^{\mathrm{obs},a} + \ell_s(v_0,a)\cdot Z_s^{\cancel{\mathrm{obs}},a} - k(v_0,a) \\
&- \frac{R_A}{2}\left(\left\|\Sigma_s^T(v_0)Z_s^{\mathrm{obs},a}\right\|^2 + \left\|Z_s^{\cancel{\mathrm{obs}},a}\right\|^2\right)\Big\} ds \\
- \int_0^t \Big(&\mu_s(v_0,a_s)\cdot Z_s^{\mathrm{obs},a} + \ell_s(v_0,a_s)\cdot Z_s^{\cancel{\mathrm{obs}},a} - k(v_0,a_s) \\
&- \frac{R_A}{2}\left(\left\|\Sigma_s^T(v_0)Z_s^{\mathrm{obs},a}\right\|^2 + \left\|Z_s^{\cancel{\mathrm{obs}},a}\right\|^2\right)\Big) ds,\ \mathbb{P}^{\nu_0}-a.s.
\end{aligned}
$$

Again, we must have $Z^{\cancel{\mathrm{obs}},a} = 0$ in order to ensure that ξ_T is $\mathscr{F}_T^{\mathrm{obs}}$-measurable, so that we have

$$
A_t^a = \int_0^t \left(G_{d_0}^{\mathrm{obs}}(s, Z_s^{\mathrm{obs}}, 0) - g_{d_0}^{\mathrm{obs}}(s, Z_s^{\mathrm{obs}}, 0, v_0, a_s)\right) ds,\ \mathbb{P}^{\nu_0}-a.s.,
$$

which means that Assumption 7.1 is satisfied. $\square$

Step 4: Proof of statement (iii) in Theorem 4.2.

Let us start by giving an alternative representation formula for the value function of the agent $V^{A,w}$. Define first

$$Y_t^{A,w} := -\frac{\log(-V_t^{A,w})}{R_A},\ t \in [0,T].$$

Let us also define the following set for any $t \in [0,T]$ and any symmetric positive $d \times d$ matrix α

$$\Delta(t,\alpha) := \left\{ \nu \in \mathscr{V},\ \begin{pmatrix} \Sigma_s(\nu)\Sigma_s(\nu)^T & 0_{d_0+1,d-d_0-1} \\ 0_{d-d_0-1,d_0+1} & I_{d-d_0-1} \end{pmatrix} = \alpha \right\}.$$

We claim that $Y^{A,w}$ solves the following 2BSDE, defined for any probability measure $\mathbb{P} \in \mathscr{P}_m$

$$Y_t^{A,w} = \xi_T + \int_t^T \overline{F}(s, Z_s^{\text{obs}}, Z_s^{\cancel{\text{obs}}}, \widehat{\alpha}_s)ds - \int_t^T ((Z_s^{\text{obs}})^T, (Z_s^{\cancel{\text{obs}}})^T)^T \cdot dB_s + \int_t^T dK_s^{\mathbb{P}},\ \mathbb{P}-a.s.,$$

where

$$\overline{F}(t, z^{\text{obs}}, z^{\cancel{\text{obs}}}, \alpha) := \sup_{\nu \in \Delta(t,\alpha)} \left\{ \mu_t(\nu, a_0) \cdot z^{\text{obs}} + \ell_t(\nu, a_0) \cdot z^{\cancel{\text{obs}}} - k(\nu, a) \right\}$$
$$- \frac{R_A}{2}(z^{\text{obs}})^T \alpha_{d_0+1,d_0+1} z^{\text{obs}} - \frac{R_A}{2} \left\| z^{\cancel{\text{obs}}} \right\|^2,$$

where α_{d_0+1,d_0+1} denotes the submatrix of α consisting of the first d_0+1 rows and columns and where the family $(K^{\mathbb{P}^\nu})_{\nu \in \mathscr{V}}$ satisfies the minimality condition

$$K_t^{\mathbb{P}^\nu} = \underset{\nu' \in \mathscr{V}, \nu_s = \nu'_s,\ s \leq t}{\operatorname{essinf}^{\mathbb{P}^\nu}} \mathbb{E}_t^{\mathbb{P}^{\nu'}} \left[K_T^{\mathbb{P}^{\nu'}} \right].$$

Since a_0 is fixed, it is clear that the sup in the definition of $\overline{F}$ is actually Lipschitz continuous in $(z^{\text{obs}}, z^{\cancel{\text{obs}}})$ in the sense that for some constant $C > 0$

$$\left| \overline{F}(t, z^{\text{obs}}, z^{\cancel{\text{obs}}}, \widehat{\alpha}_t) - \overline{F}(t, z^{\text{obs},\prime}, z^{\cancel{\text{obs}},\prime}, \widehat{\alpha}_t) \right| \leq C \left(\left\| \Sigma_t(\nu^{\mathbb{P}})(z^{\text{obs}} - z^{\text{obs},\prime}) \right\| + \left\| \Sigma_t^{\perp}(\nu^{\mathbb{P}})(z^{\cancel{\text{obs}}} - z^{\cancel{\text{obs}},\prime}) \right\| \right).$$

Since the remaining terms in $\overline{F}$ are purely quadratic, the generator $\overline{F}$ is actually concave in $(z^{\text{obs}}, z^{\cancel{\text{obs}}})$, so that by Briand and Hu (2008), we know that the corresponding BSDEs with generator $\overline{F}$ and terminal condition ξ_T admit a unique solution (recall that ξ_T is assumed to have exponential moments of any order, under any of the measures $\mathbb{P} \in \mathscr{P}_m$). Since these solutions can be approximated by the solutions to Lipschitz BSDEs, we can use Proposition 2.1 and Remark 4.2 of Possamaï, Tan and Zhou (2015) to conclude to the wellposedness of the above 2BSDE. Furthermore, thanks to Remark 4.1 of Possamaï, Tan and Zhou (2015), we can also aggregate the family $K^{\mathbb{P}}$ into a universal process K.

With this in hand, the proof that $Y_t^{A,w}$ does solve the above 2BSDE is the same as the proof of Proposition 4.5 in Cvitanić, Possamaï and Touzi (2015). In particular, it turns out that a contract ξ_T belongs to $\overline{\mathfrak{C}}$ if and only if there is some probability measure $\mathbb{P}^* \in \mathscr{P}_m$ such that $K_T = 0$, $\mathbb{P}^* - a.s.$

Since such a measure may not be unique, let us denote by $\mathscr{P}^*(\xi_T)$ the corresponding set. Let us then define the value function of the principal

$$U_0^{P,w} := \sup_{\xi_T \in \overline{\mathfrak{C}}} \sup_{\mathbb{P}^* \in \mathscr{P}^*(\xi_T)} \mathbb{E}^{\mathbb{P}^*}\left[U_P(B_T^1 - \xi_T)\right]. \tag{7.17}$$

The remainder of the proof is then exactly the same as the proof of Theorem 4.6 in Cvitanić, Possamaï and Touzi (2015). □

Proposition 7.1. (Sufficient conditions for the existence of optimal v in the non-contractible case.) *Suppose $a = 0$ and B^1 cannot be contracted upon. Let $\mathscr{I}$ be the subset of $\{1,\dots,d\}$ such that for every $j \in \mathscr{I}$, $\beta_j = \min_i \beta_i$ and assume that $b_j \neq 0$ for $j \in \mathscr{I}$. If either of the following holds*

(i) $\mathrm{Card}(\mathscr{I}) > 1$ *and there is at least one pair $(i,j) \in \mathscr{I} \times \mathscr{I}$ such that $\alpha_j\beta_j/b_j \neq \alpha_i\beta_i/b_i$.*

(ii) $\mathrm{Card}(\mathscr{I}) \geq 1$, *for all $(i,j) \in \mathscr{I} \times \mathscr{I}$, $i \neq j$, $\alpha_j\beta_j/b_j = \alpha_i\beta_i/b_i =: \eta$ and*

$$\frac{(1+\eta)^2\left(\sum_{i\in\mathscr{I}} b_i\right)^2}{2\left(\mathrm{Card}(\mathscr{I})R_A\eta^2 + \mathrm{Card}(\mathscr{I})R_P(1-\eta)^2 + \sum_{i\in\mathscr{I}}\beta_i\right)} + \sum_{i\notin\mathscr{I}}(b_i+\alpha_i\beta_i)\frac{-\eta b_i + \alpha_i\beta_i}{\beta_i - \min_j\beta_j}$$
$$-\frac{1}{2}\sum_{i\notin\mathscr{I}}\frac{(-\eta b_i+\alpha_i\beta_i)^2}{(\beta_i-\min_j\beta_j)^2}\left(R_A\eta^2 + R_P(1-\eta)^2+\beta_i\right) \leq \sum_{i=1}^d \frac{(b_i+\alpha_i\beta_i)^2}{2(\beta_i+R_A)}.$$

Then, there exists at least one couple $(Z^*,\Gamma^*) \in \mathbb{R}\times(-\infty,\min_i\beta_i)$ attaining the maximum in the principal's problem [¶¶¶] .

Proof of Proposition 7.1. If for some i, we have $\Gamma - \beta_i > 0$, or $\Gamma - \beta_i = 0$, but $Z \neq 0$, it is easily verified that the agent chooses optimally $|v_i^*(Z,\Gamma)| = \infty$, and that this cannot be optimal for the principal. Thus, we can optimize under the constraint $\Gamma - \min_j \beta_j \leq 0$.

In the case in which $\Gamma - \beta_i = 0$ for some i and $Z = 0$, the principal has to maximize over all admissible values of v_i, because the agent is indifferent among those. It can be verified that for $Z = 0$ and in that case there exists an optimal v^* for the problem (5.2).

The only remaining case is $\Gamma - \beta_i < 0$ for all i. Then, the contract is incentive compatible for

$$v_i^*(Z,\Gamma) = \frac{b_iZ + \alpha_i\beta_i}{\beta_i - \Gamma}, \quad i = 1,\dots,d.$$

From (5.2), the principal's problem is then to maximize

$$v^*\cdot b - \frac{1}{2}\|v^*\|^2\left(R_AZ^2 + R_P(1-Z)^2\right) - \frac{1}{2}\sum_{i=1}^d \beta_i\left(v_i^* - \alpha_i\right)^2,$$

[¶¶¶] Notice that the left-hand side in (ii) above can be made negative, if for instance $\mathrm{Card}(\mathscr{I}) \leq d-1$ and $\min_{i\notin\mathscr{I}}\beta_i$ is sufficiently close to $\min_i\beta_i$, so that the condition can, indeed, be satisfied in examples.

which is the same as maximizing

$$f(Z,\Gamma) := \sum_{i=1}^{d}(b_i+\alpha_i\beta_i)\frac{b_iZ+\alpha_i\beta_i}{\beta_i-\Gamma} - \frac{1}{2}\sum_{i=1}^{d}\frac{(b_iZ+\alpha_i\beta_i)^2}{(\beta_i-\Gamma)^2}\left(R_AZ^2+R_P(1-Z)^2+\beta_i\right). \qquad (7.18)$$

We will show that the maximization of (7.18) over all $(Z,\Gamma)\in\mathbb{R}\times(-\infty,\min_i\beta_i)$ can be reduced to a maximization over a compact set strictly included in $\mathbb{R}\times(-\infty,\min_i\beta_i)$. First, notice that by taking $(Z,\Gamma)=(1,-R_A)$, (7.18) becomes

$$\sum_{i=1}^{d}\frac{(b_i+\alpha_i\beta_i)^2}{2\,(\beta_i+R_A)}\geq 0,$$

which implies that the maximum in (7.18) is non-negative.

Let us now show that the maximum in (7.18) can never be achieved on the boundary of the domain $\mathbb{R}\times(-\infty,\min_i\beta_i)$. Let us distinguish several cases:

(i) Z goes to $\pm\infty$ and Γ remains bounded and does not go to $\min_i\beta_i$. Then, it is easily seen that f goes to $-\infty$. This is therefore suboptimal.

(ii) Z remains bounded and Γ goes to $-\infty$. Then, f goes to 0 which is again suboptimal.

(iii) Z goes to $\pm\infty$ and Γ goes to $-\infty$. Then, f can either go to $-\infty$ or 0, depending on whether Z/Γ remains bounded or not.

(iv) Γ goes to $\min_i\beta_i$ and Z goes to $\pm\infty$. Then, f goes to $-\infty$.

(v) Γ goes to $\min_i\beta_i$ and Z remains bounded. Then, we have to distinguish between three sub-cases.

- If Z is fixed and $Z\neq-\alpha_j\beta_j/b_j$, for every $j\in\mathscr{I}$, then, f goes to $-\infty$.
- If Z goes to $-\alpha_j\beta_j/b_j$ for some $j\in\mathscr{I}$, $\mathrm{Card}(\mathscr{I})>1$ and there is at least one $j_0\in\mathscr{I}\backslash\{j\}$ such that $\alpha_{j_0}\beta_{j_0}/b_{j_0}\neq\alpha_j\beta_j/b_j$. Then, f still goes to $-\infty$.
- If Z goes to $-\alpha_j\beta_j/b_j$ for some $j\in\mathscr{I}$, $\mathrm{Card}(\mathscr{I})\geq 1$ and for all $(i,j)\in\mathscr{I}\times\mathscr{I}$, $i\neq j$, $\alpha_j\beta_j/b_j=\alpha_i\beta_i/b_i=:\eta$, then, if $\frac{b_jZ+\alpha_j\beta_j}{\beta_j-\Gamma}$ goes to $\pm\infty$, then, f goes to $-\infty$. If $\frac{b_jZ+\alpha_j\beta_j}{\beta_j-\Gamma}$ remains bounded, then, the maximum value that can be achieved by f is

$$\begin{aligned}&\sup_{u\in\mathbb{R}}\left\{(1+\eta)\sum_{i\in\mathscr{I}}b_iu-\frac{1}{2}u^2\left(\mathrm{Card}(\mathscr{I})R_A\eta^2+\mathrm{Card}(\mathscr{I})R_P(1-\eta)^2+\sum_{i\in\mathscr{I}}\beta_i\right)\right\}\\&+\sum_{i\notin\mathscr{I}}^{d}(b_i+\alpha_i\beta_i)\frac{-\frac{\alpha_j\beta_jb_i}{b_j}+\alpha_i\beta_i}{\beta_i-\beta_j}\\&-\frac{1}{2}\sum_{i\notin\mathscr{I}}^{d}\frac{(-\frac{\alpha_j\beta_jb_i}{b_j}+\alpha_i\beta_i)^2}{(\beta_i-\beta_j)^2}\left(R_A\left(\frac{\alpha_j\beta_j}{b_j}\right)^2+R_P\left(1-\frac{\alpha_j\beta_j}{b_j}\right)^2+\beta_i\right),\end{aligned}$$

which is equal to

$$\frac{(1+\eta)^2\left(\sum_{i\in\mathscr{I}} b_i\right)^2}{2\left(\mathrm{Card}(\mathscr{I})R_A\eta^2+\mathrm{Card}(\mathscr{I})R_P\left(1-\eta\right)^2+\sum_{i\in\mathscr{I}}\beta_i\right)}+\sum_{i\notin\mathscr{I}}(b_i+\alpha_i\beta_i)\frac{-\eta b_i+\alpha_i\beta_i}{\beta_i-\min_j\beta_j}$$
$$-\frac{1}{2}\sum_{i\notin\mathscr{I}}\frac{(-\eta b_i+\alpha_i\beta_i)^2}{(\beta_i-\min_j\beta_j)^2}\left(R_A\eta^2+R_P\left(1-\eta\right)^2+\beta_i\right). \tag{7.19}$$

However, by assumption, this is again sub-optimal.

□

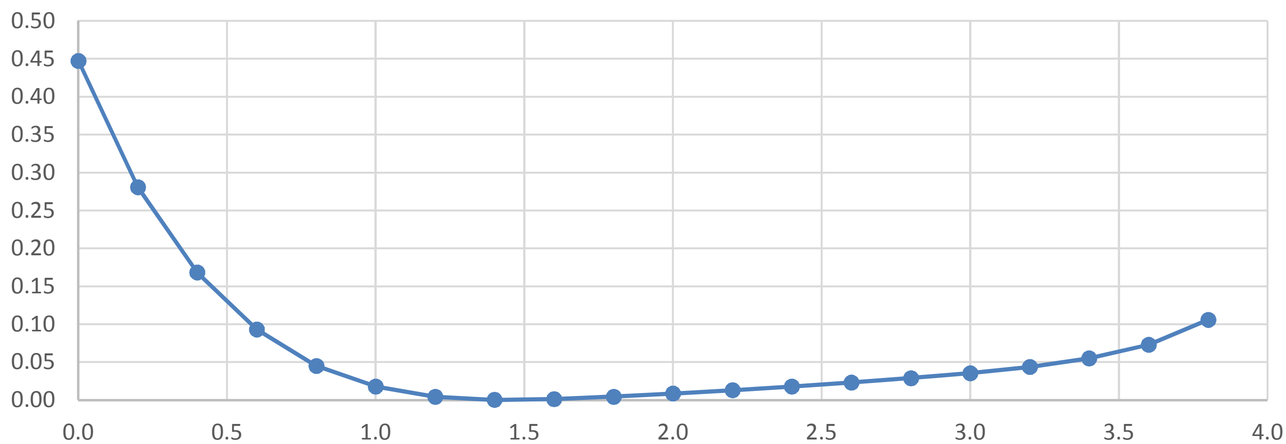

Figure 1: Percentage loss in principal's certainty equivalent relative to first best, as function of α_2.

Parameter values: R_A=10, R_P=0.58, α_1=0.5, β_1=0.4, β_2=1, b_1=0.4, b_2=1, B_0=0.

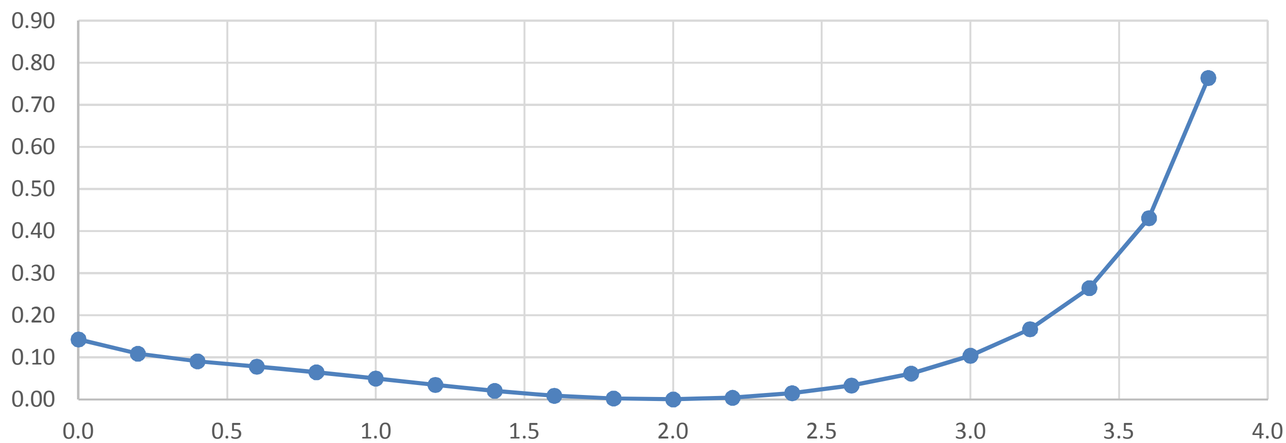

Figure 2: Percentage loss in principal's certainty equivalent when not using quadratic variation, as function of α_2.
Parameter values: R_A=10, R_P=0.58, α_1=0.5, β_1=0.4, β_2=1, b_1=0.4, b_2=1, B_0=0.

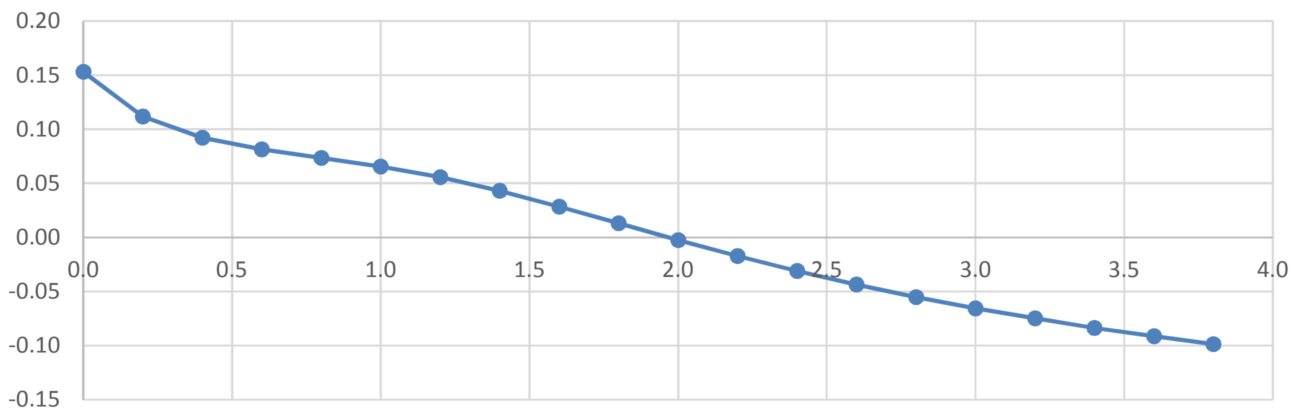

Figure 3: Optimal contract's sensitivity to quadratic variation, as function of α_2. Parameter values: R_A=10, R_P=0.58, α_1=0.5, β_1=0.4, β_2=1, b_1=0.4, b_2=1, B_0=0.