BEM 105, HMWK 5, Due Monday, Mar 13 2006 (hand in to Patricia Hamad, Baxter 112 or put in Cvitanic mailbox)

1. The one-month and two-month interest rates are 4% and 4.1%, respectively. Our model of the term structure says that one month from now the one-month interest rate will be either 3.5% or 4.5%. Compute the price of an interest rate derivative that pays \$1 in one month if the one-month interest rate is 4.5% and \$0.5 if the one-month interest rate is 3.5%. (**NOTE:** the interest rates quoted are the spot rates that have been annualized using compounding.)

2. Consider the HJM model with

$$\sigma(t,T) = \sigma e^{-a(T-t)} \quad ,$$

for given positive constants σ and a. (In this model the forward rates volatility goes down with shorter time to maturity.) Show that we can write

$$dr(t) = m'(t)dt + d\left(e^{-at}\int_0^t \sigma e^{au}dW(u)\right) = \left[m'(t) - a(r(t) - m(t))\right]dt + \sigma dW(t)$$

for some deterministic function m. This is recognized as a Hull-White model.

3. Consider the Hull-White model

$$dr = [b(t) - ar]dt + \sigma dW.$$

Show that the bond price is given as

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)}$$

where

$$B(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}], \quad A(t,T) = \int_t^T [\frac{\sigma^2}{2}B^2(s,T) - b(s)B(s,T)]dt$$

4. In the previous problem, which expression for volatility would you use in the Black-Scholes formula in order to price the call option $[P(T_1,T_2) - K]^+$ on the bond in this model?

5. Denote $P = P(T_{i-1}, T_i)$ the bond price at the time T_{i-1}, maturing at T_i, and let f be a given function. Show that the time-t value of the payoff of $C = f(P)$ dollars paid at time T_i is the same as the time-t value of the payoff of $P \cdot C$ dollars paid at time T_{i-1}. Is this statement still true if we replace $P = P(T_{i-1}, T_i)$ with $P = P(T_i, T_{i+1})$? You can either use the expectation formula (8.1), or a no-arbitrage argument.